Directional derivatives

The **directional derivative** \( D_u f(a) \) of a function \( f : \mathbb{R}^n \to \mathbb{R} \) at a point \( a \) in a given direction (unit vector) \( u \) is defined by

\[
D_u (a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}
\]

Notice that for a function \( f : \mathbb{R}^2 \to \mathbb{R} \) we have

\[
f_x(a, b) = D_1 f(a, b) \quad \text{and} \quad f_y(a, b) = D_2 f(a, b)
\]

Fun fact: if \( f \) is differentiable at \( a \), then \( D_u f(a) = \| \nabla f(a) \| \cos \theta \) (so \( D_u = u \cdot \nabla \)). In particular:

\[
D_u f(a) = \| \nabla f(a) \| \cos \theta
\]

where \( \theta \) is the angle between \( \nabla f(a) \) and \( u \) (with \( 0 \leq \theta \leq \pi \)).

Some fun consequences:

(i) \( D_u f(a) \) is maximised when \( u \) points in the same direction as \( \nabla f(a) \)—thus \( \nabla f(a) \) points in the direction of fastest increase of \( f \);

(ii) \( D_u f(a) \) is minimised when \( u \) points in the opposite direction from \( \nabla f(a) \)—thus \( -\nabla f(a) \) points in the direction of fastest decrease of \( f \);

(iii) \( D_u f(a) = 0 \) when \( u \perp \nabla f(a) \).

In fact, (iii) implies that:

- For a function \( f : \mathbb{R}^2 \to \mathbb{R} \) and a point \( (a, b) \), the vector \( \nabla f(a, b) \) is perpendicular to (the tangent line to) the level curve of \( f \) at \( (a, b) \);

- For a function \( f : \mathbb{R}^3 \to \mathbb{R} \) and a point \( (a, b, c) \), the vector \( \nabla f(a, b, c) \) is perpendicular to (the tangent plane to) the level surface of \( f \) at \( (a, b, c) \).
1. Compute the directional derivative of the function $f(x, y) = x^2y + y^2x$ at $(1, 2)$ in the direction of the vector $(-2, 1)$.

2. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function, let $\mathbf{u} = (u, v)$ be a unit vector in $\mathbb{R}^2$, and let $\mathbf{a} = (a, b)$ in $\mathbb{R}^2$. Assuming $f$ is differentiable at $\mathbf{a}$, use the chain rule to show that $D_{\mathbf{u}}f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a})$. 
3. Find the direction(s) in which the function \( f(x,y) = e^{x+y}(x^2 + y^2) \) is increasing most rapidly at the point \((1, 2)\).

4. Find an equation for the tangent plane to the surface \( 2x^2 + y^2 - z^2 = 4 \) at the point \((2, 0, 2)\).