Math 300 Class 19
Friday 22nd February 2019

Strategy (Addition principle)
Let $X$ be a finite set. In order to compute $|X|$, it suffices to find a partition $U_1, U_2, \ldots, U_n$ of $X$; it then follows that $|X| = \sum_{k=1}^{n} |X_k|$.

Strategy (Multiplication principle)
Let $X$ be a finite set. In order to compute $|X|$, it suffices to find a step-by-step procedure for specifying elements of $X$, such that:

- Each element is specified by a unique sequence of choices;
- The number of choices at each step is constant, even if the choices themselves depend on choices made in previous steps.

If there are $n$ steps and $m_k$ possible choices in the $k^{th}$ step, then $|X| = \prod_{k=1}^{n} m_k$.

Example 1
Let $m, n \in \mathbb{N}$. Prove that $|\mathcal{P}([n])| = 2^n$ and $|[n]^m| = n^m$, where $Y^X$ is the set of functions $X \to Y$.

- Procedure for specifying an element $U \in \mathcal{P}([n])$ (i.e. $U \subseteq [n]$):
  - Step 1: Decide if $1 \in U$ or $1 \notin U \leftarrow 2$ choices
  - Step 2: Decide if $2 \in U$ or $2 \notin U \leftarrow 2$ choices
  - \vdots
  - Step $n$: Decide if $n \in U$ or $n \notin U \leftarrow 2$ choices

  By MP, $|\mathcal{P}([n])| = \frac{2 \times 2 \times \cdots \times 2}{n \text{ times}} = 2^n$

- Procedure for specifying an element $f \in [n]^m$ (i.e. $f : [m] \to [n]$):
  - $m$ steps; at step $k$, choose the value of $f(k) \in [n]$;
  - There are $n$ choices for each $k \in [m]$

  \( \Rightarrow \) by MP, $|[n]^m| = \frac{n \times n \times \cdots \times n}{m \text{ times}} = n^m$. \( \square \)
Definition 2 — Factorials (recursive definition)
Let \( n \in \mathbb{N} \). The factorial of \( n \) is defined by
\[
 n! = |\{ f : [n] \to [n] \mid f \text{ is a bijection} \}| 
\]

\[ \text{for all } x, y, \text{ where } \left| [x] \right| = \left| [y] \right| = n \]

Example 3
Prove that \( n! = \prod_{k=1}^{n} k \).

Procedure for specifying a bijection \( f : [n] \to [n] \):

\begin{align*}
\text{Step 1} & \quad \text{Choose } f(1) \in [n] \quad \leftarrow \ n \text{ choices} \\
\text{Step 2} & \quad \text{Choose } f(2) \in [n] \setminus \{f(1)\} \quad \leftarrow \ n-1 \text{ choices} \\
\vdots & \quad \text{Choose } f(k) \in [n] \setminus \{f(1), \ldots, f(k-1)\} \quad \leftarrow \ n-k \text{ choices} \\
\text{Step n} & \quad \text{Choose } f(n) \in [n] \setminus \{f(1), \ldots, f(n-1)\} \quad \leftarrow \ 1 \text{ choice} \\
\end{align*}

By MP, \( n! = n \times (n-1) \times \cdots \times 1 = \prod_{k=1}^{n} k \).

\*Note: \( n! = |\{ f : X \to Y \mid f \text{ is a bijection} \}| \) for all \( X, Y \) with \( \left| X \right| = \left| Y \right| = n \).

Example 4
Let \( n, k \in \mathbb{N} \). Prove that the number of injections \( [k] \to [n] \) is \( \binom{n}{k} \cdot k! \).

Procedure for specifying an injection \( f : [k] \to [n] \):

\begin{align*}
\text{Step 1} & \quad \text{Since } f \text{ will be injective, it will take exactly } k \text{ values in } [n] \quad \text{so choose } U \in \binom{[n]}{k} \text{ to be the set of values of } f \quad \leftarrow (\binom{n}{k}) \text{ choices} \\
\text{Step 2} & \quad \text{Choose a bijection } [k] \to U : \text{ this will then determine an injection } [k] \to [n] \text{ with image } U. \\
& \quad \text{Since } |U| = k \text{, there are } k! \text{ choices.} \\
\end{align*}

By MP, \( (\text{# injections } [k] \to [n]) = \binom{n}{k} \cdot k! \).
Strategy (Double counting)

In order to prove that two expressions involving natural numbers are equal, it suffices to define a set \( \mathcal{X} \) and devise two counting arguments to show that \( |\mathcal{X}| \) is equal to both expressions.

Example 5

Let \( n, k, \ell \in \mathbb{N} \) with \( \ell \leq n \) and \( \ell \leq k \). Prove that \( \binom{n}{k} \binom{k}{\ell} = \binom{n}{\ell} \binom{n-\ell}{k-\ell} \).

Let \( \mathcal{X} = \{ (A, B) \mid B \subseteq A \subseteq \mathcal{N}, \quad |A| = k, \quad |B| = \ell \} \).

Procedure 1

* Step 1
  Choose \( A \in \binom{\mathcal{N}}{k} \) \( \leftarrow \binom{n}{k} \) choices

* Step 2
  Choose \( B \in \binom{A}{\ell} \) \( \leftarrow \binom{k}{\ell} \) choices \( \vdots |A| = k \)

Then \( B \subseteq A \subseteq \mathcal{N}, \quad |A| = k \) \& \( |B| = \ell \), so \( |\mathcal{X}| = \binom{n}{k} \binom{k}{\ell} \) by MP.

Procedure 2

* Step 1
  Choose \( B \in \binom{\mathcal{N}}{\ell} \) \( \leftarrow \binom{n}{\ell} \) choices

* Step 2
  Choose \( A' \in \binom{\mathcal{N} \setminus B}{k-\ell} \) \( \leftarrow \binom{n-\ell}{k-\ell} \) choices \( \vdots |\mathcal{N} \setminus B| = n-\ell \)

This determines \( (A, B) \in \mathcal{X} \) by letting \( A = A' \cup B \) — then certainly \( B \subseteq A \subseteq \mathcal{N} \); we have \( |B| = \ell \) by construction, and since \( A' \subseteq \mathcal{N} \setminus B \), we have \( A' \cap B = \emptyset \), so

\[
|A| = |A'| + |B| = (k-\ell) + \ell = k
\]

as required.

So by MP, \( |\mathcal{X}| = \binom{n}{k} \binom{k}{\ell} \Rightarrow \binom{n}{\ell} \binom{\ell}{n-\ell} = \binom{n}{\ell} \binom{n-\ell}{k-\ell} \).
Strategy (Double counting)

In order to prove that two expressions involving natural numbers are equal, it suffices to define a set \( X \) and devise two counting arguments to show that \( |X| \) is equal to both expressions.

Example 5

Let \( n, k, \ell \in \mathbb{N} \) with \( \ell \leq n \) and \( \ell \leq k \). Prove that \( \binom{n}{k} \binom{k}{\ell} = \binom{n}{\ell} \binom{n-\ell}{k-\ell} \).

Let \( X \) be the set of all possible appointments of a \( k \)-person committee, with an \( \ell \)-person executive subcommittee, from a population of size \( n \).

Procedure 1

- **Step 1**: Choose \( k \) people from the population to serve on the committee \( \left\langle \binom{n}{k} \right\rangle \) choices

- **Step 2**: Choose \( \ell \) people from the committee to serve on the executive subcommittee \( \left\langle \binom{k}{\ell} \right\rangle \) choices

By MP, \( |X| = \binom{n}{k} \binom{k}{\ell} \).

Procedure 2

- **Step 1**: Choose \( \ell \) people from the population to serve on the executive subcommittee \( \left\langle \binom{n}{\ell} \right\rangle \) choices

- **Step 2**: Fill the remaining \( k-\ell \) committee positions from the remaining \( n-\ell \) people in the population \( \left\langle \binom{n-\ell}{k-\ell} \right\rangle \) choices

By MP, \( |X| = \binom{n}{\ell} \binom{n-\ell}{k-\ell} \).

By double counting, \( \binom{n}{k} \binom{k}{\ell} = \binom{n}{\ell} \binom{n-\ell}{k-\ell} \).
Example 6

Prove that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \) and that \( \sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \).

\[
\text{Let } X = \mathcal{P}([n])
\]

Procedure 1

Choose \( U \in X \) (i.e. \( U \subseteq [n] \)) \( \leftarrow \) \( 2^n \) choices. Done!

So \( |X| = 2^n \).

Procedure 2

Let \( X_k = \{ U \subseteq [n] \mid |U| = k \} = \binom{[n]}{k} \), \( \text{for } 0 \leq k \leq n \).

The sets \( X_k \) partition \( X \) since if \( U \subseteq [n] \) then there is a unique \( 0 \leq k \leq n \) s.t. \( |U| = k \).

By AP, \( |X| = \sum_{k=0}^{n} |X_k| = \sum_{k=0}^{n} \binom{n}{k} \), so \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).

\[
\text{Let } X = \{ (A, a) \mid A \subseteq [n], \ a \in A \}.
\]

Procedure 1

Split into cases based on \( k = |A| \) — note \( 0 \leq k \leq n \).

So define \( X_k = \{ (A, a) \mid A \subseteq [n], \ a \in A \} \). The sets \( X_k \) partition \( X \) (as above) \( \Rightarrow |X| = \sum_{k=0}^{n} |X_k| \), by AP.

For fixed \( 0 \leq k \leq n \), we can specify \( (A, a) \in X_k \) by:

- Step 1 Choose \( A \subseteq [n] \) \( \leftarrow \binom{n}{k} \) choices
- Step 2 Choose \( a \in A \) \( \leftarrow k \) choices

So \( |X| = \sum_{k=0}^{n} k \binom{n}{k} \).

Procedure 2

- Step 1 Choose \( a \in [n] \) \( \leftarrow n \) choices
- Step 2 Choose \( A' \subseteq [n] \setminus \{a\} \) \( \leftarrow 2^{[n] \setminus \{a\}} = 2^{n-1} \) choices

Then let \( A = A' \cup \{a\} \). This determines \( (A, a) \in X \)

\( \Rightarrow \ |X| = n \cdot 2^{n-1} \) by MP.

So \( \sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \). \( \square \)
Example 6
Prove that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \) and that \( \sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \).

- Let \( X \) be the set of all possible committees (of any size) from a population of size \( n \).

Procedure 1: Select a subset of the population to serve on the committee \( \leftarrow 2^n \) choices.

Procedure 2: For \( 0 \leq k \leq n \), let \( X_k \) be the set of committees with exactly \( k \) members. These sets partition \( X \), since the size \( k \) of any committee is in the range \( 0 \leq k \leq n \).

\[ |X| = \sum_{k=0}^{n} |X_k| \quad \text{by AP} \]

To specify an element of \( X_k \) for fixed \( 0 \leq k \leq n \), just choose \( k \) people from the population to serve on the committee \( \leftarrow \binom{n}{k} \) choices.

\[ |X_k| = \binom{n}{k} \quad \text{for all } 0 \leq k \leq n \]

\[ |X| = \sum_{k=0}^{n} |X_k| = \frac{2^n}{k=0} \binom{n}{k}. \quad \square \]

- Now let \( X \) be the set of committees from a population of size \( n \), with a distinguished chairperson.

Procedure 1: Step 1: Choose chair from population \( \leftarrow n \) choices

Step 2: Fill remaining positions from the remaining \( \frac{n}{n-1} \) people in the population \( \leftarrow 2^{n-2} \) choices

\[ |X| = n \cdot 2^{n-2} \quad \text{by MP}. \]

Procedure 2: For \( 0 \leq k \leq n \), let \( X_k \) be the set of possible committees, where there are \( k \) people on the committee. As argued above, the sets \( \{X_k\} \) \( 0 \leq k \leq n \) partition \( X \).

To specify an element of \( X_k \): Step 1: Choose \( k \) people from the population \( \leftarrow \binom{n}{k} \) choices

4. Step 2: Choose chair from the \( k \) committee members \( \leftarrow k \) choices

\[ |X_k| = \binom{n}{k} \cdot k \quad \text{by MP}, \text{ so by AP}, \quad n \cdot 2^{n-1} = |X| = \sum_{k=0}^{n} |X_k| = \sum_{k=0}^{n} \binom{n}{k} \cdot k. \quad \square \]
Example 7

Let \( n, k \in \mathbb{N} \). Prove that \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

We prove \( n! = \binom{n}{k} \cdot k! \cdot (n-k)! \) by double counting.

Let \( X = \{ \text{ordered lists of the numbers } 1, 2, \ldots, n \} \) s.t. each number appears exactly once in the list.

\[ |X| = n! \]

**Procedure 1**

Choose a bijection \( f: [n] \to [n] \). \( (n! \text{ choices}) \)

This determines such a list — for \( 1 \leq k \leq n \), the value \( f(k) \) is exactly the \( k \text{th} \) number on the list.

\[ |X| = n! \]

**Procedure 2**

* Step 1 Choose which elements of \([n]\) will be the first \( k \) to appear in the list \( \leftarrow \binom{n}{k} \text{ choices} \)

* Step 2 Choose the order that the first \( k \) numbers in the list appear. This amounts to specifying a bijection \([k] \to \{ \text{first } k \text{ el'pts on list}\} \leftarrow k! \text{ choices} \)

* Step 3 Choose the order that the last \( n-k \) numbers in the list appear. This amounts to specifying a bijection \([n-k] \to \{ \text{last } k \text{ el'pts on list}\} \leftarrow (n-k)! \text{ choices} \)

This completely determines a list of the numbers \( 1, 2, \ldots, n \) s.t. each el'pt appears exactly once.

By MP, \( |X| = \binom{n}{k} \cdot k! \cdot (n-k)! \)

\[ n! = \binom{n}{k} \cdot k! \cdot (n-k)! \quad \Rightarrow \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \Box \]