Partial derivatives

Given a function \( f : \mathbb{R}^n \to \mathbb{R} \), we can find the rate of change of \( f(x_1, x_2, \ldots, x_n) \) as just one of its variables \( x_i \) varies by differentiating \( f \) with respect to \( x_i \) and holding all other variables constant.

The resulting function \( \mathbb{R}^n \to \mathbb{R} \) is called the partial derivative of \( f \) with respect to \( x_i \), and can be written as \( \frac{\partial f}{\partial x_i} \) or simply \( f_{x_i} \). For example, if \( f \) is a function \( \mathbb{R}^2 \to \mathbb{R} \):

- \( \frac{\partial f}{\partial x}(a, b) \) gives the slope of the curve \( z = f(x, b) \) at \( x = a \);
- \( \frac{\partial f}{\partial y}(a, b) \) gives the slope of the curve \( z = f(a, y) \) at \( y = b \).

Note that the lines containing \((a, b, f(a, b))\) and parallel to the vectors \((1, 0, f_x(a, b))\) and \((0, 1, f_y(a, b))\), respectively, are both tangent to the surface \( z = f(x, y) \) at the point \((a, b, f(a, b))\).

These lines lie on a plane, called the tangent plane to the graph of \( f \) at \((a, b)\).

The normal vector to the tangent plane is given by \((1, 0, f_x(a, b)) \times (0, 1, f_y(a, b)) = (f_x(a, b), f_y(a, b), -1)\), and so the equation of the tangent plane is

\[
f_x(a, b)(x - a) + f_y(a, b)(y - a) - (z - f(a, b)) = 0
\]

or equivalently

\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

The gradient vector

Given a function \( f : \mathbb{R}^n \to \mathbb{R} \) and a point \( \vec{a} \) in \( \mathbb{R}^n \), the gradient vector of \( f \) at \( \vec{a} \) is the vector \( \nabla f \) defined by

\[
\nabla f(\vec{a}) = (f_{x_1}(\vec{a}), f_{x_2}(\vec{a}), \ldots, f_{x_n}(\vec{a}))
\]

For example if \( f : \mathbb{R}^2 \to \mathbb{R} \) and \((a, b)\) is a point in \( \mathbb{R}^n \), then

\[
\nabla f(a, b) = \left( \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right)
\]

This gives us a nice expression for the tangent plane of \( f \), namely

\[
z = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})
\]

where we have written \( \vec{a} = (a, b) \) and \( \vec{x} = (x, y) \).
1. Find the partial derivatives of the following functions.

(a) \( f(x,y) = x^2 + y^2 \)

(b) \( g(x,y) = \sin(x+y)\cos(x-y) \)

(c) \( h(x,y) = e^{xy^2+yz^2+zx^2} \)
2. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ and points $(a, b)$ in $\mathbb{R}^2$, find the equation of the tangent plane to the graph of $f$ at $(a, b)$. Draw a sketch if you can.

(a) $f(x, y) = x^2 + y^2$; $(a, b) = (1, 1)$.

(b) $f(x, y) = x \cos y - y \sin x$; $(a, b) = (0, \frac{\pi}{4})$. 
3. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, sketch the level curves of its graph and indicate the direction of $\nabla f(a, b)$ at a few points $(a, b)$ of your choosing.

(a) $f(x, y) = x^2 + y^2$

(b) $f(x, y) = x^2 - y^2$