Definition 1
A function \( f : X \rightarrow Y \) is...

- ...injective if, for all \( a, b \in X \), if \( f(a) = f(b) \), then \( a = b \);
- ...surjective if, for all \( c \in Y \), there exists \( a \in X \) such that \( f(a) = c \);
- ...bijective if it is injective and surjective.

For the next couple of examples, it will be helpful to remark that, for a function \( f : X \rightarrow Y \) and elements \( x \in X \) and \( y \in Y \), we have

\[
x \in f^{-1}(\{y\}) \iff f(x) \in \{y\} \iff f(x) = y
\]

That is, \( f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\} \).

Example 2
Let \( f : X \rightarrow Y \) be a function. Prove that \( f \) is surjective if and only if, for all \( y \in Y \), the set \( f^{-1}(\{y\}) \) has at least one element.
Example 3
Let $f : X \to Y$ be a function. Prove that $f$ is injective if and only if, for all $y \in Y$, the set $f^{-1}(\{y\})$ has at most one element.

Example 4
Prove that there does not exist a surjection $[2] \to [3]$, where $[n] = \{1, 2, \ldots, n\}$. 
Definition 5
An inverse for a function \( f : X \rightarrow Y \) is a function \( g : Y \rightarrow X \) such that \( g \circ f = \text{id}_X \) and \( f \circ g = \text{id}_Y \).

Theorem 6
A function \( f : X \rightarrow Y \) is a bijection if and only if it has an inverse.

Proof
(\( \Rightarrow \)) Suppose \( f : X \rightarrow Y \) is a bijection. Define \( g : Y \rightarrow X \) as follows. Given \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \) since \( f \) is surjective. Moreover this element \( x \) is unique, since \( f \) is injective. So define \( g(y) = x \) for the unique \( x \in X \) for which \( f(x) = y \). Then

- Given \( x \in X \), we have \( g(f(x)) \) is the unique \( a \in X \) such that \( f(a) = f(x) \), so \( g(f(x)) = x \).
- Given \( y \in Y \), let \( x \in X \) be such that \( y = f(x) \). Then we have \( f(g(y)) = f(g(f(x))) = f(x) = y \).

So \( g \circ f = \text{id}_X \) and \( f \circ g = \text{id}_Y \), and so \( g \) is an inverse for \( f \).

(\( \Leftarrow \)) Assume \( f \) has an inverse \( g : Y \rightarrow X \). Then

- \( f \) is injective. Let \( a, b \in X \) and assume that \( f(a) = f(b) \). Then \( a = g(f(a)) = g(f(b)) = b \).
- \( f \) is surjective. Let \( c \in Y \) and define \( a = g(c) \). Then \( f(a) = f(g(c)) = c \).

So \( f \) is a bijection. \( \square \)

Example 7
Recall that the function \( f : (0,1) \rightarrow (a,b) \) defined by \( f(t) = a + t(b-a) \) is a bijection. Find an inverse for \( f \).
Example 8
Find a bijection $f : [0, 1] \rightarrow [0, 1)$.

Summary of proof strategies for *jections

Strategy (Proving a function is injective)
In order to prove that a function $f : X \rightarrow Y$ is injective, it suffices to fix $a, b \in X$, assume that $f(a) = f(b)$, and then derive $a = b$. ▷

Strategy (Proving a function is surjective)
To prove that a function $f : X \rightarrow Y$ is surjective, it suffices to take an arbitrary element $y \in Y$ and, in terms of $y$, find an element $x \in X$ such that $f(x) = y$.

In order to find such a value of $x$, it is often useful to start from the equation $f(x) = y$ and derive some possible values of $x$. But be careful—in order to complete the proof, it is necessary to verify that the equation $f(x) = y$ is true for the chosen value of $x$. ▷

Strategy (Proving a function is bijective)
To prove that a function $f : X \rightarrow Y$ is bijective, it suffices to either:

- Prove that $f$ is injective, and that $f$ is surjective; or

- Find an inverse $g : Y \rightarrow X$ for $f$, and verify that $g(f(x)) = x$ for all $x \in X$ and that $f(g(y)) = y$ for all $y \in Y$. ▷