Definition 1 — Countably infinite, countable and uncountable sets
A set $X$ is countably infinite if there is a bijection $\mathbb{N} \rightarrow X$. A set is countable if it is finite or countably infinite, and is uncountable if it is not countable.

Exercise 2
Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.
**Theorem 3 — Some facts about countability**

(i) If there is an injection $X \rightarrow \mathbb{N}$, then $X$ is countable.

(ii) If there is a surjection $\mathbb{N} \rightarrow X$, then $X$ is countable.

(iii) Properties (i) and (ii) remain true if $\mathbb{N}$ is replaced by any countably infinite set.

(iv) If $X$ and $Y$ are countable, then $X \times Y$ is countable.

(v) The union of countably many countable sets is countable.

**Example 4**

Prove that $\mathbb{Q}$ is countable by defining a surjection from a countable set to $\mathbb{Q}$. 
**Example 5**
Prove that \( \mathbb{Q} \) is countable by expressing it as a union of countably many countable sets.
Example 6

Prove that $\binom{\mathbb{N}}{2}$, the set of all subsets of $\mathbb{N}$ of size 2, is countably infinite.