Directional derivatives

The **directional derivative** $D_u f(a)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $a$ in a given direction (unit vector) $u$ is defined by

$$D_u(a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}$$

Notice that for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$$f_x(a, b) = D_1 f(a, b) \quad \text{and} \quad f_y(a, b) = D_2 f(a, b)$$

Fun fact: if $f$ is differentiable at $a$, then $D_u f(a) = u \cdot \nabla f(a)$ (so ‘$D_u = u \cdot \nabla$’). In particular:

$$D_u f(a) = \|\nabla f(a)\| \cos \theta$$

where $\theta$ is the angle between $\nabla f(a)$ and $u$ (with $0 \leq \theta \leq \pi$).

Some fun consequences:

(i) $D_u f(a)$ is maximised when $u$ points in the same direction as $\nabla f(a)$—thus $\nabla f(a)$ points in the direction of fastest increase of $f$;

(ii) $D_u f(a)$ is minimised when $u$ points in the opposite direction from $\nabla f(a)$—thus $-\nabla f(a)$ points in the direction of fastest decrease of $f$;

(iii) $D_u f(a) = 0$ when $u \perp \nabla f(a)$.

In fact, (iii) implies that:

- For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $(a, b)$, the vector $\nabla f(a, b)$ is perpendicular to (the tangent line to) the level curve of $f$ at $(a, b)$;

- For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a point $(a, b, c)$, the vector $\nabla f(a, b, c)$ is perpendicular to (the tangent plane to) the level surface of $f$ at $(a, b, c)$.
1. Find the direction(s) in which the function \( f(x, y) = e^{x+y}(x^2 + y^2) \) is increasing most rapidly at the point \((1, 2)\).

2. Find an equation for the tangent plane to the surface \( 2x^2 + y^2 - z^2 = 4 \) at the point \((2, 0, 2)\).
3. Define functions \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( h : \mathbb{R}^2 \to \mathbb{R} \) by

\[
g(x, y) = \left( \frac{e^{x+y} + e^{x-y}}{2}, \frac{e^{x+y} - e^{x-y}}{2} \right) \quad \text{and} \quad h(s, t) = 2st
\]

Find \( D_{(-1,2)} f(2, 1) \), where \( f(x, y) = h(g(x, y)) \).
4. Peeve the guinea pig is standing on a steep Andes mountain. Conveniently, the mountain looks just like the elliptic paraboloid $3x^2 + 7y^2 + 5z = 20$. Owing to her short legs, the steepest grade that Peeve can climb up the mountain is $\frac{1}{5}$.

Given that Peeve’s $(x, y)$-coordinates are $(1, 1)$ and she is facing in the direction of steepest ascent, find the smallest angle that Peeve must turn in order to be able to ascend the mountain.