Theorem 1 — Some properties of size

(a) If $Y$ is finite and there is an injection $X \rightarrow Y$, then $X$ is finite and $|X| \leq |Y|$;

(b) If $X$ is finite and there is a surjection $X \rightarrow Y$, then $Y$ is finite and $|X| \geq |Y|$;

(c) If $X$ and $Y$ are finite, then $X \times Y$ is finite and $|X \times Y| = |X| \cdot |Y|$;

(d) If $X$ and $Y$ are finite and $X \cap Y = \emptyset$, then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$.

Definition 2 — Binomial coefficients (combinatorial definition)

Let $n, k \in \mathbb{N}$. The set $^{[n]} \binom{k}{i}$ is defined by

$$\left( ^{[n]} \binom{k}{i} \right) = \{ U \subseteq [n] \mid |U| = k \}$$

The binomial coefficient $\binom{n}{k}$ is defined by $\binom{n}{k} = |^{[n]} \binom{k}{i}|$.

Example 3

Compute $^{[3]} \binom{k}{i}$ for all $k \in \mathbb{N}$.

$$\left( ^{[3]} \binom{0}{i} \right) = \emptyset \emptyset \emptyset \Rightarrow \left( ^{[3]} \binom{0}{0} \right) = 1$$

$$\left( ^{[3]} \binom{1}{i} \right) = \{ \{1,3\}, \{2,3\} \} \Rightarrow \left( ^{[3]} \binom{1}{1} \right) = 2$$

$$\left( ^{[3]} \binom{2}{i} \right) = \{ \{1,2,3\}, \{1,3\}, \{2,3\} \} \Rightarrow \left( ^{[3]} \binom{2}{2} \right) = 3$$

$$\left( ^{[3]} \binom{3}{i} \right) = \{ \{1,2,3\} \} \Rightarrow \left( ^{[3]} \binom{3}{3} \right) = 1$$

$$\left( ^{[3]} \binom{k}{i} \right) = \emptyset \text{ for all } k \geq 3 \Rightarrow \left( ^{[3]} \binom{k}{k} \right) = 0 \text{ for all } k \geq 3.$$
Parts (a) and (b) of Theorem 1 combine to give the following useful proof technique.

**Strategy (Bijective proof)**
In order to prove that finite sets $X$ and $Y$ have the same size, it suffices to find a bijection $X \to Y$.

**Example 4**
Prove that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ with $k \leq n$.

Define $f: \binom{[n]}{k} \to \binom{[n]}{n-k}$ by

$$f(U) = [n] \setminus U \text{ for all } U \subseteq [n] \text{ with } |U| = k$$

Note $f$ is well-defined: by Ex 10 from yesterday:

$$|\binom{[n]}{k} | = |[n]| - |U| = n - k$$

$$\Rightarrow [n] \setminus U \in \binom{[n]}{n-k} \text{ as claimed.}$$

& $f$ is a bijection — it has an inverse

$$g: \binom{[n]}{n-k} \to \binom{[n]}{k}$$

defined by $g(V) = [n] \setminus V \text{ for all } V \subseteq [n] \text{ with } |V| = n-k$

(Note: $[n] \setminus ([n] \setminus U) = U \text{ for all } U \subseteq [n]$)

In general, $y \setminus (y \setminus x) = x \cap y$.

Since there is a bijection $\binom{[n]}{k} \to \binom{[n]}{n-k}$,

we have $|\binom{[n]}{k}| = |\binom{[n]}{n-k}|$

$$\Rightarrow \binom{n}{k} = \binom{n}{n-k}.$$
Definition 5
A partition of a finite set $X$ is a family $U_1, U_2, \ldots, U_n$ of (inhabited) subsets of $X$ such that:

(i) $\bigcup_{i=1}^{n} U_i = X$; and

(ii) $U_i \cap U_j = \emptyset$ if $i \neq j$ (that is to say that $U_1, \ldots, U_n$ are pairwise disjoint).

[\text{In the current context, we will additionally allow the sets } U_i \text{ to be empty.}]

Theorem 6 — Addition principle
Let $X$ be a finite set and $U_1, U_2, \ldots, U_n$ be a partition of $X$. Then $|X| = \sum_{i=1}^{n} |U_i|$. \hfill $\square$

Strategy 7
In order to count the elements of a set $X$, it suffices to partition $X$ into subsets $U_1, \ldots, U_n$ and add up the sizes of the sets in the partition.

Example 8
Prove that, for all $n, k \in \mathbb{N}$, we have $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Let $X = \binom{\mathbb{N} + 1}{k+1}$ and define

\[
\begin{align*}
U_1 &= \{ A \subseteq X \mid n+1 \notin A \} \\
U_2 &= \{ A \subseteq X \mid n+1 \in A \}
\end{align*}
\]

Then $U_1, U_2$ is a partition of $X$; for all $A \subseteq \mathbb{N} + 1$ we must have $n+1 \in A$ or $n+1 \notin A$, but not both.

Moreover:

- Each $A \in U_1$ is $A' \cup \{n+1\}$ for a unique $A' \subseteq \mathbb{N}$ with $|A'| = k$ (i.e. $A' \in \binom{\mathbb{N}}{k}$) \Rightarrow $|U_1| = \binom{n}{k}$

- Each $A \in U_2$ is precisely a subset of $\binom{\mathbb{N}}{k}$ of size $k+1$ \Rightarrow $|U_2| = \binom{n}{k+1}$

By the addition principle,

\[
\binom{n+1}{k+1} = |\binom{\mathbb{N} + 1}{k+1}| = |U_1| + |U_2| = \binom{n}{k} + \binom{n}{k+1}. \hfill \square
\]
Theorem 9 — Multiplication principle
Fix $m,n \in \mathbb{N}$. Let $X$ be a finite set with $|X| = m$, and for each $a \in X$, let $Y_a$ be a finite set with $|Y_a| = n$. Then
\[ |\{(a,b) \mid a \in X, \ b \in Y_a\}| = mn \]

The pair $(a,b)$ is called a dependent pair, because the set that $b$ belongs to depends on the value of $a$. This generalises (by induction!) to sets of dependent $n$-tuples—the precise statement is ugly.

Strategy 10
Given a finite set $X$, in order to compute $|X|$, it suffices to devise a step-by-step procedure for uniquely specifying an element of $X$—each step may depend on the last, but

Example 11
|\{A\} = k
Compute the size of the set $X = \{(A,a) \mid A \subseteq \{n\}, \ a \in A\}$ in two ways:

(a) Specify $(A,a) \in X$ by first choosing $A$ and then choosing $a$.

- **Step 1** Choose $A \subseteq \{n\}$ with $|A| = k$. There are $\binom{n}{k}$ choices.
- **Step 2** Choose $a \in A$. Since $|A| = k$, there are $k$ choices.

By MP, $|X| = \binom{n}{k} \cdot k$.

(b) Specify $(A,a) \in X$ by first choosing $a$ and then choosing $A$.

- **Step 1** Choose $a \in \{n\}$. There are $n$ choices.
- **Step 2** Choose the remaining $k-1$ elements of $A$ from $\{n\} \setminus \{a\}$. Since $|\{n\} \setminus \{a\}| = n-1$, there are $\binom{n-1}{k-1}$ choices.

By MP, there are $n \cdot \binom{n-1}{k-1}$ elements in $X$.

Observe that (a) & (b) imply that $\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$.

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set $X$ and devise two counting proofs to show that $|X|$ is equal to both expressions.