Definition 1 — Random variable
Let \((\Omega, \mathbb{P})\) be a discrete probability space and let \(E\) be a set. An \(E\)-valued random variable on \((\Omega, \mathbb{P})\) is a function \(X : \Omega \rightarrow E\).

The set \(E\) is called the state space of \(X\).

We think of \(X\) as being a ‘variable’ element of \(E\) depending on the outcome of a random process—if the outcome of the random process is \(\omega\), then the value of \(X\) is \(X(\omega)\).

Example 2
Let \(X\) be a real-valued random variable on a discrete probability space \((\Omega, \mathbb{P})\). Express the following events as subsets of \(\Omega\).

(a) The event that \(X = 0\);

(b) The event that \(X \in \mathbb{Z}\);

(c) The event that \(e^{X^2} > 3X + 4\).
**Exercise 3**
A biased coin, which shows heads with probability $0 < p < 1$, is flipped $n$ times, where $n$ is some natural number. Let $N$ be the number of heads that show. Describe a probability space $(\Omega, \mathbb{P})$ which models this random process, give an explicit definition of $N$ as a function $\Omega \to E$ (for an appropriate choice of state space $E$), and compute $\mathbb{P}\{N = k\}$ for each $k \in E$.

**Exercise 4**
Let $(\Omega, \mathbb{P})$ be a discrete probability space and let $X$ be a random variable on $(\Omega, \mathbb{P})$. Prove that the events $\{X = e\}$ for $e \in E$ are mutually exclusive (i.e. pairwise disjoint).
Definition 5
Let \((\Omega, \mathbb{P})\) be a discrete probability space and let \(X : \Omega \to E\) be a random variable. The \textbf{probability mass function} of \(X\) is the function \(f_X : E \to [0, 1]\) defined by \(f_X(e) = \mathbb{P}\{X = e\}\) for all \(e \in E\).

Example 6
Let \(E\) be a set and let \(X\) be an \(E\)-valued random variable on a probability space \((\Omega, \mathbb{P})\). Prove that \(X_\ast \mathbb{P}\) is a probability measure on \(E\), where \((X_\ast \mathbb{P})(A) = \mathbb{P}\{X \in A\}\) for all \(A \subseteq E\).
**Definition 7**
Let \((\Omega, \mathbb{P})\) be a probability space. A family of events \(\{A_i \mid i \in I\}\) is **mutually independent** if
\[
\mathbb{P}\left( \bigcap_{i \in I} A_i \right) = \prod_{i \in I} \mathbb{P}(A_i)
\]
In particular, events \(A_1, A_2, \ldots, A_n\) are mutually independent if and only if
\[
\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \cdots \times \mathbb{P}(A_n)
\]

**Exercise 8**
Let \(p \in [0, 1]\) and suppose that \(X_1, X_2, \ldots, X_n\) are \(\{0, 1\}\)-valued with
\[
f_{X_k}(i) = \begin{cases} 
1 - p & \text{if } i = 0 \\
p & \text{if } i = 1 
\end{cases}
\]
for each \(k \in [n]\). Assuming the events \(\{X_k = i\}\) are mutually independent for \(k \in [n]\) and \(i \in \{0, 1\}\), prove that the \(\{0, 1, \ldots, n\}\)-valued random variable \(X = X_1 + \cdots + X_n\) satisfies
\[
f_X(r) = \binom{n}{r} p^r (1 - p)^{n-r}
\]
for each \(r \in \{0, 1, \ldots, n\}\).