Quadric surfaces

A quadric surface is a surface in $\mathbb{R}^3$ of the form $x^T A x + b \cdot x + c = 0$, where $A$ is a $3 \times 3$ symmetric matrix, $b$ is a vector in $\mathbb{R}^3$ and $c$ is a scalar. That is:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + b_1x + b_2y + b_3z + c = 0$$

Some special cases include the following. [See supplemental handout for sketches.]

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Elliptic cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ — [think about what a 'hyperbolic cone' would be]
- Elliptic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- Hyperbolic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Quadratic forms revisited (not in Colley)

Every quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ defines a quadric surface. If $q(x,y,z) = x^T A x$, with $A$ symmetric, then we obtain

$$q(x,y,z) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $A$ and $(c_1, c_2, c_3)$ are the coordinates of $(x,y,z)$ with respect to the orthonormal eigenbasis $u_1, u_2, u_3$.

We can then sketch $q(x,y,z) = 1$ on $(c_1, c_2, c_3)$-axes, just as we did for ellipses and hyperbolae in two dimensions. Note that a surface of the form $x^T A x = 1$ will either be an ellipsoid, a hyperboloid of one or two sheets, or an (elliptic or hyperbolic) cylinder (this occurs when one of the eigenvalues is zero).
1. Describe (and try to sketch) the following surfaces:

(a) \( \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \)

   Ellipse w/ x-intercepts ±2, y-intercepts ±3 & z-intercepts ±1.

(b) \( \frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1 \)

   Hyperboloid of one sheet along z-axis w/ x-intercepts ±2 & y-intercepts ±3

(c) \( \frac{x^2}{4} - \frac{y^2}{9} - z^2 = 1 \)

   Hyperboloid of two sheets along x-axis w/ x-intercepts ±2
(d) \[ \frac{x^2}{4} - \frac{y^2}{9} + z^2 = 1 \]

Hyperboloid of one sheet along y-axis. x-intercepts \( \pm 2 \) & z-intercepts \( \pm 1 \)

(e) \[ \frac{x^2}{4} + \frac{y^2}{9} - z^2 = 0 \quad \leftrightarrow \quad z = \left( \frac{x}{2} \right)^2 + \left( \frac{y}{3} \right)^2 \]

Elliptic paraboloid along z-axis.

(f) \[ \frac{x^2}{4} - \frac{y^2}{9} + z^2 = 0 \quad \leftrightarrow \quad x = \left( \frac{y}{3\sqrt{2}} \right)^2 - \left( \frac{z}{\sqrt{12}} \right)^2 \]

Hyperbolic paraboloid. Intersections with \( xy \)-plane are hyperbolae, intersections with planes parallel to the \( xz \)- & \( yz \)-planes are parabolae.
(g) \[ x^2 + y^2 + z^2 + 2x - 4y + 6z = 2 \]
\[ \equiv (x + 1)^2 + (y - 2)^2 + (z + 3)^2 = 16 \]
\[ \equiv \left(\frac{x + 1}{4}\right)^2 + \left(\frac{y - 2}{4}\right)^2 + \left(\frac{z + 3}{4}\right)^2 = 1 \]
\[ \text{Sphere of radius 4 centred at } (-1, 2, -3) \]

(h) \[(x, y, z) \cdot (y, z, x) = 1 \]
\[ \equiv xy + yz + zx = 1 \]
\[ \equiv x^T A x = 1 \quad \text{where } A = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \]

Orthonormal eigenbasis of \(A\):
\[ B = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \]

Eigenvalues: \[1, \quad -\frac{1}{2}, \quad -\frac{1}{2} \]

So in \(B\)-coordinates, the surface is described by
\[ c_1^2 - \frac{1}{2} c_2^2 - \frac{1}{2} c_3^2 = 1 \]

Hyperboloid of 2 sheets intersecting \(c_1\)-axis at \(\pm 1\).