1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \cos x \cos y$. Find the second-order Taylor polynomial of $f$...

(a) ... at $(0,0)$:

\[
\begin{align*}
\hat{f}(0,0) &= 1 \times 1 = 1 \\
\nabla f(0,0) &= \left(-\sin x \cos y, -\cos x \sin y\right) \bigg|_{(0,0)} = (0,0) \\
Hf(0,0) &= \begin{pmatrix}
-\cos x \cos y & \sin x \sin y \\
\sin x \sin y & -\cos x \cos y
\end{pmatrix} \bigg|_{(0,0)} = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\end{align*}
\]

So the 2nd-order Taylor polynomial of $f$ at $(0,0)$ is

\[
Q(x,y) = 1 + (0,0) \cdot (x,y) + \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
= 1 - \frac{1}{2} x^2 - \frac{1}{2} y^2
\]

[Note $z = Q(x,y)$ is an elliptic paraboloid!]

(b) ... at \(\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\):

\[
\begin{align*}
\hat{f}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) &= 0 \\
\nabla f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) &= (0,0) \\
Hf\left(\frac{\pi}{2}, \frac{\pi}{2}\right) &= \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\end{align*}
\]

\[
\Rightarrow Q(x,y) = 0 + (0,0) \cdot (x-\frac{\pi}{2}, y-\frac{\pi}{2}) + \frac{1}{2} \begin{pmatrix} x-\frac{\pi}{2} \\ y-\frac{\pi}{2} \end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix} x-\frac{\pi}{2} \\ y-\frac{\pi}{2} \end{pmatrix}
= \frac{1}{2} \cdot 2 \left( x-\frac{\pi}{2} \right) \left( y-\frac{\pi}{2} \right)
= \left( x-\frac{\pi}{2} \right) \left( y-\frac{\pi}{2} \right).
\]

[Note $z = Q(x,y)$ is a hyperbolic paraboloid canted at \(\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\).]
2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y) = 3 - 2x + 4y - z$.

(a) Find the first-order Taylor polynomial of $f$ at $(0,0,0)$.

$$f(0,0,0) = 3, \quad \nabla f(0,0,0) = (-2, 4, -1)$$

$$\Rightarrow L(x,y,z) = 3 + (-2)(x-0) + 4(y-0) + (-1)(z-0)$$

$$= 3 - 2x + 4y - z$$

$$= f(x,y,z) \ldots \text{ (oh...)}$$

(b) Find the second-order Taylor polynomial of $f$ at $(0,0,0)$.

$$Hf = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ since } f_x, f_y, f_z \text{ are constant}$$

$$\Rightarrow Q(x,y,z) = L(x,y,z) + \underbrace{x \nabla Hf(0,0,0) \cdot x}_{=0}$$

$$= 3 - 2x + 4y - z$$

$$= f(x,y,z) \ldots \text{ (oh...)}$$

(c) What's going on?

$f$ is a linear polynomial!

So the first-order Taylor polynomial is already as good an approximation to $f$ as we could hope for.

($\Rightarrow$ All higher-order Taylor polynomials are zero.)
3. For each of the following statements, determine whether it is true or false.

(a) If \( f : \mathbb{R}^2 \to \mathbb{R} \) is differentiable at \((0,0)\), and the second-order Taylor polynomial of \( f \) is the constant zero function, then \( f \) is the constant zero function.

\[
\text{False! \ Let } f(x,y) = x^3. \text{ Then } f(0,0) = 0, \quad \nabla f(0,0) = (3x,0) \big|_{(0,0)} = (0,0) \\
\text{and } Hf(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \big|_{(0,0)} = (0,0) \\
\Rightarrow Q(x,y) = 0 \text{ for all } x, y.
\]

(b) If \( Q(x,y,z) \) is the second-order Taylor polynomial of a function \( f : \mathbb{R}^3 \to \mathbb{R} \) at a point where \( f \) is differentiable, then \( \frac{\partial^3 Q}{\partial x \partial y \partial z} = 0. \)

\[
\text{True! } Q \text{ is a polynomial of degree 2 so when differentiated 3 times we obtain a value of 0.}
\]

(c) If \( L(x,y) \) and \( Q(x,y) \) are the first- and second-order Taylor polynomials of a differentiable function \( f : \mathbb{R}^2 \to \mathbb{R} \) at a point \((a,b)\), then the graph \( z = L(x,y) \) is the tangent plane to the graph \( z = Q(x,y) \) at \((x,y) = (a,b)\).

\[
\text{True! The tangent plane to } z = Q(x,y) \text{ at } (a,b, Q(a,b)) \text{ is } L(x,y) = f(a,b) \\
\text{is } z = Q(a,b) + \nabla Q(a,b) \cdot (x-a, y-b) \\
= f(a,b) + \nabla f(a,b) \cdot (x-a, y-b) \text{ since } Q_x = f_x \\
\text{and } Q_y = f_y \text{ at } (a,b).
\]