Math 300 Class 17
Monday 18th February 2019

**Definition 1**
Given \( n \in \mathbb{N} \), the set \([n]\) is defined by \([n] = \{ k \in \mathbb{N} \mid 1 \leq k \leq n \}\).

**Definition 2** — Finite and infinite sets
A set \( X \) is **finite** if there exists a bijection \( f : [n] \rightarrow X \) for some \( n \in \mathbb{N} \); the function \( f \) is called an **enumeration** of \( X \). If \( X \) is not finite we say it is **infinite**.

**Theorem 3** — Uniqueness of size
Let \( X \) be a finite set and let \( f : [m] \rightarrow X \) and \( g : [n] \rightarrow X \) be enumerations of \( X \). Then \( m = n \).

The proof of this ‘obvious’ fact is a surprisingly complicated induction argument—you can read all about it in §3.2 of the book.

**Definition 4** — Size of a finite set
Let \( X \) be a finite set. The **size** of \( X \), written \( |X| \), is the unique natural number \( n \) for which there exists a bijection \([n] \rightarrow X\).

**Example 5**
Prove that \([n]\) is finite and \( |[n]| = n \) for all \( n \in \mathbb{N} \).
Example 6
Prove that every inhabited finite subset of \( \mathbb{N} \) has a greatest element.

Theorem 7
\( \mathbb{N} \) is infinite.

Proof
Suppose \( \mathbb{N} \) is finite. Then \( \mathbb{N} \) is an inhabited finite subset of \( \mathbb{N} \), so by Example 6, \( \mathbb{N} \) has a greatest element, say \( n \). But then \( n + 1 \in \mathbb{N} \) and \( n + 1 > n \), contradicting maximality of \( n \). So \( \mathbb{N} \) is infinite. \( \square \)
**Theorem 8 — Some properties of size**

(a) If $Y$ is finite and there is an injection $X \to Y$, then $X$ is finite and $|X| \leq |Y|$;

(b) If $X$ is finite and there is a surjection $X \to Y$, then $Y$ is finite and $|X| \geq |Y|$;

(c) If $X$ and $Y$ are finite, then $X \times Y$ is finite and $|X \times Y| = |X| \cdot |Y|$;

(d) If $X$ and $Y$ are finite and $X \cap Y = \emptyset$, then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$.

---

**Example 9**

Prove that if $X$ is a finite set and $U \subseteq X$, then $U$ is finite and $|U| \leq |X|$.

---

**Example 10**

Prove that if $X$ is a finite set and $U \subseteq X$, then $X \setminus U$ is finite and $|X \setminus U| = |X| - |U|$.
**Strategy** (Bijective proof)
In order to prove that finite sets $X$ and $Y$ have the same size, it suffices to find a bijection $X \rightarrow Y$.

**Example 11**
Let $X$ be a finite set. Prove that $|\mathcal{P}(X)| = |\{0, 1\}^X|$, where $\{0, 1\}^X$ is the set of functions $X \rightarrow \{0, 1\}$.