RATIONAL HOMOTOPY THEORY

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Contents

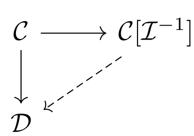
These are notes from John Francis' lectures on rational homotopy theory at Northwestern during the Winter quarter of 2019.

1. January 14

There will be no class on the 16th, but instead we will have class on the 18th, same place same time.

The first main theorem we will discuss in this class is this big theorem of Quillen from either 1968 or 1969 which got the whole subject started. The idea is that if you think about spaces purely rationally, topology should reduce to algebra.

Consider the category $\mathsf{Top}_*^{\geq 2}$ of pointed topological spaces that are 1-connected where the morphisms are homotopy classes of maps. Consider the class of rational homotopy equivalences $\mathcal{W}$: a map $f : X \to Y$ is in this class if it induces isomorphisms on either rational homotopy groups $\pi_i \otimes \mathbb{Q}$ or rational homology groups. Recall now that given a subcategory $\mathcal{I} \subset \mathcal{C}$, one defines the localization $\mathcal{C} \to \mathcal{C}[\mathcal{I}^{-1}]$ which has the universal property that if the vertical map inverts maps in $\mathcal{I}$ then there is a induced dashed map:



The objects in this localized categories are the objects of $\mathcal{C}$ and the morphisms are iterated zig-zags of morphisms. This is in general impossible to describe nicely.

Theorem 1 (Quillen)**.** *There is an equivalence*

$$\mathsf{Top}_*^{\geq 2}[\mathcal{W}^{-1}] \simeq \mathsf{coAlg}_{com}^{\geq 2}[\mathcal{W}^{-1}]$$

where the right hand side is cocommutative coalgebras with cohomology in degrees greater than 1 localized at quasi-isomorphisms.

What's interesting is that this paper has like 363 or citations but only like 4 in the first couple years afterward. The paper seems to have landed with a bit of a thud... So I looked into why. If you look at the MathSciNet review written by Adams you'll see insert quote here. Typical Adams. So why are people so impressed now? Well to prove this result Quillen developed the theory of model categories, in part because describing these localizations is so difficult unless you have good control over certain subcategories. But there's another reason that this proved to be so unpopular. Often the way homotopy theory is formulated is that X a space is a "problem" and the "solution" is its homotopy groups. Keep this in mind as I now discuss what Quillen did.

First, Quillen factored the equivalence of the theorem above through

$$\mathsf{LieAlg}_{\mathbb{Q}}^{\geq 1}[\mathcal{W}^{-1}]$$

where $\mathcal{W}$ are quasi-isomorphisms. The dg Lie algebra structure associated to a space X is $\pi_* X \otimes \mathbb{Q}[-1]$ with the Whitehead bracket. Notice that this is not a very natural because there are no differentials in sight. So here's what Quillen did instead. Since topological spaces are very technically inconvenient he passed first to the category of reduced simplicial sets $\mathsf{sSet}^{\geq 2}$. There is an adjunction between $\mathsf{Top}_*^{\geq 2}$ and simplicial sets via the geometric realization and (a slightly modified) singular simplicial set functor. Next he looked at the adjunction between simplicial sets and simplicial groups sGrp via the simplicial loops functor G which satisfies $|GX_\bullet| \simeq \Omega |X_\bullet|$. Next we pass to $\mathsf{sHopf}^{\geq 1}$ which are simplicial dg Hopf algebras over $\mathbb{Q}$ via the usual homology for simplicial sets (free abelian group). Next we pass to $\mathsf{sLieAlg}$ simplicial Lie algebras via the primitives construction, which are equivalent to LieAlg dg Lie algebras. Here we did two things that sorta cancelled – we took loops at some point but then realized that the Hopf algebra was a universal enveloping algebra of some Lie algebra. In particular in turns out these are all equivalences. Notice that this is technically inconvenient because you get functors both ways but not adjoints (since you're composing left with right adjoints etc.)

This complicated picture is maybe one of the reason why this theory wasn't immediately popular or accessible. Another problem is that we're kinda skipped to the "solution" by looking at the homotopy groups and the Whitehead bracket in the first place. Anyway, so ten years passed, and Sullivan kinda reproves the same thing but in a much better way, and the theory takes off.

Let's put it this way. We want to understand simply connected spaces. We've seen that apparently one way to understand spaces is through Lie algebras

$$\begin{array}{ccc} & (\mathsf{Top}_*^{\geq 2})_{\mathbb{Q}} & \\ \swarrow & & \searrow^{\Omega^*_{PL}} \\ \mathsf{LieAlg}_{\mathbb{Q}}^{\geq 1} & \xrightarrow{\text{Koszul duality}} & \mathsf{Alg}_{\mathrm{com},\mathbb{Q}}^{\mathrm{aug,\ op},\leq 2} \end{array}$$

Sullivan's map on the right can't quite be singular cochains since they're not commutative on the nose. Instead he came up with PL de Rham forms (since we are working in characteristic zero one expects something like the de Rham complex which is commutative on the nose). Sullivan than proved a similar equivalence as long as the homology is finite dimensional. Then there is a purely algebraic relationship between Lie algebras and augmented commutative algebras.

Recall that an augmented commutative algebra is a map of commutative dgas $A \to \mathbb{Q}$. Intuitively we think of this as a map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} A$ (where maybe the right is a dg-scheme or something, let's not worry about that). Roughly speaking the functor from right to left is

$$\varepsilon : A \to \mathbb{Q} \mapsto (T_\varepsilon \operatorname{Spec} A)[-1] \simeq \mathsf{Der}_A(\mathbb{Q}, \mathbb{Q})$$

How do we go back? The Chevalley-Eilenberg complex

$$\mathfrak{g} \mapsto C^*(\mathfrak{g}) \simeq \mathrm{Hom}_{\mathrm{Rep}\,\mathfrak{g}}(\mathbb{Q}, \mathbb{Q}).$$

Everywhere everything is derived. This doesn't quite give you an equivalence unless you restrict to finite dimensional and the relevant degree vanishing conditions above.

So when you use the Sullivan theory it turns out that computations become fairly straightforward – we will describe how to read off rational homotopy groups of a space given a commutative differential graded algebra over $\mathbb{Q}$. So the first thing we do is we'll prove Sullivan's equivalence. Two, we will use Koszul duality to compute the rational homotopy groups of a space. In particular there's a large class of spaces that are called formal – as dgas they are equivalent to their cohomology. For formal spaces this computation will be super easy! Compact Kähler manifolds are good examples of formal spaces.

One might find these degree bounds (and simply connected conditions) rather annoying. Indeed, one might consider all Lie algebras. Then the natural generalization of commutative dg algebras is that of formal moduli problems over $\mathbb{Q}$ which are roughly describing deformations of something, i.e.

$$\mathsf{FModuli}_{\mathbb{Q}} \subset \mathrm{Fun}(\mathsf{Artin}_{\mathbb{Q}}, \mathsf{Spaces}).$$

The basic example you should have in mind is that of deformations of an algebraic variety Y and e.g.

$$F = \mathsf{Def}_Y(R) = \{\tilde{Y}/R \text{ such that } \tilde{Y} \times_R \cong Y\}.$$

Given a commutative algebra just consider the functor that it corepresents. Now the generalized version of Koszul duality is what for years and years people have used the motto that formal moduli are "controlled" by a dg Lie algebra. The Maurer-Cartan functor is a way of going back from the Lie algebra to the formal moduli problem. This makes formal this motto (done by Jacob Lurie). Roughly speaking think of those formal moduli problems coming from So we will next discuss Kähler manifolds and formality and then finally moduli problems.

2. January 23

Recall the theorem of Quillen from last time. The first step is to show that, taking spaces with finite dimensional homology

$$\mathsf{Top}_*^{\geq 2}[\mathcal{W}^{-1}] \simeq \mathrm{ho}(\mathsf{Spaces}_{*,\mathbb{Q}}^{\geq 2}).$$

where on the right we have not a localization at quasi-isomorphisms but actually a quotient where we identify homotopic maps. Moreover objects on the right are (nice enough) spaces Y such that $\pi_* Y \in \mathsf{Vect}_{\mathbb{Q}}$.

Since we may not go into detail on the model category side of things let's just make sure this is a reasonable assertion. In particular, let's try to understand something simpler:

$$\mathsf{Ab}[\mathcal{W}^{-1}] \simeq \mathsf{Vect}_{\mathbb{Q}}.$$

Roughly speaking we are only considering the Eilenberg-MacLane spaces on the left. On the right hand side we know that this should correspond to maps $[K(A \otimes Q, n), K(B \otimes Q, n] = \mathrm{Hom}_{\mathsf{Ab}}(A \otimes \mathbb{Q}, B \otimes \mathbb{Q})$. some argument about zig zags Quillen's theory is roughly an upgrade of this for arbitrary spaces built out of these Eilenberg-Maclane spaces.

Now the second step is to construct a functor

$$\mathsf{Spaces}_{\mathbb{Q}}^{\geq 2} \xrightarrow{A} \mathsf{CAlg}_{\mathbb{Q}}^{\leq -2, \mathrm{op}}.$$

Intuitively this functor should be equivalent to simply taking singular cochains $C^*(X, \mathbb{Q})$ together with its multiplication. In particular A should be some strictly commutative refinement of singular cochains. The most important property that A should have, however, is that it should send homotopy pullbacks to homotopy pushouts. Well okay the functor sending everything to $\mathbb{Q}$ satisfies this so let's put in another requirement. We ask that

$$A(K(\mathbb{Q}, n)) \simeq \mathrm{Sym}(\mathbb{Q}[-n]).$$

More generally,

$$A(K(V, n)) \simeq \mathrm{Sym}(V^\vee[-n]).$$

Once we have this we will construct a functor backwards called the space of augmentations $\mathrm{Maps}(-, \mathbb{Q})$. We will show that (for simply connected X with finite dimensional homology)

$$X \simeq \underset{\mathsf{CAlg}_{\mathbb{Q}}}{\mathrm{Maps}}(A(X), \mathbb{Q})$$

and that

$$\mathrm{Maps}(X, Y) \simeq \underset{\mathsf{CAlg}_{\mathbb{Q}}}{\mathrm{Maps}}(AX, AY)$$

Actually okay there is one more requirement that we need: we need our spaces to have Postnikov towers.

Now given this theoretical framework, how do we actually compute some rational homotopy groups? That will be step three.

Okay, so let's look at our requirements for the functor A. What are homotopy pullbacks in spaces?

Definition 2. We say that **the homotopy pullback** of a span is

$$\begin{array}{ccc} X \times_Y Y^{[0,1]} \times_Y Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

the space of triples of points in X and Y' together with a path in Y between the images.

We define **a homotopy pullback** to be the actual pullback of the diagram where we have replaced the map $X \to Y$ by a Serre fibration $\bar{X} \to Y$. In the above case $\bar{X} = X \times_Y Y^{[0,1]}$. Notice that we are asking that $\bar{X}$ is homotopy equivalent to X over Y.

Lemma 3. *The homotopy type of the homotopy pullback depends only on* $[f]$ *and* $[g]$.

Proof. Homework 1. □

This is the advantage of the homotopy pullback. For instance if we were to take the actual pullback of the diagram $* \to D^2 \xleftarrow{*}$. Depending on the maps you may find that the resulting pullback is different. However the homotopy pullback will only care about the homotopy classes of these maps!

The most basic type of homotopy pullback is the case in which the base is a point, i.e.

$$\begin{array}{ccc} X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

where we just recover the product. What about on the side of commutative algebras? Well the Künneth theorem tells us that

$$C^*(X \times Y) \cong C^*(X) \otimes C^*(Y).$$

Lemma 4. *The pushout in* CAlg_k

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A' \sqcup_A B \end{array}$$

is just $A \sqcup_A B = A' \otimes_A B$.

Proof. Homework 2. □

Recall the map $\mathsf{sAb} \xrightarrow{N} \mathsf{Ch}$. The Alexander-Whitney map

$$NR \otimes NS \to N(R \otimes S)$$

given by

$$d = \sum (-1)^i d^{\text{front}} \otimes d^{\text{back}}.$$

Because we only use front and back you can see that this won't be commutative. In particular consider applying $\mathbb{Z}\mathsf{Sing}$ to $X \times X \times X$. This yields

$$\mathbb{Z}\mathsf{Sing}X \to \mathbb{Z}(\mathsf{Sing}X \times \mathsf{Sing}X).$$

Applying N and dualizing yields the cup product.

Now there is a parameterized Künneth theorem where given a homotopy pullback $X' = X \times_Y Y'$ (with some conditions) then we obtain a homotopy pushout $A(X') \simeq A(X) \otimes_{A(Y)} A(Y')$. This is exactly the statement that the Eilenberg-Moore spectral sequence converges under the conditions.

Let's now look at condition three.

Theorem 5. *Every simply-connected space X has a Postnikov tower i.e. a tower*

$$\cdots P_n X \to P_{n-1} X \to \cdots \to P_0 X$$

with maps $X \to P_n X$ for each n inducing

$$\pi_* X \xrightarrow{\cong} \pi_* P_n X, \quad * \leq n$$

and moreover such that

$$\pi_* P_n X = 0, \quad * > n.$$

Finally we ask that there is a homotopy pushout diagram

$$\begin{array}{ccc} P_n X & \longrightarrow & * \\ \downarrow & & \downarrow \\ P_{n-1} X & \longrightarrow & K(\pi_n X, n+1) = BK(\pi_n X, n) \end{array}$$

What is that last condition asking? Roughly the first two conditions are asking that we have a fiber bundle

$$\begin{array}{ccc} K(\pi_n X, n) & \longrightarrow & P_n X \\ \downarrow & & \downarrow \\ * & \longrightarrow & P_{n-1} X \end{array}$$

The third condition is actually requiring that we get a principal bundle. Actually the first two things you can get without the simply connected condition, but the third is really what gives Postnikov towers their power.

Finally let's look at condition 2 for our functor A. Note that there is a homotopy pullback square expressing $K(V, n) \simeq \Omega K(V, n+1)$.

$$\begin{array}{ccc} K(V, n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(V, n+1) \end{array}$$

This is the most fundamental example of a homotopy pullback. The claim is that this square, assuming that $V \in \mathsf{Vect}_{\mathbb{Q}}$, is taken to the square fill this in. Let's check that this is reasonable: well what is $\mathbb{Q} \otimes^L_{\mathrm{Sym}(\mathbb{Q}[k])} \mathbb{Q}$ (here we're assuming $V = \mathbb{Q}$ for simplicity say). Let's do this in coordinates – we have $\mathbb{Q} \otimes^L_{\mathbb{Q}[x]} \mathbb{Q} = \mathbb{Q}[x, y \mid dy = x] \otimes_{\mathbb{Q}[x]} \mathbb{Q}$. This simplifies to $\mathbb{Q}[y]$ where $|y| = k+1$. But general V is just a direct sum of copies of $\mathbb{Q}$ whence this computation shows that all we get is a shift of our generators. So this computation tells us that at least the first two conditions of our functor A are compatible.

Once we have a functor A for which our three conditions are satisfied we are done. Let's return to our setup from before. We need to define what we mean by a space of maps between two commutative dg algebras. By the Yoneda lemma it suffices to understand maps into it.

Definition 6. Let $R, S \in \mathsf{CAlg}_{\mathbb{Q}}$. We define $\mathrm{Maps}(R,S)$ such that maps $K \to \mathrm{Maps}(R,S)$ is in bijection with maps $R \to S \otimes A(k)$. In particular a homotopy between two maps $f, g : R \to S$ is a map $R \to S \otimes A([0,1])$, which is equipped with two maps to $S \otimes A(*) = S \otimes \mathbb{Q}$.

Alternatively we can use simplicial sets and define the maps from R to S as the spaces $|\operatorname{Hom}(R, S \otimes A(\Delta^\bullet))|$.

Now we would like to say that $\mathrm{Maps}(X,Y) \simeq \mathrm{Maps}(A(Y), A(X))$ to show fully faithful. Essential surjectivity will come from the fact that commutative algebras are generated by polynomial algebras (under colimits). The first case will by $Y = K(V,n)$ and case two will be induction on the Postnikov tower of Y.

3. January 28

What we'd like to show is that if we have X and Y spaces all of whose homotopy groups are rational finite-dimesnional vector spaces (and they're simply connected) then

$$\mathrm{Maps}(X,Y) \xrightarrow{\sim} \underset{\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{aug}}}{\mathrm{Maps}}(A(Y),A(X)).$$

We actually don't need that many properties of A to prove this — we will just need what we discussed last time. The proof will be by induction on Postnikov towers and by showing that the map induces isomorphisms on homotopy groups.

To induct, we need somewhere to start. In particular take $X = K(\mathbb{Q},n)$ and $Y = K(\mathbb{Q},m)$ as our base case. Let's compute the homotopy groups of the space of basepoint-preserving maps

$$\pi_i \,\mathrm{Maps}(K(\mathbb{Q},n),K(\mathbb{Q},m))$$

By adjunction

$$\begin{aligned}\pi_i \,\mathrm{Maps}(K(\mathbb{Q},n),K(\mathbb{Q},m)) &= \pi_0 \,\mathrm{Maps}(K(\mathbb{Q},n),\Omega^i K(\mathbb{Q},m))\\ &= \pi_0 \,\mathrm{Maps}(K(\mathbb{Q},n),K(\mathbb{Q},m-i))\\ &= H^{m-i}(K(\mathbb{Q},n),\mathbb{Q})\\ &= H_{i-m}\,\mathrm{Sym}(\mathbb{Q}[-n])\end{aligned}$$

Here in the last step we give ourselves a computation that we will come back to later. Now let's look at the right hand side. We compute

$$\begin{aligned}\pi_i \underset{\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{aug}}}{\mathrm{Maps}}(A(K(\mathbb{Q},m)),A(K(\mathbb{Q},n)) &= \pi_i \,\mathrm{Maps}(\mathrm{Sym}(\mathbb{Q}[-m]),\mathrm{Sym}(\mathbb{Q}[-n]))\\ &= \pi_0 \,\mathrm{Maps}(\mathrm{Sym}(\mathbb{Q}[-m]),\mathrm{Sym}(\mathbb{Q}[-n])\otimes A(S^i)).\end{aligned}$$

Here we are using the enrichment that we defined last time. Recall that there is an adjunction between chain complexes and augmented commutative algebras via Sym and the forgetful functor (by which we mean take the kernel of the augmentation). Moreover between chain complexes and simplicial abelian groups we have the Dold-Kan equivalence of adjoint functors N and Γ (homework: figure out which are the left and right adjoints).

Exercise 7. Learn the Dold-Kan equivalence.

This allows us to define the space of maps

$$\underset{\mathsf{Ch}}{\mathrm{Maps}}(A,B) = |\Gamma\,\mathrm{Hom}_{\mathsf{Ch}}(A,B)|.$$

This has the property that

$$\pi_i \,\mathrm{Maps}(A,B) = H_i(\mathrm{Hom}(A,B)), \qquad i \geq 0.$$

Our enrichment of commutative algebras is compatible with this enrichment of chain complexes. In particular we now have

$$\pi_0 \underset{\mathsf{Ch}}{\mathrm{Maps}}(\mathbb{Q}[-m],\mathrm{Sym}(\mathbb{Q}[-n])\otimes A(S^i)).$$

Using the fact that $A(K) \simeq C^*(K)$ and that the symmetric algebra is free, we obtain

$$\begin{aligned}\pi_0 \operatorname{Maps}(\mathbb{Q}[-m], \operatorname{Sym}(\mathbb{Q}[-n]) \otimes \tilde{C}^*(S^i, \mathbb{Q})) &= \pi_0 \operatorname{Maps}(\mathbb{Q}[-m] \otimes \tilde{C}_*(S_i, \mathbb{Q}), \operatorname{Sym} \mathbb{Q}[-n]) \\ &= \pi_0 \operatorname{Maps}(\mathbb{Q}[i-m], \operatorname{Sym}(\mathbb{Q}[-n])) \\ &= H_{i-m} \operatorname{Sym} \mathbb{Q}[-n].\end{aligned}$$

So that's the base case. Now we induct on a Postnikov tower for Y. Assume that the map

$$\operatorname{Maps}(X, P_n Y) \to \operatorname{Maps}(A(P_n Y), A(X))$$

is a weak homotopy equivalence. We would like to show the same equivalence for $n+1$. Recall that the key fact about maps in the Postnikov tower is that the square

$$\begin{array}{ccc} P_{n+1}Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ P_n Y & \longrightarrow & K(\pi_{n+1}Y, n+2) \end{array}$$

is a homotopy pullback square. It follows that

$$\begin{array}{ccc} \operatorname{Maps}(X, P_{n+1}Y) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \operatorname{Maps}(X, P_n Y) & \longrightarrow & \operatorname{Maps}(X, K(\pi_{n+1}Y, n+2)) \end{array}$$

is a homotopy pullback square. Likewise, by hypothesis on A we have that

$$\begin{array}{ccc} A(K(\pi_{n+1}, n+2)) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ A(P_n Y) & \longrightarrow & A(P_{n+1}Y) \end{array}$$

is a homotopy pushout in commutative algebras. We are interested in

$$\operatorname{Maps}(X, P_{n+1}Y) \to \operatorname{Maps}(A(P_{n+1}Y), A(X))$$

and each of these objects is written as a homotopy pullback. On the left it's one of the homotopy pullback above and on the right it's

$$\begin{array}{ccc} \operatorname{Maps}(A(P_{n+1}Y, A(X)) & \longrightarrow & \operatorname{Maps}(Q, A(X)) = * \\ \downarrow & & \downarrow \\ \operatorname{Maps}(A(P_n Y), A(X)) & \longrightarrow & \operatorname{Maps}(A(K(\pi_{n+1}, n+2), A(X)) \end{array}$$

(recall we are working augmented). So we have a map of diagrams and the above map is the induced map on homotopy pullbacks. Now it is enough to show that all the maps (between these diagrams) are equivalences by the homotopy invariance of the homotopy pullback. But this follows from our base case (proven slightly more generally where we take X to be anything but there's nothing new there) and our induction step.

Lastly we now show

$$\operatorname{Maps}(X, Y) \simeq \operatorname{Maps}(A(Y), A(X)).$$

We have shown it for each Postnikov stage of Y. Since $Y = \operatorname{holim} P_n Y$ we have that $\operatorname{Maps}(X, Y) \simeq \operatorname{holim}_{n\to\infty} \operatorname{Maps}(X, P_n Y)$. Similarly $\operatorname{Maps}(A(Y), A(X)) \simeq$

$\mathrm{Maps}(\mathrm{hocolim}\, A(P_nY), A(X)) \simeq \mathrm{holim}\, \mathrm{Maps}(A(P_nY), A(X))$. We've just proved above that these are equivalent. Note that here we are using that $A(Y) = \mathrm{hocolim}\, A(P_nY))$.

So what's next? Well we should probably define A and show that it has the properties that we would like. We should also show that nice enough spaces have Postnikov towers. This would complete this proof.

After that we'd like to understand how to obtain the rational homotopy groups from the augmented commutative algebra $A(X)$. Let me outline how that will go. . . it could not be simpler. Or maybe it could, but it'd just be wrong. Whatever.

Definition 8. Given $V \in \mathsf{Mod}_R$ we obtain an object in $\mathsf{CAlg}_k^{\mathrm{aug}}$ in the following silly way. Take $k \oplus V$ and define the map

$$(k \oplus V) \otimes (k \oplus V) \to k \oplus V$$

as follows. Rewrite the left as

$$k \oplus k \otimes V \oplus V \otimes k \oplus V \otimes V.$$

Define the map on k to be the identity into k. Define the map on $k \otimes V$ and $V \otimes k$ to map into V by the identity. Finally send $V \otimes V$ to zero. In the case where $V = k$ we obtain $k \oplus V = k[t]/(t^2)$.

This functor has a left-adjoint known as the functor of indecomposables. Unless we're working on the symmetric algebra we should probably take the derived functor of indecomposables instead. We will call this the cotangent space, $L = \mathbb{L} \circ \mathsf{indecomp}$. More concretely this means we resolve R via $R_\bullet = \mathrm{Sym}(V_\bullet)$ and then $LR = |\mathsf{indecomp} R_\bullet|$.

It will turn out that

$$\pi_* X_{\mathbb{Q}} = H_*(LA(X)^\vee).$$

We can see that this is reasonable because $\mathsf{indec}\, \mathrm{Sym}\, V = V$ and it gives us the right answer for an Eilenberg-MacLane space.

4. February 6

In the proofs from last time we made use of some assumptions. The first was that we could construct a strictly commutative model for cochains in characteristic zero and the second was a type of parameterized Künneth theorem. Let's talk about the latter. do we need some finiteness conditions here?

Theorem 9. *Suppose*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a homotopy pullback square. Suppose moreover that Y is simply-connected (actually we will need something weaker, which I'll leave to you). Then the natural map

$$C^*X \otimes^L_{C^*Y} C^*Y' \xrightarrow{\sim} C^*X'$$

is a quasi-isomorphism (this will be true with whatever coefficients you choose).

Before proving this let me remind you a bit about local systems and locally constant sheaves. Fix a space X. We have classically that

$$\mathsf{Shv}^{\mathrm{lc}}_{\mathsf{Set}}(X) \simeq \mathrm{Fun}(\pi_{\leq 1}X, \mathsf{Set}).$$

But if we take sheaves of spaces we might ask that it be locally constant only up to homotopy. Here's a prototypical example. Let $E \to X$ be a fiber bundle. Then we can consider the sheaf $\mathcal{E}$ of sections of $E \to X$, sending $U \subset X$ open to the space of sections on U equipped with the usual compact-open topology. Notice that this is not a locally constant sheaf – say X is a manifold. If we take open sets $U \subset U'$ such that U' deformation retracts to U then

$$\Gamma(U', E|_{U'}) \xrightarrow{\sim} \Gamma(U, E|_U).$$

Homework: Check that this is locally constant as a sheaf of homotopy types. Then

$$\mathsf{Shv}^{\mathrm{lc}}_{\mathsf{Spaces}}(X) \, \mathrm{im} \, \mathrm{Fun}(\mathsf{Sing}(X), \mathsf{Spaces}).$$

The moral is that any time our target category has homotopy theory the notion of local systems is slightly more subtle than the classical case. We are in particular interested in

$$\mathsf{Shv}^{\mathrm{lc}}_{\mathsf{Ch}}(X) \simeq \mathrm{Fun}(\mathsf{Sing}(X), \mathsf{Ch}).$$

Example 10. Given any map $f : X \to Y$ of spaces we obtain

$$\mathsf{Opens}(Y) \xrightarrow{f^{-1}} \mathsf{Spaces} \xrightarrow{C^*} \mathsf{Ch}.$$

This gives us a local system L_0 associated to $X \to Y$ where

$$L_0|_y = C^*X_y, \qquad X_y = \{y\} \times^h_Y X.$$

Now notice that

$$C^*X \simeq C^*(Y, L_0).$$

this is because there is an adjunction insert diagram here So the cohomology can be computed by first pushing forward to Y and then pushing forward to the point. But $L_0 = f_*\mathbb{Z}_X$ is the pushforward of the constant sheaf on X.

Returning now to the theorem above we have local systems L_0, L_1 on Y coming from $X \to Y$ and $Y' \to Y$ respectively. If we call the local system L associated to X' then (by locality and the ordinary Künneth theorem) we obtain that $L = L_0 \otimes L_1$. Thus the map in the theorem becomes

$$C^*(Y, L_0) \otimes^L_{C^*Y} C^*(Y, L_1) \to C^*(Y, L_0 \otimes L_1).$$

We can now forget about the original topology and ask: when is this map an equivalence?

Suppose for a moment that $L_0 = \mathbb{Z}_Y$ the constant local system. Then on the left we have

$$C^*(Y) \otimes_{C^*(Y)} C^*(Y, L_1) \to C^*(Y, \mathbb{Z} \otimes L_1)$$

but the two sides are the same. More generally we say that any local system such that this is an equivalence is "good."

Definition 11. We say that a local system L is **good** if for every local system L_1 the map

$$C^*(Y, L) \otimes_{C^*(Y)} C^*(Y, L_1) \to C^*(Y, L \otimes L_1)$$

is an equivalence.

Consider now the full subcategory of good local systems in $\mathsf{Shv}^{\mathsf{lc}}_{\mathsf{Ch}}(Y)$. Observe that good local systems are closed under

- shifts of chain complexes, since it's true for $\mathbb{Z}$ and its shifts;
- finte direct sums, since the derived tensor product commutes with finite direct sums as does taking pushforward to a point
- extensions $0 \to L' \to L \to L'' \to 0$, i.e. if L', L'' are good then L good. This follows because derived tensor products and taking pushforwards to a point preserve exact sequences

Proposition 12. *For Y simply-connected, all local systems (concentrated stalkwise in non-negative degrees with finite-dimesional stalks) on Y are good.*

Proof. Let $\mathcal{F}$ be such a local system on Y. Then we write $\mathcal{F}$ as the co?limit of its truncations

$$\mathcal{F} = \lim \tau^{\leq k} \mathcal{F}$$

Here $\tau^{\leq k}\mathcal{F} \to \mathcal{F}$ induces an isomorphism on cohomology on stalks in degrees $\leq k$. Now we induct on k.

For $k = 0$ we have $\tau^{\leq 0}\mathcal{F} = A[0]$ which is locally constant again (because $\tau^{\leq 0}$ preserves cohomology). But actually since Y is simply-connected it's constant, hence good.

For the inductive step we have an extension

$$\tau^{\leq k-1}\mathcal{F} \to \tau^{\leq k}\mathcal{F} \to \underline{A}[k]$$

Now by the fact that good local systems are closed under extension (and again because Y is simply-connected) we conclude that $\tau^{\leq k}\mathcal{F}$ is good.

But now $C^*(\operatorname{colim} \tau^{\leq k} L) \simeq \operatorname{colim} C^*(\tau^{\leq k} L)$ so we are done. Notice that this equivalence uses crucially the type of colimit that we have here where we are tacking on something in one degree. □

So what was the trick? The point is on a simply-connected space you can't have interesting local systems of abelian groups but you can have interested local systems of chain complexes. The key somehow is that this argument applies to nilpotent things i.e. extensions by trivial local systems (this is a general useful thing to remember when doing mathematics).

Now returning to our proof of the theorem, L_0 the local system associated to $X \to Y$ is good if Y is simply connected and has finite rank cohomology groups (and X has finite rank cohomology groups). This concludes the proof of the theorem.

Now we turn to the problem of constructing a functor $A : \mathsf{Spaces}^{\mathrm{op}} \to \mathsf{cAlg}_{\mathbb{Q}}$ that is equivalent to $C^*(-, \mathbb{Q})$. There are two approaches, formal and constructive (PL de Rham):

- In the formal construction one notes that cochains are $\mathcal{E}_\infty$
$$C^*(-, R) : \mathsf{Spaces}^{\mathrm{op}} \to \mathcal{E}_\infty - \mathsf{Alg}_R$$
Instead of having a multiplication map $(S \otimes_R S)_{\Sigma_2} \to S$ you instead have a map $C_*(\Sigma_2, S \otimes_R S) \to S$. This is more general as it may not factor through coinvariants. However, it turns out that in characteristic zero $\mathcal{E}_\infty - \mathsf{Alg}_R \simeq \mathsf{CAlg}_R$ essentially because symmetric groups have no higher homology in characteristic zero. Thus one can use formally this equivalence to obtain a strictly commutative algebra from singular cochains.
- The constructive approach is Sullivan's PL de Rham theory. In particular one defines it on the standard simplex $\Delta[n]$ and then extends it to all simplicial sets in the obvious way.

5. February 11

So we want to construct this commutative dga refining the cup product on cohomology. Here's the basic construction. First we will construct a functor

$$\Delta^{\mathrm{op}} \xrightarrow{A} \mathsf{CAlg}_k$$

and then two, extend to all simplicial sets. In particular, we define

$$A(\Delta[p]) = \mathrm{Sym}(x_0, \dots, x_p, dx_0, \dots, dx_p)/(\sum_{i=0}^{p} x_i = 1, \sum_{i=0}^{p} dx_i = 0)$$

where $|x_i| = 0$ and $|dx_i| = 1$. Notice that here we are using cohomological grading. This is known as the PL de Rham forms on the standard p-simplex. What happens on the face and degeneracy maps? Well just the usual things – on the degeneracy maps, add, and on the face maps substitute zero.

Now for any simplicial set X we define the PL de Rham complex of X to be

$$A(X) = \mathrm{Hom}(X, A)$$

where on the right we are taking natural transformations of simplicial sets. In degree p, for instance, we have a map $X_p \to A(\Delta[p])$. Another way of thinking about this is as a limit,

$$A(X) = \lim_{\Delta[p]\to X} A(\Delta[p]),$$

i.e. a right Kan extension

$$\begin{array}{ccc} \Delta^{\mathrm{op}} & \longrightarrow & \mathsf{CAlg}_k \\ \downarrow & \nearrow & \\ \mathsf{sSet}^{\mathrm{op}} & & \end{array}$$

Let's compare with singular cochains. First a quick sanity check. Changing variables to get rid of redundancy we obtain

$$\begin{aligned} A(\Delta[p]) &= \mathrm{Sym}(y_1, \dots, y_p, dy_1, \dots, dy_p) \\ &= A(\Delta[1]) \otimes \cdots \otimes A(\Delta[1]). \end{aligned}$$

Hence it is enough to check the statement for the standard one-simplex. In this case we have

$$0 \to k[y] \xrightarrow{d} k[dy] \to 0.$$

Notice that $d(y^l) = ly^{l-1}dy$, so if $l = 0$ in k then this sequence is not exact. This is why we need to be in characteristic zero, $\mathbb{Q} \subset k$. In this case the cohomology is just k in degree zero. Using the Künneth formula we obtain the result for the standard p-simplex.

Theorem 13. *If k characteristic zero then*

$$A(X) \simeq C^*(X, k).$$

Let's see how we obtain usual singular cochains analogously to the construction of A above. Consider

$$\Delta^{\mathrm{op}} \xrightarrow{N} \mathsf{Ch}_k$$

given as the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\mathbb{Z}} \mathsf{sAb} \xrightarrow{(-)^\vee} \mathsf{cAb} \xrightarrow{N} \mathsf{Ch}_k.$$

Notice that we actually land in algebras as the simplicial abelian group is a coalgebra via the diagonal map and since the Eilenberg-Zilber map is monoidal. This is then extended as above via the right Kan extension to all simplicial sets.

We would now like to construct a natural transformation between these functors that induces for each object X a quasi-isomorphism.

Definition 14. We say that a functor $\Delta^{\mathrm{op}} \xrightarrow{R} \mathsf{Alg}_k$ is **extendable** if for each $K \subset L$ subsimplicial set then $R(L) \to R(K)$ is surjective.

Lemma 15. *Suppose we are given a map $\phi : R \to S$ of extendable simplicial dgas such that $R(\Delta[p]) \to S(\Delta[p])$ is a quasi-isomorphism for all p. Then $R(X) \simeq S(X)$ for each $X \in \mathsf{sSet}$.*

Suppose the lemma for now. The proof has a few steps:

(1) show that A is extendable
(2) show that N is extendable
(3) show that $A \otimes_k N$ is extendable (this means take the tensor product on simplices and then extend to all simplicial sets)
(4) consider the diagram $A \to A \otimes N \leftarrow N$ and apply the lemma to show that these maps are quasi-isomorphisms

Remark 16. You can also construct a direct de Rham theorem type map from A to N, but this approach is probably a bit easier.

Naively, the way you want to think is that there is a model category of simplicial diagrams in dg algebras. If things were fibrant then an ordinary limit would be homotopy invariant. It'd be great if extendable meant fibrant, but actually extendability is a bit weaker, which turns out to be all you need for homotopy invariance in this case. In particular for the proof of them lemma we will need to open up the black box of the proof of homotopy invariance given fibrancy but use extendability instead.

Remember the only way to prove anything is by using filtrations. We need the skeletal filtration of simplicial sets. We have

$$\begin{array}{c} \Delta^{\mathrm{op}}_{\leq k} \\ \big\downarrow \scriptstyle\mathrm{res} \\ \Delta^{\mathrm{op}} \xrightarrow{X} \mathsf{Set} \end{array}$$

and we define the k-skeleton of X to be the left Kan extension

$$\mathrm{sk}_k X = \mathrm{res}_! \circ \mathrm{res}^* X.$$

We say that X is k-skeletal if $\mathrm{sk}_k X = X$. Now one obtains a pushout diagram

$$\begin{array}{ccc} \coprod_{X_k \text{ nd}} \partial\Delta[k] & \longrightarrow & \mathrm{sk}_{k-1} X \\ \big\downarrow & & \big\downarrow \\ \coprod_{X_k \text{ nd}} \Delta[k] & \longrightarrow & \mathrm{sk}_k X \end{array}$$

Then

$$X = \operatorname{colim} \mathrm{sk}_k X.$$

Proof of lemma. We induct on the skeletal filtration. The base case is when we are 0-skeletal, i.e. just a set of points I. In this case

$$R(I) = \mathrm{Hom}(I, R) = \prod_I R(\Delta[0])$$

and similarly for $S(I)$. The map $\prod_I R(\Delta[0]) \to \prod_I S(\Delta[0])$ is now a quasi-isomorphism by our hypothesis.

For the induction step assume that $R(K) \to S(K)$ is a quasi-isomorphism for all K that are $(k-1)$-skeletal. Define $R(X,Y)$ to be the kernel

$$0 \to R(X,Y) \to R(X) \to R(Y) \to 0$$

for $Y \subset X$. We now have a commutative diagram

$$\begin{array}{ccccc} R(\mathrm{sk}_k X, \mathrm{sk}_{k-1} X) & \longrightarrow & R(\mathrm{sk}_k X) & \longrightarrow & R(\mathrm{sk}_{k-1} X) \\ \downarrow & & \downarrow & & \downarrow \\ S(\mathrm{sk}_k X, \mathrm{sk}_{k-1} X) & \longrightarrow & S(\mathrm{sk}_k X) & \longrightarrow & S(\mathrm{sk}_{k-1} X) \end{array}$$

and the right hand vertical map is a quasi-isomorphism by the induction hypothesis. It is now enough to show that the left hand vertical map is a quasi-isomorphism, using the extendability and the five lemma.

Notice that R, S send colimits to limits, as

$$\mathrm{Hom}(X, R) = \mathrm{Hom}(\operatorname*{colim}_J F, R) = \lim_{J^{\mathrm{op}}} \mathrm{Hom}(F, R)$$

where $X = \mathrm{colim}_J F$. Thus we obtain a pullback diagram

$$\begin{array}{ccc} R(\mathrm{sk}_k X) & \longrightarrow & \prod R(\Delta[k]) \\ \downarrow & & \downarrow \\ R(\mathrm{sk}_{k-1} X) & \longrightarrow & \prod R(\partial\Delta[k]) \end{array}$$

where the products are over the nondegenerate simplices. Hence since fibers are same in a homotopy pullback, we find that

$$R(\mathrm{sk}_k X, \mathrm{sk}_{n-1} X) = \prod_{X_k \text{ nd}} R(\Delta[k], \partial\Delta[k])$$

It is now enough to show that

$$R(\Delta[k], \partial\Delta[k]) \to S(\Delta[k], \partial\Delta[k])$$

is a quasi-isomorphism. But notice that

$$\begin{array}{ccc} R(\Delta[k], \partial\Delta[k]) & \longrightarrow & S(\Delta[k], \partial\Delta[k]) \\ \downarrow & & \downarrow \\ R(\Delta[k]) & \longrightarrow & S(\Delta[k]) \\ \downarrow & & \downarrow \\ R(\partial\Delta[k]) & \longrightarrow & S(\Delta[k]) \end{array}$$

□

6. February 13

It remains to show that A is extendable. We will do this in two parts.

Lemma 17. *The following are equivalent for $B_\bullet$ a simplicial abelian group:*

(1) For any inclusion $K \subset L$ of simplicial sets the induced map $\mathrm{Hom}(L, B) \to \mathrm{Hom}(K, B)$ is surjective (levelwise);
(2) The homology $H_*(B) = 0$.

Proof. To show that (1) implies (2) we will only need the case in which $K = \partial\Delta[n] \hookrightarrow \Delta[n] = L$ (the terminology is that these are generating cofibrations and its enough to check it on them). Given any cocycle $[x] \in H_{n-1}(B)$ we can choose a map $\partial\Delta[n] \to B$ (in particular choose x to be normalized such that $d_i x = 0$). We can extend this to a map to B by surjectivity whence $[x] = 0$.

Next (2) implies (1). Using Dold-Kan we can write equivalently that

$$\mathrm{Hom}(L, B) = \mathrm{Hom}(N\mathbb{Z}L, NB)$$
$$\mathrm{Hom}(K, B) = \mathrm{Hom}(N\mathbb{Z}K, NB).$$

Now taking the kernel we obtain

$$0 \to \mathrm{Hom}(N\mathbb{Z}L/N\mathbb{Z}K, NB) \to \mathrm{Hom}(N\mathbb{Z}L, NB) \to \mathrm{Hom}(N\mathbb{Z}K, NB)$$

Since NB has zero homology and $N\mathbb{Z}L/N\mathbb{Z}K$ is free (and homs out of free things preserve quasi-isomorphism) we see that the kernel has zero homology. Thus the map of interest is a surjective quasi-isomorphism. But a surjective quasi-isomorphism is surjective on 0-cycles. □

Corollary 18. *The functor $A : \Delta^{op} \to \mathsf{CAlg}_k$ of PL de Rham forms is extendable if and only if for any n the homology of*

$$A^n : \Delta^{op} \to \mathsf{CAlg}_k \xrightarrow{\textit{degree } n} \mathsf{Mod}_k$$

is zero, i.e. the homology of the simplicial k-module $A^n_\bullet$ is zero.

Proof. Apply the lemma above for $B_\bullet = A^n_\bullet$ for each n. □

So now we are left with the following check.

Lemma 19. *For A the PL de Rham forms, the homology $H_*(A^n) = 0$ for all n.*

Proof. Notice that A^0 is a commutative ring and A^n is an A_0-module for all n. Hence it suffices to show that $H_*(A^0) = 0$ (since all rings are unital here). Likewise is suffices to show that $H_0(A^0) = 0$. But this follows from the simplicial structure: $\mathrm{coker}(d_0 - d_1) = 0$. □

Hopefully this delivers on everything I promised. Now we turn to the question of computing $\pi_* X \otimes \mathbb{Q}$ from $A(X)$.

Definition 20. The indecomposables functor I is the left adjoint to the trivial algebra functor $k \oplus - : \mathsf{Ch}_k \to \mathsf{CAlg}_k^{\mathrm{aug}}$.

What is this? Well $I(A)$ we can write as

$$I(A) = \bar{A}/\bar{A}^2$$

. Note that there is an equivalence (isomorphism) of categories $\mathsf{Alg}_k^{\mathrm{aug}} \simeq \mathsf{CAlg}_k^{\mathrm{nu}}$ with nonunital algebras given by taking the kernel of the augmentation and the other way is adding in a unit $k \oplus -$. Here

$$\bar{A}^2 = \{x \in \bar{A} \mid x = yz, y, z \in \bar{A}\}$$

which you might think of as the decomposables. It's immediate to see that this has the correct universal property: to give a ring map $A \to k \oplus V$ is the same as to give a nonunital ring map $\bar{A} \to V$. Since the product on V is trivial we see that it must factor through $\bar{A}/\bar{A}^2$ to be a ring map. Unfortunately I doesn't preserve quasi-isomorphisms.

Definition 21. The cotangent space $L : \mathsf{CAlg}_k^{\mathrm{aug}} \to \mathsf{Ch}_k$ is the left derived functor of I. Write $T : \mathsf{CAlg}_k^{\mathrm{aug}} \to \mathsf{Ch}_k$ to L postcomposed with dualization. This is the tangent space.

In other words it has the behavior defined above on free objects and it preserves homotopy colimits. Notice in particular that on symmetric algebras we have

$$L(\mathrm{Sym}\, V) = I(\mathrm{Sym}\, V) = V.$$

Theorem 22. *For X simply connected of finite type*

$$\pi_* X_{\mathbb{Q}} = H_*(TA(X)).$$

Let's check this in one example. Suppose X is an Eilenberg-Maclane space $X = K(V, n)$ with V a finite dimensional rational vector space. Then

$$H^*(K(V, n)) = \mathrm{Sym}(V^\vee[-n]).$$

Actually we have more.

Definition 23. We say that $R \in \mathsf{CAlg}$ is formal if there exists a zig-zag of quasi-isomorphisms between R and H^*R. We say that a space X is formal if $A(X)$ is formal.

Lemma 24. *$K(V, n)$ is formal.*

So in particular we have

$$A(K(V, n)) \sim K(V^\vee[-n])$$

whence

$$T(\mathrm{Sym}(V^\vee[-n]) = (V^\vee[-n])^\vee = V[n].$$

So we see indeed that

$$H_* V[n] = \pi_* K(V, n).$$

In general finding the cotangent space can be rather involved. For instance given R we can find a simplicial resolution by free (symmetric) algebras. For instance one might take the functorial (monadic) one $\mathrm{Sym}(\mathrm{Sym}(R)) \to \mathrm{Sym}(R) \to R$ which is extremely large in general. Now one constructs LR as

$$LR = |I(R_\bullet)|.$$

One can check that this construction does indeed preserve quasi-isomorphisms. But this is too big to actually compute with. Instead we can work with a large class of cdgas for which we still have homotopy invariance.

Usually when you build a dga you do it in terms of generators and relations. So let's say we want to try to understand the cotangent space $(R[x], d)$ with $dx = r \in$

R. How can we calculated L of this from L of R? Well this is a certain homotopy pushout (or derived tensor product)

$$\begin{array}{ccc} k[y] & \xrightarrow{y\mapsto r} & R \\ \downarrow & & \downarrow \\ k & \longrightarrow & R[x] \end{array}$$

Let's modify this to an ordinary pushout

$$\begin{array}{ccc} k[y] & \xrightarrow{\quad y\mapsto r \quad} & R \\ \downarrow & & \downarrow \\ k[x,y \mid dx = y] & \longrightarrow & R[x] = R \otimes_{k[y]} k[x,y \mid dx = y] \end{array}$$

since the left vertical map is now a cofibration. Now L preserves homotopy pushouts so we obtain

$$\begin{array}{ccc} k[-|y|] = Lk[y] & \longrightarrow & LR \\ \downarrow & & \downarrow \\ 0 = Lk & \longrightarrow & L(R[x]) \end{array}$$

Thus we find that

$$L(R[x \mid dx = r]) = \operatorname{coker}(k[-|x| - 1] \to LR)$$

Thus to prove the theorem we will now use Postnikov towers and the pushouts contained therein together with these calculations we have just done.

7. ?

Last time we were going to prove the following central result.

Theorem 25. *Let X be finite type (rationally) and simply connected. Then the rational homotopy groups of X are computed as*

$$\pi_*(X,*) \otimes \mathbb{Q} \cong H_*(TA(X))$$

for $* \in X$.

Lemma 26. *Spheres and Eilenberg-Maclane spaces are formal.*

Proof. Note that the map classifying the nontrivial generator of the cohomology of S^{2n+1} is a rational equivalence, $S^{2n+1} \to K(\mathbb{Q}, 2n+1)$. We have that $A(S^{2n+1})$ is the free graded commutative algebra on a generator in degree $2n+1$ and so there is an obvious quasi-isomorphism $H^*(S^{2n+1}) = \mathbb{Q}[x_{2n+1}] \to A(S^{2n+1})$.

On the other hand this doesn't work for even dimensional spheres since the EM space will have polynomial whereas the sphere will have truncated polynomial. In particular $H^*(S^{2n}, \mathbb{Q}) = \mathbb{Q}[x_{2n}]/(x^2)$. So we can't map directly out of this since it isn't free. Instead we consider $\mathbb{Q}[x_{2n}, y_{4n-1} \mid dy = x^2]$ which maps to the cohomology by sending $x_{2n} \mapsto x_{2n}$ and $y \mapsto 0$. Moreover there is a map to $A(S^{2n})$: there exists z such that $[z] = x$ but since $[z^2] = 0$ there exists w such that $dw = z^2$. Thus we obtain a zig-zag of quasi-isomorphisms.

Similarly even Eilenberg-Maclane spaces are formal: we have a map $\mathbb{Q}[x] = H^*(K(\mathbb{Q}, n)) \to A(K(\mathbb{Q}, n))$. □

Proof of theorem. We are interested in understanding pointed maps $\mathrm{Maps}(S^n_{\mathbb{Q}}, X_{\mathbb{Q}})$.

$$\begin{aligned}\mathrm{Maps}(S^n_{\mathbb{Q}}, X_{\mathbb{Q}}) &\simeq \mathop{\mathrm{Maps}}_{\mathsf{CAlg}^{\mathrm{aug}}_{\mathbb{Q}}}(A(X), A(S^n)) \\ &\simeq \mathrm{Maps}(A(X), \mathbb{Q} \oplus \mathbb{Q}[-n]) \\ &\simeq \mathop{\mathrm{Maps}}_{\mathsf{Ch}_{\mathbb{Q}}}(LA(X), \mathbb{Q}[-n]).\end{aligned}$$

Hence

$$\begin{aligned}\pi_0 \mathrm{Maps}(S^n_{\mathbb{Q}}, X_{\mathbb{Q}}) &\simeq \pi_0 \mathop{\mathrm{Maps}}_{\mathsf{Ch}_{\mathbb{Q}}}(LA(X), \mathbb{Q}[-n]) \\ &\cong H_0(TA(X)[-n]) \\ &\cong H_n TA(X).\end{aligned}$$

Here we have used the fact that quasi-isomorphisms preserve mapping spaces up to equivalence, see the two lemma below. □

Definition 27. We say that $(\mathrm{Sym}\, V, d)$ is a Sullivan algebra if there exists a filtration $V = \cup_I V_\alpha$ over some totally ordered set I such that for all $x_\alpha \in V_\alpha$, $d_\alpha \in \mathrm{Sym}(V_{<\alpha})$.

Here's the standard non-example of a Sullivan algebra. consider $\mathbb{Q}[x, y, z]/(dx = yz, dy = xz, dz = xy)$ where the generators are all degree 1. One checks that there's no such filtration.

Definition 28. We will say that a Sullivan algebra is simply connected if V is concentrated in degrees greater than 1 and minimal if $\mathrm{im}\, d \subset \mathrm{Sym}^{\geq 2} V$.

Lemma 29. *If $A \xrightarrow{\simeq} B$ is a map of Sullivan algebras then for all C*

$$\mathrm{Maps}(B, C) \xrightarrow{\simeq} \mathrm{Maps}(A, C)$$

is a weak homotopy equivalence.

Proof. Homework. □

As an example consider a map $A = \mathbb{Q}[x_{2n}, y_{4n-1} \mid dy = x] \xrightarrow{\simeq} \mathbb{Q}[x_{2n}]/(x^2) = B$. Now let $A = C$ and the map from A to C be the identity. Clearly the space of maps from A to C has two components but the space of maps B to C does not hit the component of the identity (the other component is just coming from the augmentation).

Lemma 30. *Say A is Sullivan and $C \to D$ is a quasi-isomorphism. Then we have an equivalence*

$$\mathrm{Maps}(A, C) \xrightarrow{\simeq} \mathrm{Maps}(A, D).$$

Proof. Homework. □

In terms of the model categories this can be viewed as saying that Sullivan algebras are cofibrant and everything is fibrant.

Recall from usual point-set topology that if we have a map $X \to Y$ of topological spaces that is a homotopy equivalence or weak homotopy equivalence, we might ask when

$$\mathrm{Maps}(Y, Z) \to \mathrm{Maps}(X, Z)$$

is an equivalence. The homotopy equivalence case is easy and is always true. For the weak homotopy equivalences it's a bit trickier: we will have a weak homotopy equivalence if for instance X and Y have the homotopy type of CW complexes. Let's look at how this could go wrong. Let $X = S^1$ and $Y = \sqcup_4 *$ given the poset topology coming from a diamond shaped poset. The map $X \to Y$ sending the upper and lower semicircles to the obvious points is a weak homotopy equivalence. If we let $Z = X$ be the identity one finds that there's no inverse map, just as in our commutative algebra example. Again this can be reflected model categorically in saying that CW complexes are cofibrant and everything is fibrant.

Now let's calculate the rational homotopy groups of spheres. First let n be odd. Then $S^{2k+1}_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2k+1)$ whence $\pi_*(S^{2k+1}, *) \otimes \mathbb{Q}$ is $\mathbb{Q}$ when $* = 2k+1$ and 0 otherwise. Next let n be even. Then $A(S^{2k}) \simeq \mathbb{Q}[x_{2k}, y_{4k-1} \mid dy = x^2]$ so $TA \simeq \mathbb{Q}[2k] \oplus \mathbb{Q}[4k-1]$ from which we see that $\pi * S^{2k} \otimes \mathbb{Q}$ is $\mathbb{Q}$ in degrees $2k$ and $4k-1$ but 0 otherwise.

Corollary 31. *If $A = (\mathrm{Sym}(V), d)$ is a minimal simply connected Sullivan algebra then $TA = V^\vee$ so if X is fintie type then writing $A = A(X)$ then $A\pi_* X \otimes \mathbb{Q} = H_*(V^\vee)$.*

This should not be too surprising. Recall that we have the Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2$$

which gives a geometric model for $S^3 \to S^2$ the homotopy class that we find algebraically. Similarly we have

$$S^3 \hookrightarrow S^7 \to S^4$$

and

$$S^7 \hookrightarrow S^{15} \to S^8$$

but no higher due to the classification of division algebras. In general it's not clear that these representatives are nontrivial but at least for the first one we can use the long exact sequence for fibrations.

8. February 25

Last time I said something slightly wrong. We were discussing when a homotopy equivalence or weak homotopy equivalence of spaces $X \to Y$ would induce weak equivalences on mapping spaces. Let's walk through this a bit more carefully. If we have a homotopy equivalence then it is easy to see that $\mathrm{Maps}(K, X) \to \mathrm{Maps}(K, Y)$ is an equivalence. Say we have $X \times [0,1] \xrightarrow{f_t} Y$. Then precomposition by f_t yields a map

$$\mathrm{Maps}(K, X) \times [0,1] \to \mathrm{Maps}(K, Y)$$

which will be the desired homotopy. The second easiest case is say mapping out of X and Y with the data of this homotopy equivalence. We want to say that we induce the same restriction maps $\mathrm{Maps}(Y, K) \to \mathrm{Maps}(X, K)$. Well notice that by adjointness we have a map $X \to Y^{[0,1]}$ and thus maps

$$\mathrm{Maps}(Y, K) \times [0,1] \to \mathrm{Maps}(Y^{[0,1]}, K) \to \mathrm{Maps}(X, K)$$

where the first map is precomposition with evaluation. This gives us our homotopy. Okay for these adjunctions and everything to be nice we need to make sure we're working with point-set theoretically decent spaces like locally compact Hausdorff or something. Otherwise the mapping spaces will not be correct.

Now suppose the map $X \to Y$ is only a weak homotopy equivalence. This means roughly that maps from spheres to these things are equivalent. Given a CW complex K (or even just a space homotopy equivalent to a CW complex) we obtain a weak homotopy equivalence $\mathrm{Maps}(K, X) \to \mathrm{Maps}(K, Y)$ by induction on the skeleton of K (using say the homotopy invariance of homotopy pullbacks). It's a little trickier to think about mapping out of X and Y: for some K do we get a weak homotopy equivalence $\mathrm{Maps}(Y, K) \to \mathrm{Maps}(X, K)$? This should remind you of the Yoneda lemma, but is quite difficult. I don't know know a great shortcut around the theory of model categories. In particular the category of topological spaces is a model category where the weak equivalences are the weak homotopy equivalences, fibrations are the Serre fibrations, and cofibrations are Serre cofibrations. In particular CW complexes are fibrant objects. This implies that $\mathrm{Maps}(X, K) \simeq \mathrm{Maps}(Y, K)$ if $X \simeq Y$ are cofibrant and K is fibrant. Quillen might not say exactly this but it follows from his identification of homotopy classes of maps as the same as maps in the category localized at weak equivalences. But the result at the level of spaces just follows from Quillen's result where we work with all suspensions of X (the higher homotopy groups of the mapping space).

Let's get back to computing examples in rational homotopy theory. Consider $\mathbb{C}P^n$ whose cohomology is

$$H^*_{\mathbb{Q}}\mathbb{C}P^n = \mathbb{Q}[x]/(x_2^{n+1}).$$

We have formality as we have a map from $\mathbb{C}[x, y_{2n+1} \mid dy = x^{n+1}]$ which is a quasi-isomorphism. Now we have a map from this cdga to $A(\mathbb{C}P^n)$ sending x to what it must send x to and y to anything such that d of it is the $n+1$st power of what we send x to. This implies that the rational homotopy groups of $\mathbb{C}P^n$ are $\mathbb{Q}$ in degrees 2 and $2n+1$. The degree 2 thing is just the sphere $\mathbb{C}P^1$ sitting inside $\mathbb{C}P^n$. The degree $2n+1$ thing is the quotient map $S^{2n+1} \to \mathbb{C}P^n$ where we take $S^{2n+1} \subset \mathbb{C}^{2n+1} \setminus \{0\}$ with the action of the circle. Of course this fibration could exactly have given us this result without using rational homotopy theory:

we get a long exact sequence on homotopy of the homotopy fiber sequence (after rationalization)

$$S^{2n+1} \to \mathbb{C}P^n \to BS^1 = \mathbb{C}P^\infty.$$

We know that $BS^1_{\mathbb{Q}}$ is a $K(\mathbb{Q}, 2)$ and S^{2n+1} is a $K(\mathbb{Q}, 2n+1)$. Hence the homotopy groups fall out immediately. Notice that after rationalization $\mathbb{C}P^n$ does not just become (homotopy equivalent to) the product $K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 2n+1)$. This is because A of the product is $\mathbb{Q}[x_2] \otimes \mathbb{Q}[y_{2n+1}]$ which is definitely not the same as the cohomology ring of $\mathbb{C}P^n$.

As a next example let's think of unitary groups. We have a sequence

$$SU(n) \to U(n) \xrightarrow{det} U(1).$$

There is a section $U(1) \to U(n)$ which gives us a splitting $U(n) \cong S^1 \times SU(n)$ (depending on our choice of section). We can compute the cohomology ring

$$H^*(SU(n), \mathbb{Q}) = \mathbb{Q}[x_3, x_5, \ldots, x_{2n-1}].$$

by using the fiber sequence

$$SU(k-1) \to SU(k) \to S^{2k-1}.$$

This cohomology ring is a symmetric algebra and thus formal. This implies that the homotopy groups of $SU(n)$ are $\mathbb{Q}$ concentrated in degrees $3, 5, \ldots, 2n-1$. Notice that we have a map

$$SU(n)_{\mathbb{Q}} \to \prod_{k=2}^{n} K(\mathbb{Q}, 2k-1)$$

and applying A we see that this map is an equivalence. We conclude that

$$U(n)_{\mathbb{Q}} \simeq \prod_{k=1}^{n} K(\mathbb{Q}, 2k-1).$$

In general actually any abelian topological group is a product of Eilenberg Maclane spaces. Why is this true? Well there is an adjunction between topological abelian groups and simplicial abelian groups. Likewise we have an equivalence between simplicial abelian groups and non-negatively graded chain complexes (by Dold-Kan). But since each chain complex is quasi-isomorphic to a chain complex with zero differential and now pull backwards along these adjunctions: Dold-Kan will take direct sums to direct products. The moral is that in the world of topological abelian groups we're never too far from being a product of Eilenberg-MacLane spaces.

Yajit asked a question about computing rational mapping spaces.

Lemma 32. *Rationalization preserves homotopy pullbacks.*

This need not be true about localizations in true. Now one might ask whether there is a relation between $\text{Maps}(K, X)_{\mathbb{Q}}$ and $\text{Maps}(K, X_{\mathbb{Q}})$. Well it turns out that $\text{Maps}(K, X_{\mathbb{Q}}) \simeq \text{Maps}(K_{\mathbb{Q}}, X_{\mathbb{Q}})$. But in general that's all you can say. Indeed, consider $K = K(\mathbb{Q}, n)$ and $X = K(\mathbb{Z}, n)$. Then (say $n \geq 1$ so that we can utilise the adjoint equivalence between loop spaces and simply-connected pointed spaces),

$$\text{Maps}(K(\mathbb{Q}, n), K(\mathbb{Z}, n)) \simeq \text{Hom}_{\mathsf{Grp}}(\mathbb{Q}, \mathbb{Z}) = \{0\}.$$

But if we rationalize X first well then there we have maps from $\mathbb{Q}$ to $\mathbb{Q}$, which is $\mathbb{Q}$.

Formality was extremely important in our examples. But we will not have formality as soon as we have nontrivial Massey products. So there are two directions we could go: understand Massey products or understand which spaces in general formal. These two directions are not equally good in some sense.

In general the extra tool that you have for manifolds or varieties is Hodge theory. For instance it is an important theorem that all simply connected Kähler manifolds are formal. This is what we'll look towards now (as well as the Lie algebra structure on homotopy groups and Koszul duality).

9. March 1

A few asides or remarks.

Last time we left off with this issue about the integral cohomology of an Eilenberg-Maclane space. For $n > 0$ let's check that $H^n(K(\mathbb{Q}, n), \mathbb{Z}) = 0$. By the universal coefficient theorem, writing $X = K(\mathbb{Q}, n)$, we have a short exact sequence

$$\mathrm{Ext}^1(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X, \mathbb{Z}), \mathbb{Z}).$$

Now $H_n(X, \mathbb{Z}) = \pi_n K(\mathbb{Q}, n) = \mathbb{Q}$ for $n \geq 1$. Hence the first term is zero but so is the last term. Thus $H^n(X, \mathbb{Z}) = 0$. This is a very important fact: it means that if we consider the homotopy category of spaces then discrete sets sit inside spaces in the obvious way, and discrete groups sit inside via $K(-, 1)$. Likewise abelian groups sit inside via $K(-, n)$ for $n \geq 1$. One good way to think of a homotopy type of a space is as a big layering via these various basic objects inside the homotopy category. Thus we have this impulse to think of $H_n(X) \otimes \mathbb{Q}$ as isomorphic to $H_n(X) \otimes \mathbb{Q}$ (up to some finite-dimensional issues) but of course this need not true as in this example since $H^n(K(\mathbb{Q}, n), \mathbb{Q}) = \mathbb{Q}$. It is only true if we have a finite CW complex.

Here's a second aside. Recall that the homotopy type of a circle can be thought of as weak homotopy equivalent to a certain poset containing four elements. The problem is that there's no inverse – not many maps out of this space. In fact this will not be a cohomology equivalence. There's a simpler thing that happens with simplicial sets. One might take a simplicial model for the circle and include it into a simplicial model where we've inverted all the one-simplices. The simplicial model might be a nerve of a poset while the larger thing is the nerve of the smallest groupoid generated by this poset. In this case you might find that this is a cohomology equivalence but not a weak homotopy equivalence (the problem being that we may not be working with Kan complexes). Passing from spaces to simplicial sets is of course achieved by using the functor Sing.

With these asides out of the way, let's have an extremely short review of Hodge theory.

Theorem 33. *For M a closed Riemannian manifold there exists a decomposition*

$$\Omega^k(M) = \Omega^{k-1}(M) \oplus \mathcal{H}^k(M) \oplus d^*\Omega^{k+1}(M)$$

where d maps $d^*\Omega^{k+1}(M)$ isomorphically to $d\Omega^k(M)$. In particular one obtains that $H^k(M, \mathbb{R}) \cong \mathcal{H}^k(M)$.

Recall that we have the Hodge star $* : \Omega^k(M) \xrightarrow{\sim} \Omega^{n-k}(M)$ coming from the inner product induced on forms and $d^* = \pm *^{-1} d*$. Now $\mathcal{H}(M) = \ker(\Delta_d)$ where $\Delta_d = d^*d + dd^*$.

Suppose now that M is moreover almost complex, i.e. we have an operator $J : TM \to TM$ such that $J^2 = -1$. Now define

$$d_c = J^{-1}dJ.$$

Let's assume the following conditions are true:

(1) d and d_c anti-commute
(2) d and d_c^* anti-commute
(3) $\Delta_d = \Delta_{d_c^*}$.

Theorem 34 (Deligne, Griffiths, Morgan, Sullivan)**.** *If M closed Riemannian is almost complex satisfying the conditions above the M is formal over $\mathbb{R}$ i.e. there exists a zig-zag of quasi-isomorphisms between $H^*(M, \mathbb{R})$ and $\Omega^*(M)$.*[1]

How did we get these conditions? Well notice that the first condition is implied if the almost complex structure is actually a complex structure. The next two are implied by M being Kähler (or say just a projective variety). The trick is to use the d_c-decomposition of k-forms: we can write

$$\Omega^k M = d_c \Omega^{k-1} M \oplus \mathcal{H}^k(M) \oplus d_c^* \Omega^{k+1} M.$$

Here these are the harmonic forms for Δ_c but by the condition above these are just the harmonic forms for the usual Laplacian. Now since d anti-commutes with d_c and d_c^* we see that d preserves this decomposition.

Now consider the span

$$\mathcal{H}^k M \xleftarrow{p} \ker(d_c^*) \xrightarrow{q} \Omega^* M.$$

Notice that $\ker p$ is acyclic and $\operatorname{coker} q$ is acyclic whence we obtain a zig-zag as desired.

[1]In fact we can find a zig-zag with just one roof because we can find a Sullivan representative.

10. March 4

So, Koszul duality. Let's return to some things we talked about at the very beginning of class. We said that there are (at least) two ways of getting at $\mathsf{Spaces}^{\geq 2}_{*,\mathbb{Q}}$. One is what we have already discussed: the functor $A = C^*_{\mathbb{Q}}$ which lands us in $\mathsf{CAlg}_{\mathbb{Q}}^{\leq -2, aug, \mathrm{op}}$. However, another invariant of interest might have been the homotopy groups, i.e. the functor $\pi_*(-_{[}-1])$ which lands in $\mathsf{Alg}^{\geq 1}_{\mathsf{Lie}}$. The relation between commutative algebras and Lie algebras is known as Koszul duality.

Everything will still continue to be over a field of characteristic zero.

Definition 35. A dg Lie algebra $\mathfrak{g} \in \mathsf{Alg}_{\mathsf{Lie}}$ is a chain complex with a k-linear bracket $[-,-] : \mathfrak{g}_i \otimes_{\mathbb{Q}} \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ satisfying:

(1) the bracket is graded antisymmetric:
$$[x, y] + (-1)^{|x||y|}[y, x] = 0$$
(2) the Leibniz rule
$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$$
(3) the Jacobi identity
$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

However the Koszul duality between commutative algebras and Lie algebras is actually not so easy to see. Let's start with a Koszul duality of a simpler nature. Koszul duality just for chain complexes takes
$$\mathsf{Ch}_Q \xrightarrow{(-)^\vee} \mathsf{Ch}^{\mathrm{op}}_{\mathbb{Q}}.$$
This is just linear duality. Notice that duality squares to the identity on chain complexes satisfying finiteness conditions. Notice that $\mathsf{Ch}_{\mathbb{Q}} = \mathsf{Alg}^{\mathrm{nu,op}}_{\mathcal{E}_0}$ so we might write instead
$$\mathsf{Alg}^{\mathrm{nu}}_{\mathcal{E}_0} \xrightarrow{\mathbb{D}^1} \mathsf{Alg}^{\mathrm{nu,op}}_{\mathcal{E}_0}.$$
What is this $\mathcal{E}_0$ gadget?

Definition 36. We define the topological category $\mathsf{Mfld}^{\partial}_n$ to have objects n-manifolds (possibly) with boundary. The morphisms are given $\mathrm{Emb}(M, N)$, open smooth embeddings, equipped with the compact open smooth topology (the boundary must be sent to the boundary).

Now we define the full subcategory $\mathsf{Disk}^{\partial}_n \subset \mathsf{Mfld}^{\partial}_n$ to have objects of the form $\coprod_I \mathbb{R}^n \sqcup \coprod_J \mathbb{R}^n_{\geq 0}$ (here the latter is the upper half-space and the indexing sets I and J are finite).

Similarly we define $\mathsf{Disk}^{\mathrm{rect}}_n \subset \mathsf{Mfld}^{\partial}_n$ to be the subcategory of objects of the form $\coprod (0,1)^n$ and morphisms given by open embeddings generated by translations and dilations. Likewise we define $\mathsf{Disk}^{\mathrm{rect},\partial}_n$ where we have objects $\coprod_I (0,1)^n \sqcup \coprod_J \mathbb{R}^n_{x_0 \geq 0, x_i \leq 1}$ and require the boundaries to map to boundaries.

Notice that when defining the category of disks we could also define a notion of framed disks where the morphisms are framed embeddings (it takes a bit of work to get the homotopy type of this space exactly correct). In this case we find that
$$\mathsf{Disk}^{fr}_n \simeq \mathsf{Disk}^{\mathrm{rect}}_n$$
and similarly for the categories allowing boundaries.

Definition 37. We define the category of $\mathcal{E}_n$-algebras to be the functor category

$$\mathsf{Alg}_{\mathcal{E}_n} = \mathrm{Fun}^{\otimes}(\mathsf{Disk}_n^{\mathrm{rect}}, \mathsf{Ch}).$$

An $\mathcal{E}_n$-algebra A yields

$$\coprod_I (0,1)^n \mapsto A^{\otimes I}$$

and moreover we have

$$C_* \,\mathrm{Emb}^{\mathrm{rect}}(\coprod_I (0,1)^n, \coprod_J (0,1)^n) \to \mathrm{Hom}(A^{\otimes I}, A^{\otimes J}).$$

Likewise we define augmented $\mathcal{E}_n$-algebras as the subcategory

$$\mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_n} \subset \mathrm{Fun}^{\otimes}(\mathsf{Disk}_n^{\mathrm{rect},\partial}, \mathsf{Ch})$$

sending $\mathbb{R}^n_{\geq 0} \mapsto \mathbb{Q}$. The augmentation ideal here appears as follows: let $I = J = \varnothing$. We have

$$\begin{array}{ccc} \varnothing & \longrightarrow & (0,1)^n \\ & \searrow & \downarrow \\ & & \mathbb{R}^n_{\geq 0} \end{array}$$

which goes to

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & A \\ & \searrow & \downarrow \\ & & \mathbb{Q} \end{array}$$

Notice that $\mathsf{Alg}_{\mathcal{E}_0}(\mathsf{Ch}_{\mathbb{Q}}) = (\mathsf{Ch}_{\mathbb{Q}})^{\mathbb{Q}/}$ and $\mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathsf{Ch}_{\mathbb{Q}}) = (\mathsf{Ch}_{\mathbb{Q}})^{\mathbb{Q}/}_{/\mathbb{Q}}$ (by taking the augmentation ideal).

Now let's think about when $n \neq 0$. There is a functor

$$\mathsf{Alg}^{\mathrm{aug}} \xrightarrow{\mathbb{D}^1} (\mathsf{Alg}^{\mathrm{aug}})^{\mathrm{op}}$$
$$A \mapsto R\,\mathrm{Hom}_A(\mathbb{Q}, \mathbb{Q})$$

where $R\,\mathrm{Hom}$ obtains an algebra structure from composition.

There are two features that tend to characterize Koszul duality:

(1) free algebras are sent to trivial algebras
(2) preserves homotopy colimits

Let's check this here. Notice that we have an adjunction

$$\mathsf{Mod}_A \underset{res_{A\to\mathbb{Q}}}{\overset{\mathbb{Q}\otimes^h_A(-)}{\rightleftarrows}} \mathsf{Ch}_{\mathbb{Q}}$$

In particular

$$R\,\mathrm{Hom}_A(\mathbb{Q},\mathbb{Q}) = R\,\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}\otimes^h_A \mathbb{Q}, \mathbb{Q})$$
$$= (\mathbb{Q}\otimes^h_A \mathbb{Q})^{\vee}.$$

Consider $A = TV = \oplus_{k\geq 0} V^{\otimes k}$. Then we have a map $TV \to \mathbb{Q}$ which is projection to the 0th factor. We have a short exact sequence of $A = TV$-modules.

$$\oplus_{k\geq 1} V^{\otimes k} \to TV \to \mathbb{Q}.$$

Notice that the kernel can be written as $V \otimes TV$ so we have

$$V \otimes TV \to TV \to \mathbb{Q}.$$

Now take a (derived) tensor by $\mathbb{Q}$ over TV to obtain a short exact sequence

$$V \otimes TV \otimes^h_{TV} \mathbb{Q} \to TV \otimes^h_{TV} \otimes \mathbb{Q} \to \mathbb{Q} \otimes^h_{TV} \mathbb{Q}$$

which simplifies to the exact triangle

$$V \xrightarrow{0} \mathbb{Q} \to \mathbb{Q} \otimes^h_{TV} \mathbb{Q} \to V[1]$$

But there's a projection $\mathbb{Q} \otimes^h_{TV} \to \mathbb{Q}$ whence

$$\mathbb{Q} \otimes^h_{TV} \mathbb{Q} = \mathbb{Q} \oplus V[1],$$

which is a trivial coalgebra. Hence $\mathbb{D}^1 TV = \mathbb{Q} \oplus V^\vee[-1]$.

That this construction preserves homotopy colimits is a technical matter that isn't too hard. We'll maybe leave that for later.

We actually have a commutative diagram

$$\begin{array}{ccc} \mathsf{Alg} & \xrightarrow{\mathbb{D}^1} & \mathsf{Alg}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathsf{Alg}^{nu}_{\mathcal{E}_0} & \xrightarrow{\mathbb{D}^0} & \mathsf{Alg}^{nu,\mathrm{op}}_{\mathcal{E}_0} \end{array}$$

where the vertical arrow on the left is the bar complex functor and the vertical arrow on the right is the forgetful functor. More generally this diagram extends vertically for higher n's via the bar construction and forgetting. Forgetting is coming form the functor $\mathsf{Disk}_{n-1} \to \mathsf{Disk}_n$ that is give $- \times \mathbb{R}$. By the bar construction we mean $A \mapsto \mathbb{Q} \otimes^h_A \mathbb{Q}$.

Definition 38. The functor $\mathbb{D}^n : \mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_n} \to \mathsf{Alg}^{\mathrm{aug},\mathrm{op}}_{\mathcal{E}_n}$ is defined

$$\begin{aligned} \mathbb{D}^n A &= \int_{D^n/\partial D^n} A \\ &:= \mathrm{hocolim}(\mathsf{Disk}^{fr,\partial}_{n/D^n} \to \mathsf{Disk}^{fr}_n \xrightarrow{A} \mathsf{Ch}_{\mathbb{Q}}). \end{aligned}$$

Let me tell you the fundamental calculation. There is the free algebra $\mathsf{Free}_{\mathcal{E}_n}(V) = \oplus_{k \geq 0} C_* \mathrm{Conf}_k \mathbb{R}^n \otimes_{\Sigma_k} V^{\otimes k}$. This is augmented by projecting onto the $k = 0$ term. Notice that when $n = 1$ we just have $\oplus_{k \geq 0} V^{\otimes k}$ the usual tensor algebra. The fundamental calculation is that

$$\int_{M/\partial M} \mathsf{Free}_{\mathcal{E}_n}(V) = \oplus_{k \geq 0} C_*(\mathrm{Conf}_k M/\partial) \otimes_{\Sigma_k} V^{\otimes k}$$

Here by this we mean the space where at least one point in the configuration lies in the boundary of M (this is just the boundary). Why is this trivial?

Lemma 39. *The space $\mathrm{Conf}_k D^n/\partial \simeq *$ is contractible when $k \geq 2$.*

Proof. We have

$$\partial = \{(x_1, \ldots, x_k) \in \mathrm{Conf}_k D^n \mid \exists i, x_i \in \partial D^n\} \hookrightarrow \mathrm{Conf}_k D^n \to \mathrm{Conf}_k D^n/\partial.$$

We can construct a deformation retraction as follows: given k points, fix x_1 but radiate all the other points out to the boundary until one of the other points hits the boundary. This of course requires us to have at least two points. $\square$

In the case $k = 1$ we have $\mathrm{Conf}_1 D^n/\partial = S^n$ so

$$\begin{aligned}\int_{D^n/\partial} \mathsf{Free}_{\mathcal{E}_n} V &= \bigoplus_{k\geq 0} \bar{C}_*(\mathrm{Conf}_k D^n/\partial) \otimes_{\Sigma_k} V^{\otimes k} \\ &\simeq \mathbb{Q} \oplus \bar{C}_*(\mathrm{Conf}_1 D^n/\partial) \otimes V \\ &\simeq \mathbb{Q} \oplus V[n].\end{aligned}$$

We conclude that

$$\mathbb{D}^n \mathsf{Free}_{\mathcal{E}_n} V = \mathbb{Q} \oplus V^\vee[-n].$$

So notice that we have done the same computation two different ways. One geometric via configuration spaces and one algebraic above. So now we have a picture

$$\begin{array}{ccc} \mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_n} & \xrightarrow{\mathbb{D}^n} & (\mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_n})^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathsf{bar}} & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_{n-1}} & \xrightarrow{\mathbb{D}^{n-1}} & (\mathsf{Alg}^{\mathrm{aug}}_{\mathcal{E}_{n-1}})^{\mathrm{op}} \end{array}$$

where $\mathcal{E}_n$ algebras are Koszul-self dual in a compatible way. Now we could let $n \to \infty$

$$\mathrm{holim}(\cdots \to \mathsf{Alg}_{\mathcal{E}_n} \xrightarrow{\mathsf{fgt}} \mathsf{Alg}_{\mathcal{E}_n} \to \cdots) \simeq \mathsf{Alg}_{\mathcal{E}_\infty} \simeq \mathsf{CAlg}$$

where in the last step we have used that we are in characteristic zero. Alternatively we could go to infinity along the bar construction and we get an equivalence (this is not in the literature but is probably true, and again should require characteristic zero)

$$\mathrm{holim}(\cdots \to \mathsf{Alg}_{\mathcal{E}_n} \xrightarrow{\mathsf{bar}} \mathsf{Alg}_{\mathcal{E}_n} \to \cdots) \simeq \mathsf{Alg}_{\mathsf{Lie}}.$$

Why is this reasonable? Well we should have functors from Lie algebras to $\mathcal{E}_n$-algebras for each n compatible with the bar construction? This is the $\mathcal{E}_n$-enveloping algebra functor which is given by

$$U^{\mathcal{E}_n} = C_*^{\mathrm{Lie}}(\Omega^n -).$$

Next time we will look at this Koszul duality in the more classical setting of Lie algebra cohomology $\mathsf{Alg}_{\mathsf{Lie}} \xrightarrow{C^*_{\mathsf{Lie}}} \mathsf{CAlg}^{\mathrm{op}}$. We will then see how this relates to rational homotopy.

11. March 6

Everything today is homologically graded and in characteristic zero.

Let's now discuss the direct approach of Koszul duality. First let's note that every associative algebra has an underlying Lie algebra

$$\mathsf{Alg} \to \mathsf{Alg}_{\mathsf{Lie}}$$

sending $R \mapsto (R, [-,-])$ where $[a,b] = ab - (-1)^{|a||b|}ba$. This functor preserves limits whence it has a right adjoint

$$U : \mathsf{Alg}_{\mathsf{Lie}} \to \mathsf{Alg}$$

known as the universal enveloping algebra. Just from universal properties you can see that there exists a map $T\mathfrak{g} \to U\mathfrak{g}$ and indeed you can describe $U\mathfrak{g}$ as the quotient of $T\mathfrak{g}$ by the ideal $x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y]$. Observe that $U\mathfrak{g}$ is naturally augmented as the tensor algebra is. Now the tensor algebra has a filtration by tensor powers of $\mathfrak{g}$ whence $U\mathfrak{g}$ inherits a filtration which is the image of the tensor power filtration,

$$(U\mathfrak{g})_{\leq k} = \mathrm{im}\left(\oplus_{0 \leq i \leq k} \mathfrak{g}^{\otimes i} \to U\mathfrak{g}\right).$$

This filtration is exhaustive since it is for the tensor algebra and this is a quotient map. Now the associated graded algebra can be identified, by the Poincaré-Birkhoff-Witt theorem,

$$\begin{aligned} \mathrm{Sym}^k \mathfrak{g} &\xrightarrow{\sim} (U\mathfrak{g})_{\leq k}/(U\mathfrak{g})_{\leq k-1} \\ x_1 \cdots x_k &\mapsto \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{mult} \circ \sigma. \end{aligned}$$

whence the associated graded algebra of this filtration on $U\mathfrak{g}$ is isomorphic to the symmetric algebra on $\mathfrak{g}$. Observe that $\mathsf{Rep}(\mathfrak{g}) \cong \mathsf{Mod}_{U\mathfrak{g}}$.

The Lie algebra homology of $\mathfrak{g}$ is defined to be

$$C_*\mathfrak{g} := \mathbb{Q} \otimes^h_{U\mathfrak{g}} \mathbb{Q}$$

It will be useful to have an explicit model of this homology. We will pick a nice resolution as follows. Recall that the cone on the identity map on $\mathfrak{g}$ is

$$\mathrm{cone}(\mathrm{id}_{\mathfrak{g}}) = (\mathfrak{g} \oplus \mathfrak{g}[1], d)$$

where $d(x + y[1]) = dx + y - dy[1]$. So far this hasn't used that $\mathfrak{g}$ is a Lie algebra and it is just a chain complex. The Lie algebra structure can be defined

$$[x + y[1], x' + y'[1]] = [x, x'] + ([y, x'] + (-1)^{|x|}[x, y'])[1].$$

This is of course a dg Lie algebra that is quasi-isomorphic to the trivial one. Remark that in general one cannot define this for any map of Lie algebras — the fact that we were working with the identity map was important. Indeed, the point is that the identity map is kinda giving us an ideal that we can take a pushout with (generaly pushouts we can't do this way).

With this construction in mind we define the point-set model for the derived tensor product above as

$$C_*\mathfrak{g} := U\,\mathrm{cone}(\mathfrak{g}) \otimes_{U\mathfrak{g}} \mathbb{Q}$$

It turns out that U preserves quasi-isomorphisms because you can check that Sym^k preserves quasi-isomorphisms (since taking quotients by symmetric group actions preserves quasi-isomorphisms) together with induction on the filtration:

$$\begin{array}{ccccc} U(\mathfrak{g})_{\leq k-1} & \longrightarrow & (U\mathfrak{g})_{\leq k} & \longrightarrow & \mathrm{Sym}^k \mathfrak{g} \\ \downarrow & & \downarrow & & \downarrow \\ U(\mathfrak{g}')_{\leq k-1} & \longrightarrow & (U\mathfrak{g}')_{\leq k} & \longrightarrow & \mathrm{Sym}^k \mathfrak{g}' \end{array}$$

since sequential colimits along inclusions preserve quasi-isomorphisms. Hence since $\mathrm{cone}\,\mathfrak{g} \simeq 0$ we have that $U(\mathrm{cone}\,\mathfrak{g}) \simeq U(0) = \mathbb{Q}$ whence $C_*\mathfrak{g}$ is indeed computing the derived tensor product above.

Now as a $U\mathfrak{g}$-module we can write

$$U(\mathrm{cone}\,\mathfrak{g}) \simeq (U\mathfrak{g} \otimes \mathrm{Sym}\,\mathfrak{g}[1], d)$$

since Sym takes direct sums to tensor products. Thus

$$U(\mathrm{cone}\,\mathfrak{g}) \otimes_{U\mathfrak{g}} \mathbb{Q} \simeq (\mathrm{Sym}\,\mathfrak{g}[1], d)$$

A direct computation reveals that the differential on the Lie algebra homology is defined

$$\begin{aligned} d(x_1 \cdots x_n) = & \sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_{k-1}|} x_1 \cdots dx_i \cdots x_n \\ & + \sum_{1 \leq i < j \leq n} (-1)^{|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)} x_1 \cdots \hat{x}_i \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n. \end{aligned}$$

Lemma 40. *The functor C_* preserves quasi-isomorphisms.*

Proof. This will follow from the fact that U preserves quasi-isomorphisms. □

Theorem 41. *The functor $C_* : \mathsf{Alg}_{\mathsf{Lie}} \to \mathsf{Ch}_{\mathbb{Q}}$ satisfies the following two properties:*

(1) it sends free Lie algebras to their generators, i.e. $C_*(\mathsf{Free}_{\mathsf{Lie}}(V)) \simeq \mathbb{Q} \oplus V[1]$
(2) it preserves homotopy colimits

Proof sketch. The functor C_* is given as a derived tensor product. By universal properties if $\mathfrak{g} = \mathsf{Free}_{\mathsf{Lie}}(V)$ then $U\mathfrak{g} = TV$. Thus $\mathbb{Q} \otimes^h_{TV} \mathbb{Q} \simeq \mathbb{Q} \oplus V[1]$.

Let's sketch the second point. You can build every homotopy colimit as a combination of three types: coproducts $J = \{0, 1\}$, sequential colimits $J = \mathbb{N}$, and geometric realizations $J = \Delta^{\mathrm{op}}$ (sifted colimits). This is formalized by the following statement: a functor $F : \mathcal{J} \to \mathcal{C}$ preserves all homotopy colimits if and only if F preserves these three classes of homotopy colimits, i.e. the natural map $\mathrm{hocolim}_J\, G \circ F \to G\, \mathrm{hocolim}_J\, F$ is an equivalence for each functor $G : \mathcal{C} \to \mathcal{D}$. We will assume this statement.o

Now things aren't so bad because our functor is, up to forgetting the differential, Sym. The key is that we will be able to reduce to free Lie algebras and the coproduct of two free Lie algebras is free. In particular consider the full subcategory $\mathsf{Alg}^{\mathrm{free}}_{\mathsf{Lie}} \hookrightarrow \mathsf{Alg}_{\mathsf{Lie}}$. We have

$$\begin{aligned} C_*(\mathsf{Free}_{\mathsf{Lie}} V \sqcup \mathsf{Free}_{\mathsf{Lie}} V') &= C_*(\mathsf{Free}_{\mathsf{Lie}}(V \oplus V')) \\ &= \mathbb{Q} \oplus (V \oplus V')[1]. \end{aligned}$$

as the free Lie algebra functor is left adjoint to the forgetful functor to chain complexes where the coproduct is direct sum. Now there's a trick. You can resolve any Lie algebra by a simplicial resolution. In particular write $F = \mathsf{Free}_{\mathsf{Lie}}$ and now using counit of the adjunction we have an augmented simplicial object (omitting the structure maps)

$$F^{\bullet}\mathfrak{g} = \cdots \to F^2\mathfrak{g} \to F\mathfrak{g} \to \mathfrak{g}$$

such that $\mathfrak{g} \simeq |F^{\bullet}\mathfrak{g}|$. This is a general story about algebras (and actually in generality you should look at the infinity-categorical Barr-Beck theorem).

We now resolve $\mathfrak{g} \sqcup \mathfrak{g}'$ by free algebras via this standard resolution. We obtain an augmented simplicial object

$$\cdots \to F^2\mathfrak{g} \sqcup F^2\mathfrak{g}' \to F\mathfrak{g} \sqcup F\mathfrak{g}' \to \mathfrak{g} \sqcup \mathfrak{g}'$$

that is free in each degree.

We will continue next time. □

12. March 11

We made the following claim. A functor $F : \mathsf{Alg}_{\mathsf{Lie}} \to \mathsf{Ch}_{\mathbb{Q}}$ (for us reduced Lie algebra homology) preserves homotopy colimits if and only if it preserves coproducts, realizations, and filtered colimits. The next step was to say that we don't have to check the entirety of these conditions: F preserves all three types if and only if it preserves coproducts of free algebras, realizations, and filtered colimits.

There's something special about realizations and filtered colimits. They are known as sifted colimits.

Definition 42. A nonempty category K is sifted if for any $F : K \times K \to C$ the natural map $\operatorname{hocolim}_K F \circ \Delta \to \operatorname{hocolim}_{K\times K} F$ (restriction along the diagonal) is an equivalence.

This may not look super natural at first but lots of functors preserve sifted colimits. Notice that for K sifted then $BK \simeq *$: take C to be spaces and $F = *$ the constant functor. The homotopy colimit of the constant functor is now $B(K \times K) \cong BK \times BK$. Being sifted thus implies that the diagonal is an equivalence $BK \times BK \simeq BK$ but this implies (by looking at homotopy groups) that $BK \simeq *$.

Lemma 43. *The forgetful functor* $\mathsf{Alg}_{\mathsf{Lie}} \to \mathsf{Ch}_{\mathbb{Q}}$ *preserves sifted homotopy colimits. Similarly* $\operatorname{Sym}^k : \mathsf{Ch}_{\mathbb{Q}} \to \mathsf{Ch}_{\mathbb{Q}}$ *preserves sifted homotopy colimits.*

Assuming this lemma for the moment, let's show the statement in the first paragraph from today. We need to show that F preserves arbitrary coproducts if it preserves coproducts of free algebras. First note that for any L there exists a simplicial free resolution $|L_\bullet| \simeq L$ where L_i is free. Thus now

$$\begin{aligned} F(L \sqcup L') &= F(|L_\bullet| \sqcup |L'_\bullet|) \\ &= F(|L_\bullet \sqcup L'_\bullet|) \\ &= |F(L_\bullet \sqcup L'_\bullet)| \\ &= |F(L_\bullet) \oplus F(L'_\bullet)| \\ &= |F(L_\bullet)| \oplus |F(L'_\bullet)| \\ &= F(L) \oplus F(L'). \end{aligned}$$

Now we need the following.

Lemma 44. *Reduced Lie algebra homology* $F = \bar{C}*^{\mathsf{Lie}}$ *commutes with sifted homotopy colimits and preserves coproducts of free objects.*

Proof. We have already proved the last point. To see that F preserves sifted colimits we invoke the lemmas above on the forgetful and symmetric algebra functors. A consequence of Poincaré-Birkhoff-Witt is that there is a filtration $C_*L_{\leq k}$ whose associated graded is $\operatorname{Sym}^k(L[1])$. Using this filtration we can prove that F preserves sifted colimits via induction: $C_*(-)_{\leq k}$ preserves sifted colimits.

In the base case we have $C_*(-)_{\leq 1}$ sending $L \mapsto \mathbb{Q} \oplus L[1]$. This copy of $\mathbb{Q}$ doesn't really matter so this is really just the forgetful functor (up to a shift, but shifting is an equivalence of categories on chain complexes) which preserves sifted colimits by the lemma above. Now we argue by induction. We have an exact sequence of chain complexes

$$0 \to C_*(-)_{\leq k-1} \to C_*(-)_{\leq k} \to \operatorname{Sym}^k(-[1]) \to 0.$$

The first term preserves sifted colimits by the induction hypothesis and the third functor is the composition of forgetful and symmetric algebra functors which each preserve sifted colimits by the lemma above. □

Maybe I will now say briefly why you might believe the lemma that we've taken on faith. You can see immediately that the coproduct is not preserved as $TV \oplus TW$ is quite different than $T(V \oplus W)$. Now we can try to equip

$$K \xrightarrow{F} \mathsf{Alg} \xrightarrow{\mathsf{fgt}} \mathsf{Ch}$$

$\mathrm{hocolim}_K F \circ \mathsf{fgt}$ with an algebra structure. Well we could try

$$\mathrm{hocolim}\left(K \times K \xrightarrow{F \times F} \mathsf{Ch} \times \mathsf{Ch} \xrightarrow{\otimes} \mathsf{Ch}\right).$$

Now we have

$$K \xrightarrow{\Delta} K \times K \to \mathsf{Ch} \times \mathsf{Ch} \to \mathsf{Ch}$$

and the homotopy colimit of this composite maps to the homotopy colimit of

$$K \to \mathsf{Ch}$$

since F was an algebra. It also maps via Δ to the homotopy colimit of

$$K \times K \to \mathsf{Ch} \times \mathsf{Ch} \to \mathsf{Ch}.$$

This last homotopy colimit in turn maps to

$$\mathrm{hocolim}(K \to \mathsf{Ch}) \otimes \mathrm{hocolim}(K \to \mathsf{Ch}).$$

In general this is all you have. If you asked that Δ and this last map to be equivalences then we'd be able to write an algebra structure. Thus we ask to impose some conditions on K so that we get these equivalences.

Suppose now that our Lie algebras are finite type throughout.

Corollary 45. *Dualizing, we obtain a functor*

$$C* : \mathsf{Alg}_{\mathsf{Lie}} \to (\mathsf{CAlg}_{\mathbb{Q}}^{aug})^{op}$$

that preserves homotopy colimits.

Proof. This is because $C^*\mathfrak{g}$ is equipped with an algebra functor:

$$C^*(\mathfrak{g} \xrightarrow{\Delta} \mathfrak{g} \times \mathfrak{g})$$

but $C^*(\mathfrak{g} \times \mathfrak{g}) = C^{\mathfrak{g}} \otimes C^*\mathfrak{g}$, which gives us the map back. More explicitly,

$$\phi \cdot \psi(x_1, \dots, x_n) = \sum \prod_{i<j} (-1)^{|x_i||x_j|} \phi(x_1, \dots, x_{i_m}) \psi(x_j \cdots, x_{j_{n-m}}).$$

Here the sum is over all partitions $\{1 \dots n\} = \{i_1 \dots i_m\} \sqcup \{j_1 \dots j_{n-m}\}$.

Now the functor preserves homotopy colimits because the composite with the forgetful functor to the opposite category of augmented chain complexes is the linear dual of Lie algebra homology. It is enough to check this composite by conservativity of forgetful functor (which is a right adjoint). □

Remark 46. Lie algebra cohomology sends free Lie algebras to trivial commutative algebras and sends trivial Lie algebras to completed free commutative algebras.

Indeed, if V is a trivial Lie algebra then

$$C^*V = (C_*V)^\vee = (\mathrm{Sym}(V[1]))^\vee.$$

This is the infinite product $\prod_{k\geq 0}\mathrm{Sym}^k(V[1])^\vee$. Since we are in characteristic zero there is a natural norm map

$$(V^{\otimes k})_{\Sigma_k} \xrightarrow{\simeq} (V^{\otimes k})^{\Sigma_k}$$

that is an equivalence, whence

$$\begin{aligned} C^*V &= \prod_{k\geq 0}(V[1]^{\otimes k}_{\Sigma_k})^\vee \\ &= \prod_{k\geq 0}((V[1]^{\otimes k})^\vee)^{\Sigma_k} \\ &= \prod((V^\vee[-1])^{\otimes k})^{\Sigma_k} \\ &= \prod \mathrm{Sym}^k(V^\vee[-1]), \end{aligned}$$

which is the completed free commutative algebra.

Sometimes completed symmetric algebras and symmetric algebras are the same thing. This is very important. If $V \in (\mathsf{Ch}_{\mathbb{Q}})_{\leq -1}$ or $V \in \mathsf{Ch}_{\mathbb{Q}}^{\geq 1}$ then $\mathrm{Sym}\, V = \hat{\mathrm{Sym}} V$ and $TV = \hat{T}V$. Let's just see why that's true for the tensor algebra. There's an obvious map

$$\oplus_{k\geq 0} V^{\otimes k} \to \prod_{k\geq 0} V^{\otimes k}$$

Notice that if $V \in \mathsf{Ch}^{\geq 1}$ then $V^{\otimes k} \in \mathsf{Ch}^{\geq k}$ whence there are only finite many terms that can be nonzero in a given degree. Exactly the same argument works for Sym.

Thus we see that in this case we are coming up on an equivalence between Lie algebras and commutative algebras.

13. March 13

There's one other thing I could have said for the heuristic for why a forgetful functor from algebras to chain complexes preserves sifted hocolimits. In particular, why do tensor products preserve homotopy colimits? Suppose we are given a pair of functors

$$K \times K \xrightarrow{F\times G} Ch \times Ch \xrightarrow{\otimes^h} \mathsf{Ch}.$$

Here we could just as well do $K \times K' \dots$ Then we have that $\operatorname{hocolim}_{K\times K} F \otimes G \simeq \operatorname{hocolim}_K F \otimes \operatorname{hocolim}_K G$. To see this, write $\pi_2 : K \times K$ for the second projection and consider the homotopy left Kan extension along the composite of $p \circ \pi_2$ where $p : K \to *$ is the map to the terminal object. Just along π_2 we have

$$\begin{aligned}(\pi_2)_!(F \times G)(y) &= \operatorname{hocolim}_{x\in K}(F(x) \otimes G(y))\\ &\simeq (\operatorname{hocolim}_{x\in K} F(x)) \otimes G(y)\end{aligned}$$

since $- \otimes^h G(y)$ preserves homotopy colimits. Now do the same thing along p,

$$\pi_!((\pi_2)_! F\times G) = \operatorname{hocolim}_{y\in K}((\operatorname{hocolim}_K F)\otimes G(y)) \simeq \operatorname{hocolim}_K F \otimes \operatorname{hocolim}_{y\in K} G(y).$$

In this part we did not use siftedness.

Siftedness is one of the key ideas in this landscape of mathematics, so it is important to remember these details. Another important thing to remember is that the norm map

$$V^{\otimes k} \to (V^{\otimes k})^{\Sigma_k}$$

factors through the coinvariants and is an isomorphism (we are in characteristic zero). The third crucial thing is that with V (co)connected, the tensor algebra on V is automatically complete. One last thing (maybe from a while ago) is things like convergence of Eilenberg-Moore spectral sequences: that cohomology, which a priori behaves well with respect to homotopy pushouts (excision) but actually also behaves well with respect to homotopy pullbacks.

Everything today will be finite type and homologically graded. Recall that we were contemplating the functor

$$\mathsf{Alg}^{\geq 1}_{\mathsf{Lie}} \xrightarrow{C^*} \mathsf{CAlg}^{\mathrm{aug,op}}_{\leq -2},$$

which sends free Lie algebras to trivial commutative algebras and trivial Lie algebras to completed free commutative algebras, and moreover takes homotopy colimits to homotopy limits. By (a suitable version of) the adjoint functor theorem, this functor must have a right adjoint. Today we'll probably just be making a lot of claims, with not too much proof. This right adjoint can be identified with the derived tangent space $T[-1]$. This is a suitably derived functor of derivations $TA = R\operatorname{Der}_A(\mathbb{Q}, \mathbb{Q})$.

As an aside: it's hard to make these functors adjoint on the nose for ordinary categories. This leads to some funny things in the literature. For instance bar constructions are left-adjoints in homotopy theory (∞-category theory) but sometimes right adjoints in ordinary category theory.

So we have this Koszul duality picture and the theorem is that the above functor induces an equivalence on homotopy categories or on ∞-categories. This result is essentially due to Quillen. Here's the basic idea (we'll be missing one step but that's okay). From what we've done so far, if we apply C^* then $T[-1]$ to a trivial Lie algebra then we'll get what we started. First we note that for trivial (abelian) Lie algebras V then $C^*V \simeq \operatorname{Sym}(V^\vee[-1])$. But now $T \operatorname{Sym} W = W^\vee$ from which the result follows.

How far can we get just by understanding abelian Lie algebra? Filter $\mathfrak{g}$ by its lower central series. By the connectivity condition, the Lie algebra $\mathfrak{g}$ is automatically nilpotent whence the lower central series terminates. So now we have a filtration

$$\mathfrak{g} \to \cdots \to \mathfrak{g}^{(k-1)} \to \mathfrak{g}^{(k)} \to 0$$

whence $\mathfrak{g}$ is a finite sequence of extensions by abelian Lie algebras. We now use the lemma that the Lie algebra cohomology of a nilpotent extension goes to a homotopy pushout in commutative algebras. You can think of this as some sort of parameterized Künneth theorem (it's a parameterized version of the fact that the Lie algebra cohomology of a product is a tensor product of the Lie algebra cohomologies, as long as we're in the finite-type case).

Let's return to one of our goals now which was to produce a Lie algebra from a pointed space with connectivity and finite type conditions. We had a functor $A \simeq C^*_{\mathbb{Q}}$ that landed in commutative algebras. Alternatively we could take $C_*(\Omega -, \mathbb{Q})$, which gives us a cocommutative Hopf algebra. From here there's an easy way of obtaining a Lie algebra, which is the functor of primitives. Recall that bialgebras are precisely the coalgebras in the category of algebras. How do you define the algebra structure on $A \otimes B$? Well you take

$$a \otimes b \cdot a' \otimes b' = (-1)^{|b|}(-1)^{|a'|} aa' \otimes bb'.$$

is the sign correct? If A is a bialgebra we say that $a \in A$ is primitive if $\Delta a = a \otimes 1 + 1 \otimes a$. This definition may seem a bit odd at first but notice that it is closely related to the kernel of the comultiplication. Now observe that the collection of primitives of A is a Lie subalgebra of A, with the commutator induced by that on A. Check this for homework (check that the signs work out in particular (check that the signs work out in particular). If we now throw in the functors of primitives and chains on loops into our diagram with C^* and $T[-1]$ then the whole thing will commute. There is a functor

$$\mathsf{Hopf}^{\text{cocomm}} \xrightarrow{\mathsf{Bar}(-)^\vee} \mathsf{CAlg}^{\text{aug,op},\leq -2}.$$

where $A \mapsto \mathbb{Q} \otimes^h_A \mathbb{Q}$. This is the way Quillen approaches rational homotopy theory in his paper.

A consequence of this picture is that the primitives in the homology of the loop space is equivalent as Lie algebras to the homotopy groups of X shifted by -1, rationally.

$$\mathsf{Prim}(H_*(\Omega X, \mathbb{Q})) \cong \pi_* X_{\mathbb{Q}}[-1]$$

where the right is equipped with the Whitehead bracket. What is this bracket? Consider the usual cell structure on S^p and S^q. This gives a cell structure on the product and there's an interesting attaching map

$$\begin{array}{ccc} S^{p+q-1} & \longrightarrow & D^{p+q} \\ \big\downarrow & & \big\downarrow \\ S^p \wedge S^q & \longrightarrow & S^p \times S^q \end{array}$$

Now

$$[-,-] : \pi_* X[-1] \times \pi_* X[-1] \to \pi_* X[-1]$$

is given by sending $S^p \wedge S^q \to X$ to the homotopy class of the composite

$$S^{p+q-1} \to S^p \wedge S^q \to X.$$

It is a classical result that this is a Lie algebra.

So we now have two models for rational homotopy types. But we had these connectivity conditions. Is there a better version of this story where we can get rid of these connectivity conditions? What kinds of geometric objects give you all these Lie algebras.

$$\begin{array}{ccc} \mathsf{Alg}_{\mathsf{Lie}} & & \mathsf{CAlg} \\ \downarrow & & \downarrow \\ \mathsf{Alg}_{\mathsf{Lie}} & \longrightarrow & \mathsf{FModuli} \end{array}$$

Here formal moduli problems are $\mathsf{FModuli} \subset \mathsf{Fun}(\mathsf{Artin}, \mathsf{Spaces})$. What are dg Artin things? They have finite dimensional homology, are local, and have zero cohomology far enough to the right and no negative degree cohomology. Formal moduli problems are precisely those functors which preserve homotopy pullbacks along square zero extensions

$$\begin{array}{ccc} \mathbb{Q} \otimes V & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & A' \end{array}$$

We moreover ask that $F(\mathbb{Q}) = *$.

There is a functor MC from Lie algebras to formal moduli:

$$\mathsf{MC}(\mathfrak{g})(R) := \underset{\mathsf{Lie}}{\mathrm{Maps}}(TR[-1], \mathfrak{g}).$$

The functor back is the shifted tangent space functor. These functors are equivalences (this statement is due to Lurie). Here it turns out you need no finite type conditions here. Why is this called the Maurer-Cartan equation? Well if $\mathfrak{g}$ were nilpotent, we could simply use the Maurer-Cartan simplicial set.

We could also consider the relation between this C^* and our story of Koszul duality $\mathbb{D}^n = (\int_{D^n/\partial})^\vee$. Well there is a commutative diagram

$$\begin{array}{ccc} \mathsf{Alg}_{\mathsf{Lie}} & \longrightarrow & \mathsf{CAlg}^{\mathrm{aug,op}} \\ \downarrow & & \downarrow \\ \mathsf{Alg}_{E_n}^{\mathrm{aug}} & \longrightarrow & \mathsf{Alg}_{E_n}^{\mathrm{aug,op}} \end{array}$$

where the vertical map on the left is U^{E_n} the E_n-enveloping algebra and the vertical map on the right is forgetting. There is a nice description $C_*(\Omega^n -) \simeq U^{E_n}$. Notice that Ω is just shifted $\mathfrak{g}$ and will be abelian. However there's still interesting structure — there's loops involved. This functor actually has a filtration and a higher version of the PBW theorem that tells you that $C_*(\Omega^n -) = \mathrm{Sym}(-[n-1])$. So in particular this commutative diagram in the $n = 1$ case tells you that you can think of $U\mathfrak{g} = C_*(\Omega\mathfrak{g})$ which tells you that the multiplication matches up with composition of loops.