Math 300 Class 11
Friday 1st February 2019

**Definition 1**
Let $f : X \rightarrow Y$ be a function and let $U \subseteq X$. The **image of $U$ under $f$** is the subset $f[U] \subseteq Y$ defined by

$$f[U] = \{ f(x) \mid x \in U \} = \{ y \in Y \mid \exists x \in U, y = f(x) \}$$

That is, $f[U]$ is the set of values that the function $f$ takes when given inputs from $U$.

The ‘**image of $f$**’ is the image of its entire domain, i.e. the set $f[X]$.

**Example 2**
Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. Find $f[\mathbb{R}]$ and $f[(-1, 1)]$.

**Example 3**
Define $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by $g(a, b) = \frac{a}{1 + |b|}$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$.

Find a subset $V \subseteq \mathbb{R} \times \mathbb{R}$ such that $g[V] = \mathbb{Q}$. 
Example 4
Let $f : X \to Y$ be a function and let $U, V \subseteq X$.

Prove that $f[U \cap V] \subseteq f[U] \cap f[V]$.

Give an example to show that $f[U] \cap f[V]$ need not be a subset of $f[U \cap V]$.

Definition 5
Let $f : X \to Y$ be a function and let $V \subseteq Y$. The preimage of $V$ under $f$ is the subset $f^{-1}[V] \subseteq X$ defined by

$$f^{-1}[V] = \{x \in X \mid f(x) \in V\} = \{x \in X \mid \exists y \in V, y = f(x)\}$$

Example 6
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. Find $f^{-1}[\mathbb{R}]$ and $f^{-1}[(-\infty, 4)]$. 

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Example 7
Let $f : X \to Y$ be a function. Prove that $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$ for all subsets $U, V \subseteq Y$.

Example 8
Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $W \subseteq Z$. Prove that $(g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]]$. 