Global extrema

A **global maximum** of a function \( f : \mathbb{R}^n \to \mathbb{R} \) within a region \( D \) of \( \mathbb{R}^n \) is a point \( a \) in \( D \) such that \( f(a) \geq f(x) \) for all points \( x \) in \( D \). A **global minimum** is defined likewise, and a **global extremum** is a point that is either a global maximum or a global minimum.

In general, a function might attain no global maximum or minimum value. However, the extreme value theorem tells us that if \( D \) is a region of \( \mathbb{R}^n \) that is:

- **closed** (it contains all of its boundary points); and
- **bounded** (there is an upper bound on how far apart two points of \( D \) can be);

... then \( f \) attains both a maximum and minimum value in \( D \)—that is, there are points \( a_{\text{max}} \) and \( a_{\text{min}} \) in \( D \) such that

\[
f(a_{\text{min}}) \leq f(x) \leq f(a_{\text{max}})
\]

for all \( x \) in \( D \). (The fancy name for a closed and bounded region of \( \mathbb{R}^n \) is a **compact set**.)

In order to find the global extrema of \( f \) on a compact set \( D \):

- Find the critical points of \( f \) inside \( D \).
- Find the global extrema of \( f \) on the boundary of \( D \).

‘Officially’, what you need to do here is find a parametrisation \( x(t) \) of the boundary of \( D \)—this gives rise to a new function \( g(t) = f(x(t)) \), where \( t \) ranges over some suitable region of \( \mathbb{R}^{n-1} \), whose global extrema you can find by repeating this method.

This sounds scary, but it really isn’t—it is best illustrated by example.

- Amongst the points that you found, a point where the value of \( f \) is least is a global minimum of \( f \) on \( D \), and a point where the value of \( f \) is greatest is a global maximum.

If you want to find the global extrema of \( f \) on all of \( \mathbb{R}^n \), then you should find its local extrema and make sure that the function truly does attain a local minimum and local maximum value.
1. Find the global extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2 - 2xy - y^2$ on the closed circular disc of radius 1 centred at $(0,0)$. 
2. Find the maximum and minimum values attained by the function \( f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \) on the semicircular region of \( \mathbb{R}^2 \) defined in polar coordinates by \( 0 \leq r \leq 1, \frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4} \).
3. For each of the following statements, determine whether it is true or false.

(a) If a function \( f : \mathbb{R}^n \to \mathbb{R} \) has a unique critical point \( a \), and \( a \) is a local minimum of \( f \), then \( a \) is a global minimum of \( f \).

(b) If \( D \) is a bounded region of \( \mathbb{R}^n \), and \( f \) is a bounded function on \( X \) (that is, there are real numbers \( a \) and \( b \) such that \( a \leq f(x) \leq b \) for all \( x \) in \( D \)), then \( f \) has a global minimum and a global maximum on \( D \).

(c) If \( D \) is a closed region of \( \mathbb{R}^n \), and \( f \) is a function on \( X \) that is unbounded above (that is, for any \( a > 0 \), there is some \( x \) in \( D \) such that \( f(x) > a \)), then \( D \) is unbounded.