THE DE RHAM AND MORSE A_∞-PRECATEGORIES

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Throughout, we will take (M, g) a compact oriented Riemannian manifold.

We have seen that the de Rham complex $\Omega^\bullet(M)$ contains more information than the de Rham cohomology $H^\bullet_{\mathrm{dR}}M$. In particular, when M is 1-connected, the complex picks up rational homotopy theoretic data (see [DGMS75]) that is lost after passing to cohomology (unless the manifold is "formal"). There is a fix, as described by Sean: if we consider $H^\bullet_{\mathrm{dR}}M$ as an A_∞-algebra in the natural manner, we can recover the rational homotopy theory of M. Let us see how this fits into our story about Morse theory. We will be somewhat sketchy on the Morse theory, but the next talk will fill these details in. For the details, see [Abo09].

Recall that the Morse complex $CM^\bullet(M, f)$ of a Morse(-Smale) function f computes the de Rham cohomology of M. Though this is a classical statement, Witten in [Wit82] suggested a different proof that interpolates between the de Rham and Morse complexes. Fix some real parameter $\hbar > 0$ and and consider the deformed de Rham complex,

$$\Omega^\bullet(M, f, \hbar) = (\Omega^\bullet(M), d_{f,\hbar})$$
$$d_{f,\hbar} = \exp(-f/\hbar)\hbar d \exp(f/\hbar) = \hbar d + df \wedge .$$

The "interpolation" is to be considered as follows: for $\hbar \to \infty$, the first term in the deformed differential dominates, yielding the usual de Rham differential. More precisely, multiplication by $\hbar \exp(f/\hbar)$ is an isomorphism of chain complexes $\Omega^\bullet(M, f, \hbar) \to \Omega^\bullet(M)$. On the other hand, Witten showed that there is a subcomplex of "small-eigenvalue" forms

$$\Omega^\bullet_{\mathrm{sm}}(M, f, \hbar) \subset \Omega^\bullet(M, f, \hbar)$$

that is quasi-isomorphic to the Morse complex under a map[1]

$$\phi(\hbar) : CM^\bullet(M, f) \xrightarrow{\simeq} \Omega^\bullet(M, f, \hbar)$$

taking a critical point to the corresponding small eigenform with support near that point. This gives us a chain of quasi-isomorphisms

$$CM^\bullet(M, f) \xrightarrow{\simeq} \Omega^\bullet_{\mathrm{sm}}(M, f, \hbar) \xrightarrow{\simeq} \Omega^\bullet(M, f, \hbar) \xrightarrow{\times \hbar e^{f/\hbar}} \Omega^\bullet(M)$$

inducing isomorphisms on cohomology,

$$\mathrm{H}^\bullet(CM^\bullet(M, f)) \cong \mathrm{H}^\bullet_{\mathrm{dR}}\, M,$$

as desired.

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[1] This is not quite true. This map is just an isomorphism of graded vector spaces that computes the same cohomology. This is a subtle point that I don't really understand, and will persist in the statement of the main theorem below. Understanding this is probably a job for one of the analysts here.

Now it is natural to ask whether Morse theory can also be refined using some type of A_∞-structure in order to detect rational homotopy theory, and if so, whether this isomorphism can be promoted to an isomorphism of such A_∞-structures. Naively, we would like to say that Morse cohomology can be given the structure of an A_∞-algebra, and that Witten's arguments can be extended to show that the higher multiplications on the Morse and de Rham sides agree to leading order in $\hbar$.

Unfortunately, there is no A_∞-structure on Morse cohomology. Recall that Sean constructed the A_∞-algebra structure on the de Rham cohomology by Kadeishvili's algorithm, which used the multiplicative structure on $\Omega^\bullet(M)$ as well as a bit of Hodge theory (see, for instance, [Mer98]). The same algorithm clearly will not work for Morse cohomology – it is not the cohomology of a dga, and moreover does not naturally split-include into the Morse complex. This is a hint that perhaps "A_∞-algebra" is not the correct structure to be looking for.

Fukaya in [Fuk93] noticed that the correct A_∞-structure on Morse theory is that of an A_∞-precategory. In other words, working with chain complexes is not enough to find higher multiplications – we are forced to "categorify" and replace higher multiplications with higher compositions.[2] With this in mind, let us present the definition of an A_∞-precategory (due to Kontsevich and Soibelman in [KS00]) and then give the examples which will concern us.

Definition 1. An A_∞-precategory $\mathcal{A}$ (over $\mathbb{R}$) consists of

(1) a class of objects $\mathrm{Ob}(\mathcal{A})$,
(2) for each $n \geqslant 2$, a distinguished subset of n-tuples of "transversal sequences"
$$\mathrm{Ob}^n_{\mathrm{tr}}(\mathcal{A}) \subset \mathrm{Ob}(\mathcal{A}),$$
(3) for each $(X_0, X_1) \in \mathrm{Ob}^2_{\mathrm{tr}}(\mathcal{A})$, a $\mathbb{Z}$-graded vector space
$$\mathrm{Hom}_{\mathcal{A}}(X_0, X_1),$$
(4) for each $(X_0, \dots, X_d) \in \mathrm{Ob}^{d+1}_{\mathrm{tr}}(\mathcal{A})$, a map of graded vector spaces
$$m_d : \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \to \mathrm{Hom}_{\mathcal{A}}(X_0, X_d)[2-d],$$

such that every subsequence of a transversal sequence is transversal and the maps m_d satisfy the A_∞-relations: for each $d \geqslant 1$,
$$\sum (-1)^{r+st} m_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$
where the sum runs over all decompositions $d = r + s + t$ where $s \geqslant 1$.

For some practice with the A_∞-relations it is worth working out the following.

Exercise 2. Check that the A_∞-relation for $d = 1$ enforces $m_1 \circ m_1 = 0$, implying that each hom-space (when defined) is a chain complex $(\mathrm{Hom}_{\mathcal{A}}(X_0, X_1), m_1)$. The A_∞-relation for $d = 2$ enforces that m_1 is a derivation over m_2:
$$m_1 \circ m_2 - m_2(m_1 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m_1) = 0$$
as maps $\mathrm{Hom}_{\mathcal{A}}(X_2, X_1) \otimes \mathrm{Hom}_{\mathcal{A}}(X_1, X_0) \to \mathrm{Hom}_{\mathcal{A}}(X_2, X_0)[1]$.[3]

Exercise 3. Define an A_∞-category.

[2] As a simple example of this philosophy, verify that an $\mathbb{R}$-linear category – i.e. a category where the hom-sets are $\mathbb{R}$-vector spaces and the composition $\circ$ is a bilinear map – with one object is precisely the data of an $\mathbb{R}$-algebra.

[3] I am not being particularly careful with signs or grading shifts here. I leave that as an exercise.

Notice that an A_∞-category with one object is precisely an A_∞-algebra. It is in this sense that we have categorified the notion of an A_∞-algebra. For comparison, check that a dg-category with one object is a dga.

We now discuss the (A_∞-pre)categorifications of the de Rham and Morse complexes.

Definition 4. The de Rham category $\mathrm{dR}(M,\hbar)$ is the dg-category with objects the smooth functions $\mathrm{Ob}(\mathrm{dR}(M,\hbar)) = C^\infty(M)$ and hom-spaces

$$\mathrm{Hom}_{\mathrm{dR}(M,\hbar)}(f,g) = (\Omega^\bullet(M), d_{g-f,\hbar} = \hbar d + d(g-f)\wedge).$$

Composition of morphisms is given by wedge product,

$$\mathrm{Hom}_{\mathrm{dR}(M,\hbar)}(f,g) \otimes \mathrm{Hom}_{\mathrm{dR}(M,\hbar)}(g,h) \xrightarrow{\wedge} \mathrm{Hom}_{\mathrm{dR}(M,\hbar)}(f,h)$$

and is associative by the associativity of the wedge product. We take the A_∞-categorical structure to be the trivial one; here m_1 is the deformed differential on each hom-space, m_2 is the wedge product above, and all higher multiplications are taken to be zero: $m_k = 0$ for $k \geqslant 3$. Notice that $\mathrm{Hom}_{\mathrm{dR}(M,\hbar)}(f,f)$ is the usual de Rham complex, and the identity map $\mathrm{id} : f \to f$ is the constant zero-form 1.

Exercise 5. First make sense of the wedge product of forms as composition maps and then check that they commute with the differentials (i.e. are maps of chain complexes, as is required to form a dg-category). The fact that we are considering differences $f - g$ is crucial here.

Exercise 6. Check that the A_∞-structure on any dg-category (as described in this case) satisfies the A_∞ relations, by dint of the composition being associative.

Next up is the Morse category, which we sketch the definition of.

Definition 7. The Morse A_∞-precategory $\mathrm{Morse}(M)$ has objects the smooth functions $\mathrm{Ob}(\mathrm{Morse}(M)) = C^\infty(M)$ and transversal sequences

$$\vec{f} = (f_0, \ldots, f_n) \in \mathrm{Ob}_{\mathrm{tr}}^{n+1}(\mathrm{Morse}(M))$$

those sequences for which $f_{ij} = f_j - f_i$ is Morse for each $i \neq j$. The hom-spaces are given

$$\mathrm{Hom}_{\mathrm{Morse}(M)}(f,g) = CM^\bullet(M, g-f)$$

The first multiplication is simply defined to be the Morse differential $m_1 = \delta$. We will not define the higher multiplications m_k ($k \geqslant 2$) here, as Peng will describe them in detail later. Roughly speaking, they are constructed analogously to the construction of $m_1 = \delta$ (using gradient flows), replacing critical points and functions with sequences of critical points and (transverse) functions.

It is important to note, in particular, that the construction of the higher multiplications on the Morse category are not coming from manipulations that are just algebraic in nature (as the construction of the A_∞-algebra structure on $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ was). Instead, the m_k are constructed from geometric data about gradient flows on M.

With these definitions in mind, we can now state Fukaya's conjecture.[4]

[4] There is a similar statement in the work [KS00] of Kontsevich and Soibelman, though the de Rham category there is defined differently, avoiding the Witten deformation altogether. It would be useful to understand the precise relation with the ideas here.

Theorem 8. *There is an equivalence of A_∞-precategories*

$$\mathrm{Morse}(M) \xrightarrow{\simeq} \mathrm{dR}(M, \hbar).$$

Since the category on the right is equivalent to the rather simple category $\mathrm{dR}(M)$ (just the data of $\Omega^\bullet(M)$, we conclude that Morse theory does indeed contain the data of rational homotopy theory: however, instead of finding it in an A_∞-algebra structure, we find it in an A_∞-categorical structure. The goal of this reading group is to understand the proof of this statement, which reduces fairly straightforwardly to an analytic result in [CLM14] showing that $m_k^{\mathrm{Morse}} \approx m_k^{\mathrm{dR}}$ to leading order in $\hbar$ using the WKB approximation. Here m_k^{dR} are the higher compositions on the subcategory of small eigenforms $\mathrm{dR}_{\mathrm{sm}}(M, \hbar) \hookrightarrow \mathrm{dR}(M, \hbar)$ induced from the dg-category structure on $\mathrm{dR}(M, \hbar)$ (Kontsevich and Soibelman call this procedure "homological perturbation" in [KS00]).

Proof idea. Following Witten, we define an auxiliary A_∞-category of "small" eigenforms. Define $\mathrm{dR}_{\mathrm{sm}}(M, \hbar)$ to be the subcategory of $\mathrm{dR}(M, \hbar)$ with the same objects but with morphism complexes

$$\mathrm{Hom}_{\mathrm{dR}_{\mathrm{sm}}(M,\hbar)}(f, g) \hookrightarrow \mathrm{Hom}_{\mathrm{dR}(M,\hbar)}(f, g)$$

the subcomplex of forms spanned by those with eigenvalues small enough (how small depends on $\hbar, f, g$ and the metric on M – we won't be precise here). This subcategory is actually not a category – the naive composition given by inclusion, wedge product, and then spectral projection,

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{sm}}(f_2 - f_1) \otimes \mathrm{Hom}_{\mathrm{sm}}(f_1 - f_0) \arrow[r, "(\iota,\iota)"] & \mathrm{Hom}(f_2 - f_1) \otimes \mathrm{Hom}(f_1 - f_0) \arrow[d, "\wedge"] \\ & \mathrm{Hom}(f_2 - f_0) \arrow[d, "P"] \\ & \mathrm{Hom}_{\mathrm{sm}}(f_2 - f_0) \end{tikzcd}$$

which we shall call m_2, is not associative. The obvious guess, then, is that the category of small eigenforms is an A_∞-category. Indeed, defining the differential m_1 to be inclusion followed by the differential in $\mathrm{dR}(M, \hbar)$ followed by projection, it is easy to see that m_1 is a derivation over m_2.

How do we define the higher compositions? We will use the fact that the hom-complexes in the small eigenform category are deformation retracts of the hom-complexes of the full de Rham category. The argument is exactly the argument in the proof of the Hodge decomposition: if G is the Green's operator for the Witten Laplacian Δ (associated to the function $f_{01} = f_1 - f_0$) defined to be zero on small eigenforms (multiplied by $\mathrm{id} - P$, say) then $H = d^* G$ is a homotopy from id to P on the full deformed de Rham complex:

$$dH + Hd = \mathrm{id} - P.$$

Now we define

$$m_3 : \mathrm{Hom}_{\mathrm{sm}}(f_3 - f_2) \otimes \mathrm{Hom}_{\mathrm{sm}}(f_2 - f_1) \otimes \mathrm{Hom}_{\mathrm{sm}}(f_1 - f_0) \to \mathrm{Hom}_{\mathrm{sm}}(f_3 - f_0)[-1]$$

by

$$m_3(\alpha_{23}, \alpha_{21}, \alpha_{10}) = P_{03}\left(H_{13}(\alpha_{23} \wedge \alpha_{12}) \wedge \alpha_{01} + \alpha_{23} \wedge H_{02}(\alpha_{12} \wedge \alpha_{01})\right).$$

It is a messy exercise to check that with this definition the $d = 3$ A_∞-relation is satisfied.

To see how to define the higher multiplications, notice that we can represent the m_2 and m_3 by 2- and 3− trees – see section 2.3 of [CLM14], for instance. We sum over all (topological types) of k-trees

$$m_k = \sum_T m_k^T$$

where each m_k^T is defined along the tree as follows:

(a) applying the inclusion $\iota_{i(i+1)}$ at each incoming (semi-infinite) edge;
(b) applying the wedge product at each interior vertex;
(c) applying the homotopy H_{ij} to each internal edge labelled ij;
(d) applying the projection P_{0k} to the outgoing (semi-infinite) edge.

Of course, one has to check that the resulting multiplications satisfy the A_∞-relations. One can also obtain this A_∞-structure more abstractly via the homological perturbation lemma of [KS00].

Now we have an A_∞-category $\mathrm{dR_{sm}}(M, \hbar)$ that is A_∞-equivalent to $\mathrm{dR}(M, \hbar)$, as it is a deformation retract. Since the small eigenform hom-complexes are identified (as graded vector spaces) with the Morse hom-complexes, it now suffices to show that the small eigenform and the Morse higher multiplications agree to leading order in $\hbar$.[5] Understanding this is our main goal. □

For further reading, here are a few questions that might be worth investigating:

(a) Show rigorously that Witten's map between the Morse complex and the small eigenform complex – which is a priori only a map of graded vector spaces – "induces" an isomorphism on cohomology. Argue similarly for the equivalence of A_∞-categories.
(b) It is known that every A_∞-category is A_∞-equivalent to a dg-category. This is effectively a "straightening" theorem. Is there a procedure for constructing that dg-category, and if so, what is it for the Morse category and what is the relation to the de Rham category?
(c) Is there a notion of an A_∞-nerve? In particular, is there a space we can associate these categories? What will that space be or what will it know about the homotopy type of M? For instance, what is the totalization of the cosimplicial object obtained from $\Omega^\bullet(M)$ via Dold-Kan?
(d) Compare the statement (and methods) of the theorem here to the theorem proved in [KS00] by Kontsevich and Soibelman.

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[5] Again, it is not clear to me why this is enough. See the questions below.

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