Math 300 Class 20
Monday 25th February 2019

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set \( X \) and devise two counting arguments to show that \(|X|\) is equal to both expressions.

Example 1
Let \( n, k \in \mathbb{N} \). Prove that \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

We prove \( \binom{n}{k} \cdot k! \cdot (n-k)! = n! \) by double counting.

Define \( X = \{ \text{lists of elements of } [n] \text{ with each } i \in [n] \text{ appearing exactly once on the list} \} \).

Procedure 1
Select a bijection \( f : [n] \to X \). This uniquely determines such a list: \( f(1), f(2), \ldots, f(n) \); each \( i \) of \( X \) appears exactly once \( \Rightarrow f \) is a bijection. \( \Rightarrow |X| = n! \).

Procedure 2
- **Step 1** Select \( U \subseteq \binom{[n]}{k} \). The elements of \( U \) will be the first \( k \) on the list (in some order). \( \binom{n}{k} \) choices.
- **Step 2** Put the elements of \( U \) in some order in the first \( k \) positions in the list. There are \( k! \) choices since this amounts to specifying a bijection \([k] \to U\).
- **Step 3** Put the elements of \([n] \setminus U\) in some order in the last \( n-k \) positions in the list. Again, there are \( (n-k)! \) choices.

By MP, \( |X| = \binom{n}{k} \cdot k! \cdot (n-k)! \).

So \( n! = \binom{n}{k} k! \cdot (n-k)! \)

\( \Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!} \) \( \Box \)
Example 2

Let $a, b, k \in \mathbb{N}$. Prove that $\sum_{i=0}^{k} \binom{a}{i} \binom{b}{k-i} = \binom{a+b}{k}$.

An animal rescue shelter houses $a$ cats and $b$ dogs. Let $X$ be the set of collections of $k$ animals from the shelter.

Procedure 1

Choose $k$ of the $a+b$ animals. There are \( \binom{a+b}{k} \) choices. \( \Rightarrow \) \( |X| = \binom{a+b}{k} \).

Procedure 2

For $0 \leq i \leq k$, let $X_i$ be the set of collections of $k$ animals in which exactly $i$ cats are chosen. The sets $X_i$ for $0 \leq i \leq k$ partition $X$. At least 0 cats must be chosen & $i \leq \text{animals chosen} = k$.

To count $|X_i|$ for each $0 \leq i \leq k$

1. Step 1: Choose $i$ of the $a$ cats \( \binom{a}{i} \) choices
2. Step 2: Choose $k-i$ of the $b$ dogs \( \binom{b}{k-i} \) choices

\( \Rightarrow \) \( |X_i| = \binom{a}{i} \binom{b}{k-i} \) by MP

\( \Rightarrow \) \( |X| = \sum_{i=0}^{k} \binom{a}{i} \binom{b}{k-i} \) by AP

So \( \binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i} \binom{b}{k-i} \). \( \square \)
Recall that a set $X$ is finite if there is a bijection $[n] \to X$ for some $n \in \mathbb{N}$—this captured the idea that the elements of $X$ can be listed one-by-one in such a way that the list eventually ends. Removing the requirement that the list end reveals the following definition.

**Definition 3 — Countably infinite, countable and uncountable sets**

A set $X$ is countably infinite if there is a bijection $\mathbb{N} \to X$. A set is countable if it is finite or countably infinite, and is uncountable if it is not countable.

**Exercise 4**

Prove that $\mathbb{N}$ is countably infinite.

$$\text{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \text{ is a bijection.}$$

**Exercise 5**

Prove that $\mathbb{Z}$ is countably infinite.

Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Then $f$ is a bijection—it has an inverse $g$ defined by

$$g(n) = \begin{cases} 2n & \text{if } n > 0 \\ -(2n+1) & \text{if } n < 0 \end{cases}$$
Exercise 6
Prove that \(\mathcal{P}(\mathbb{N})\) is uncountable.

We show no function \(f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})\) is surjective.

Idea Find \(B \in \mathcal{P}(\mathbb{N})\) ("bad element") such that \(B\) disagrees with (\(\Rightarrow\) is not equal to) each \(f(n) \in \mathcal{P}(\mathbb{N})\).

So let \(f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})\) be arbitrary. Define

\[
B = \{ n \in \mathbb{N} \mid n \notin f(n) \}
\]

Intuitively: \(B\) disagrees with \(f(n)\) about whether \(n\) is an element, for each \(n \in \mathbb{N}\).

Assume \(B = f(k)\) for some \(k \in \mathbb{N}\).

- If \(k \in B\), then \(k \notin f(k)\), so \(k \notin B\) \(\Rightarrow\) contradiction
- If \(k \notin B\), then \(k \in f(k)\), so \(k \in B\) \(\Rightarrow\) contradiction

In both cases we have a contradiction.

So \(B \neq f(k)\) for any \(k \in \mathbb{N}\)

\(\Rightarrow f\) is not surjective!

Since no \(f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})\) is surjective,

no \(f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})\) is bijective

\(\Rightarrow \mathcal{P}(\mathbb{N})\) is uncountable

\(\square\)