Definition 1
A discrete probability space \((\Omega, P)\) consists of a countable set \(\Omega\) and a function \(P : \mathcal{P}(\Omega) \rightarrow [0, 1]\), such that:

(i) \(P(\Omega) = 1\); and

(ii) (Countable additivity) For any family \(\{A_i \mid i \in I\}\) of pairwise disjoint subsets of \(\Omega\) indexed by a countable set \(I\), we have

\[
P\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} P(A_i)
\]

Some terminology:

- The word ‘discrete’ refers to countability of \(\Omega\);
- The set \(\Omega\) is called the sample space, and its elements \(\omega \in \Omega\) are called outcomes;
- A subset \(A \subseteq \Omega\) is called an event;
- The function \(P : \mathcal{P}(\Omega) \rightarrow [0, 1]\) is called a probability measure on \(\Omega\);
- For each event \(A\), the value \(P(A)\) is called the probability of \(A\).

Amazingly, everything we could possibly want to prove about discrete probability spaces can be derived from the two conditions in Definition 1.

Exercise 2
Prove that \(P(\Omega \setminus A) = 1 - P(A)\) for all events \(A\), and deduce that \(P(\emptyset) = 0\).
**Theorem 3** — A probability measure is uniquely determined by its values on individual events

Let $\Omega$ be a countable set. Given any subset $\{p_\omega \mid \omega \in \Omega\} \subseteq [0, 1]$, if $\sum_{\omega \in \Omega} p_\omega = 1$, then there is a unique probability measure $\mathbb{P}$ on $\Omega$ such that $\mathbb{P}(\{\omega\}) = p_\omega$ for all $\omega \in \Omega$.

**Proof (sketch)**

**Existence.** Define $\mathbb{P}(A) = \sum_{\omega \in A} p_\omega$ for all $A \subseteq \Omega$, and verify conditions (i) and (ii) from Definition 1. Condition (i) is immediate from the assumption that the numbers $p_\omega$ sum to 1. Condition (ii) follows from properties of the summation operator $\sum$.

**Uniqueness.** Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures on $\Omega$ such that $\mathbb{P}_1(\{\omega\}) = \mathbb{P}_2(\{\omega\}) = p_\omega$ for each $\omega \in \Omega$. Then by countable additivity ($\star$) we have

$$\mathbb{P}_1(A) = \mathbb{P}_1\left(\bigcup_{\omega \in A} \{\omega\}\right) = \sum_{\omega \in A} \mathbb{P}_1(\{\omega\}) = \sum_{\omega \in A} p_\omega = \sum_{\omega \in A} \mathbb{P}_2(\{\omega\}) = \mathbb{P}_2\left(\bigcup_{\omega \in A} \{\omega\}\right) = \mathbb{P}_2(A)$$

for all $A \subseteq \Omega$, where the steps marked $\star$ follow from countable additivity. So $\mathbb{P}_1 = \mathbb{P}_2$.  

**Example 4**

Define a probability space that models the roll of a fair die. Which subset of your sample space represents the event that the die roll is prime? What is the probability that this event occurs?
**Example 5**
A coin shows heads with probability \( p \in [0, 1] \), and tails otherwise. Define a probability space that models the random process of flipping the coin until it shows heads; verify that your probability measure is well-defined.

**Definition 6**
Let \((\Omega, \mathbb{P})\) be a probability space. Events \( A \) and \( B \) are **independent** if \( \mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \).

**Exercise 7**
Let \((\Omega, \mathbb{P})\) be the probability space defined by \( \Omega = [6] \times [6] \) and \( \mathbb{P}((a,b)) = \frac{1}{36} \) for each \((a,b) \in \Omega\). Find events \( A, B, C, D \subseteq \Omega \) such that \( A \) and \( B \) are independent, but \( C \) and \( D \) are not.
**Exercise 8**
When is an event independent from itself?

**Exercise 9**
Prove that the relation ‘A and B are independent’ on the set of all events in a probability space need not be transitive.