1. Find the direction(s) in which the function \( f(x,y) = e^{x+y}(x^2+y^2) \) is increasing most rapidly at the point \((1,2)\).

\[
\nabla f(1,2) = \left( e^{x+y} \left( x^2+y^2 \right) + 2xe^{x+y} , \ 2e^{x+y} \left( x^2+y^2 \right) + 2ye^{x+y} \right) \bigg|_{(1,2)}
\]

\[
= e^{x+y} \left( x^2+y^2 + 2x , \ x^2+y^2 + 2y \right) \bigg|_{(1,2)}
\]

\[
= e^3 \left( 7 , \ 9 \right)
\]

\[
\| \nabla f(1,2) \| = e^3 \sqrt{49+81} = e^3 \sqrt{130}
\]

So the direction of most rapid increase is \( \frac{4}{\sqrt{130}} \left( 7,9 \right) \).

2. Find an equation for the tangent plane to the surface \( 2x^2 + y^2 - z^2 = 4 \) at the point \((2,0,2)\).

The surface \( 2x^2 + y^2 - z^2 = 4 \) is the level surface \( (k = 4) \) of the function \( f(x,y,z) = 2x^2 + y^2 - z^2 \).

\((2,0,2)\) is a point on the tangent plane.

\( \nabla f(2,0,2) \) is \( \perp \) to the tangent plane.

\[
\nabla f(2,0,2) = \left( 8, 0, -4 \right)
\]

So the equation of the tangent plane at \((2,0,2)\) is

\[
2(x-2) + 0(y-0) + (-4)(z-2) = 0
\]

i.e. \( 2x - z = 2 \).
3. Define functions $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \left( \frac{e^{x+y} + e^{x-y}}{2}, \frac{e^{x+y} - e^{x-y}}{2} \right) \quad \text{and} \quad h(x,t) = 2xt$$

Find $D_{(-1,2)} f(2,1)$, where $f(x,y) = h(g(x,y))$.

$$D_{(-1,2)} f(2,1) = \frac{d}{dt} \left( -1,2 \right) \cdot \nabla f(2,1)$$

$$\leftarrow (-1,2) \text{ is not a unit vector}$$

By the chain rule,

$$\nabla f(1,2) = \left( f_x, f_y \right) \bigg|_{(1,2)}$$

$$= \left( \frac{\partial h}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial f}{\partial y}, \frac{\partial h}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial f}{\partial y} \right) \bigg|_{(x,y)=(1,2)}$$

$$= \left( 2t \cdot \frac{e^{x+y} + e^{x-y}}{2} + 2s \frac{e^{x+y} - e^{x-y}}{2}, 2s \frac{e^{x+y} - e^{x-y}}{2} + 2t \frac{e^{x+y} + e^{x-y}}{2} \right) \bigg|_{x=1, y=2}$$

Where $s = \frac{e^{x+y} + e^{-x-y}}{2}$ and $t = \frac{e^{x+y} - e^{-x-y}}{2}$

$$(e^x - e^y)(e^y + e^{-y}) + (e^y + e^{-y})(e^x - e^{-y}) \quad \text{and} \quad (e^x + e^{-y})(e^y - e^{-y}) + (e^y - e^{-y})(e^x - e^{-y})$$

$$= 2(e^y - e^{-y}) (1,1)$$

$$\Rightarrow \quad D_{(-1,2)} f(2,1) = \frac{d}{dt} \left( -1,2 \right) \cdot 2(e^y - e^{-y}) (1,1)$$

$$= \frac{2(e^y - e^{-y})}{\sqrt{3}} (-1,2)$$

$$= \frac{2(e^y - e^{-y})}{\sqrt{3}} \left( = \frac{2e^2}{\sqrt{3}} (e^4 - 1) \right)$$
4. Peeve the guinea pig is standing on a steep Andes mountain. Conveniently, the mountain looks just like the elliptic paraboloid $3x^2 + 7y^2 + 5z = 20$. Owing to her short legs, the steepest grade that Peeve can climb up the mountain is $\frac{1}{5}$.

Given that Peeve's $(x,y)$-coordinates are $(1,1)$ and she is facing in the direction of steepest ascent, find the smallest angle that Peeve must turn in order to be able to ascend the mountain.

\[ 3x^2 + 7y^2 + 5z = 20 \implies z = 4 - \frac{3}{5}x^2 - \frac{7}{5}y^2 \]

So the mountain is the graph of $f(x,y) = 4 - \frac{3}{5}x^2 - \frac{7}{5}y^2$.

Let $\mathbf{u}$ be a direction* that Peeve can ascend with grade $\frac{1}{5}$ — note that Peeve is currently facing in the direction of $\nabla f(1,1)$. Let $\theta$ be the angle between $\mathbf{u}$ and $\nabla f(1,1)$. Then

\[ \frac{1}{5} = D_{\mathbf{u}} f(1,1) = \| \nabla f(1,1) \| \cos \theta \]

\[ \nabla f(1,1) = -\frac{2}{5} (3x, 7y) \bigg|_{(1,1)} = -\frac{2}{5} (3, 7) \]

\[ \implies \| \nabla f(1,1) \| = \frac{2}{5} \sqrt{9 + 49} = \frac{2 \sqrt{58}}{5} \]

\[ \implies \cos \theta = \frac{1}{5\| \nabla f(1,1) \|} = \frac{1}{2 \sqrt{58}} \]

\[ \implies \theta = \arccos \frac{1}{2 \sqrt{58}} \approx 1.51 \text{ radians} \approx 86.2^\circ \]

So Peeve must turn $\approx 1.51$ rads / $86.2^\circ$ in either direction to be able to ascend the mountain.