Math 300 Class 25
Friday 8th March 2019

Definition 1 — Conditional probability
Let \((\Omega, \mathbb{P})\) be a probability space and let \(B \subseteq \Omega\) be an event with \(\mathbb{P}(B) > 0\). The conditional probability of an event \(A\) given \(B\) is defined by

\[
\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}
\]

Intuitively speaking, \(\mathbb{P}(A \mid B)\) is the updated probability of \(A\) upon receiving the knowledge that the event \(B\) has occurred.

Exercise 2
Let \((\Omega, \mathbb{P})\) be a probability space and \(B \subseteq \Omega\) with \(\mathbb{P}(B) > 0\). Prove that \(\mathbb{P}(- \mid B)\) is a probability measure on \(\Omega\).

Let \(\omega \in \Omega\). Then

\[
\mathbb{P}(\{\omega \mid B\}) = \begin{cases} 0 & \text{if } \omega \notin B : \{\omega \} \cap B = \emptyset \\ \frac{\mathbb{P}(\{\omega\})}{\mathbb{P}(B)} & \text{if } \omega \in B : \{\omega\} \subseteq B \\ \end{cases}
\]

So

\[
\sum_{\omega \in \Omega} \mathbb{P}(\{\omega \mid B\}) = \sum_{\omega \in B} \mathbb{P}(\{\omega \mid B\}) + \sum_{\omega \in \Omega \setminus B} \mathbb{P}(\{\omega \mid B\}) \quad \text{splitting up } \sum
\]

\[
= \sum_{\omega \in B} \frac{\mathbb{P}(\{\omega\})}{\mathbb{P}(B)} + 0 \quad \text{as noted above}
\]

\[
= \frac{1}{\mathbb{P}(B)} \sum_{\omega \in B} \mathbb{P}(\{\omega\}) \quad \text{factoring out } \frac{1}{\mathbb{P}(B)}
\]

\[
= \frac{1}{\mathbb{P}(B)} \mathbb{P}(B) \quad \text{countable additivity}
\]

\[
= 1
\]

\(\Rightarrow \mathbb{P}(- \mid B)\) is a probability measure on \(\Omega\).
**Theorem 3 — Bayes's theorem (simple version)**

Let $A$ and $B$ be events in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then

$$
\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \mathbb{P}(B)}{\mathbb{P}(A)}
$$

**Proof**

$$
\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B) \mathbb{P}(B)}{\mathbb{P}(A) \mathbb{P}(B)} = \frac{\mathbb{P}(A \cap B) \mathbb{P}(B)}{\mathbb{P}(A)}
$$

This form of Bayes's theorem isn't very enlightening, so we will derive a more useful version of it.

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**Theorem 4 — Bayes's theorem (slightly more useful version)**

Let $A$ and $B$ be events in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$ and $0 < \mathbb{P}(B) < 1$. Prove that

$$
\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \mathbb{P}(B)}{\mathbb{P}(A \mid B) \mathbb{P}(B) + \mathbb{P}(A \mid B^c) \mathbb{P}(B^c)}
$$

[We have written $B^c$ to denote the event $\Omega \setminus B$.]

**Proof**

Note that $A = A \cap \Omega = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$

and $(A \cap B) \cap (A \cap B^c) = A \cap (B \cap B^c) = A \cap \emptyset = \emptyset$

so by countable additivity,

$$
\mathbb{P}(A \cap B) \mathbb{P}(B) + \mathbb{P}(A \cap B^c) \mathbb{P}(B^c) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)
$$

$$
= \mathbb{P}(A \cap B) \cup (A \cap B^c)
$$

Now substitute into Thm 3.
Exercise 5
A town has 10000 inhabitants, of whom 30 are infected with Disease X. An inhabitant of the town tests positive for Disease X. Given that the test is 99% accurate, what is the probability that the person is infected with Disease X?

Informally: Let \( B = \{ \text{person is infected} \} \) (so \( B^c = \{ \text{person is not infected} \} \))
& let \( A = \{ \text{test is positive} \} \).

By Bayes's theorem:
\[
\Pr(B | A) = \frac{\Pr(A | B) \Pr(B)}{\Pr(A | B) \Pr(B) + \Pr(A | B^c) \Pr(B^c)}
\]
\[
= \frac{\frac{99}{100} \cdot \frac{30}{10000}}{\frac{99}{100} \cdot \frac{30}{10000} + \frac{1}{100} \cdot \frac{9770}{10000}}
\]
\[
= \frac{99 \cdot 30}{99 \cdot 30 + 9770}
\]
\[
= \frac{2970}{12940} \approx 0.230
\]

**Theorem 6** — Bayes's theorem (even more useful version)
Let \( A \) be an event in a probability space \( (\Omega, \mathbb{P}) \) such that \( \Pr(A) > 0 \), and let \( B_1, B_2, \ldots, B_n \) be mutually exclusive events such that \( \Pr(B_i) > 0 \) for all \( 1 \leq i \leq n \) and such that \( B_1 \cup B_2 \cup \cdots \cup B_n = \Omega \). Then
\[
\Pr(B_i | A) = \frac{\Pr(A | B_i) \Pr(B_i)}{\Pr(A | B_1) \Pr(B_1) + \Pr(A | B_2) \Pr(B_2) + \cdots + \Pr(A | B_n) \Pr(B_n)}
\]
for all \( 1 \leq i \leq n \).

**Proof.** Notice that \( A = A \cap \Omega = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \). By countable additivity,
\[
\Pr(A) = \Pr(A \cap B_1) + \Pr(A \cap B_2) + \cdots + \Pr(A \cap B_n)
\]
Now observe that \( \Pr(A \cap B_i) = \Pr(A | B_i) \Pr(B_i) \) for each \( k \in [n] \) and substitute into Theorem 3. \( \square \)
Exercise 7
A small car manufacturer, Cars N'At, makes three models of car: the Allegheny, the Monongahela and the Ohio. It made 3000 Alleghenys, 6500 Monongahelas, and 500 Ohios. In a given day, an Allegheny breaks down with probability $\frac{1}{100}$, a Monongahela breaks down with probability $\frac{1}{200}$, and the notoriously unreliable Ohio breaks down with probability $\frac{1}{5}$. An angry driver calls Cars N'At to complain that their car has broken down. Find the probability that the driver was driving an Ohio.

Let $A = \{\text{car broke down}\}$, $B_1 = \{\text{car was an Allegheny}\}$, $B_2 = \{\text{car was a Monongahela}\}$, $B_3 = \{\text{car was an Ohio}\}$. \(\text{Note: these are mutually exclusive & cover all possibilities} \implies B_1 \cup B_2 \cup B_3 = \Omega\)

$$P(B_3 | A) = \frac{P(A | B_3) P(B_3)}{P(A | B_1) P(B_1) + P(A | B_2) P(B_2) + P(A | B_3) P(B_3)}$$

$$= \frac{\frac{1}{20} \cdot \frac{500}{10000}}{\frac{1}{100} \cdot \frac{3000}{10000} + \frac{1}{200} \cdot \frac{6500}{10000} + \frac{1}{20} \cdot \frac{500}{10000}}$$

$$= \frac{10.5}{230 + 1.65 + 10.5}$$

$$= \frac{50}{60 + 6.5 + 50}$$

$$= \frac{10}{12 + 13 + 10}$$

$$= \frac{10}{35}$$

$$= \frac{2}{7} \approx 29\%$$