Definition 1
A function \( f : X \to Y \) is...

- \textbf{injective} if, for all \( a, b \in X \), if \( f(a) = f(b) \), then \( a = b \);
- \textbf{surjective} if, for all \( c \in Y \), there exists \( a \in X \) such that \( f(a) = c \);
- \textbf{bijective} if it is injective and surjective.

Example 2
For each of the following diagrams, determine whether the function it represents is: (B) bijective, (I) injective and not surjective, (S) surjective and not injective, or (N) neither injective nor surjective.

Example 3
Let \( a, b \in \mathbb{R} \) with \( a < b \). Find a bijection \( (0, 1) \to (a, b) \).
Example 4
Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that if $f$ and $g$ are injective, then $g \circ f$ is injective. [Note: the same is true with ‘injective’ replaced by ‘surjective’ or ‘bijective’.

Definition 5
An inverse for a function $f : X \to Y$ is a function $g : Y \to X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Example 6
Find an inverse for the function you defined in Example 3.
Theorem 7
A function $f : X \to Y$ is a bijection if and only if it has an inverse.

Proof
$(\Rightarrow)$ Suppose $f : X \to Y$ is a bijection. Define $g : Y \to X$ as follows. Given $y \in Y$, there exists $x \in X$ such that $f(x) = y$ since $f$ is surjective. Moreover this element $x$ is unique, since $f$ is injective. So define $g(y) = x$ for the unique $x \in X$ for which $f(x) = y$. Then

- Given $x \in X$, we have $g(f(x))$ is the unique $a \in X$ such that $f(a) = f(x)$, so $g(f(x)) = x$.
- Given $y \in Y$, let $x \in X$ be such that $y = f(x)$. Then we have $f(g(y)) = f(g(f(x))) = f(x) = y$.

So $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$, and so $g$ is an inverse for $f$.

$(\Leftarrow)$ Assume $f$ has an inverse $g : Y \to X$. Then

- $f$ is injective. Let $a, b \in X$ and assume that $f(a) = f(b)$. Then $a = g(f(a)) = g(f(b)) = b$.
- $f$ is surjective. Let $c \in Y$ and define $a = g(c)$. Then $f(a) = f(g(c)) = c$.

So $f$ is a bijection. □

Summary of proof strategies for *jections

**Strategy** (Proving a function is injective)
In order to prove that a function $f : X \to Y$ is injective, it suffices to fix $a, b \in X$, assume that $f(a) = f(b)$, and then derive $a = b$.

**Strategy** (Proving a function is surjective)
To prove that a function $f : X \to Y$ is surjective, it suffices to take an arbitrary element $y \in Y$ and, in terms of $y$, find an element $x \in X$ such that $f(x) = y$.

In order to find such a value of $x$, it is often useful to start from the equation $f(x) = y$ and derive some possible values of $x$. But be careful—in order to complete the proof, it is necessary to verify that the equation $f(x) = y$ is true for the chosen value of $x$.

**Strategy** (Proving a function is bijective)
To prove that a function $f : X \to Y$ is bijective, it suffices to either:

- Prove that $f$ is injective, and that $f$ is surjective; or
- Find an inverse $g : Y \to X$ for $f$, and verify that $g(f(x)) = x$ for all $x \in X$ and that $f(g(y)) = y$ for all $y \in Y$. □