NOTES FROM SMS 2018

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Contents

These are very rough notes from the lectures at the SMS conference in Toronto. A few of the lectures are missing and many of them are missing diagrams and the like. Corrections very welcome!

1. June 11, 2018 – Jacob Lurie

In this first lecture I would like to give a historical introduction to some number theoretic problems, which we will later use factorization homology to study. We'll start with quadratic forms in two variables:

$$x^2+y^2 \quad x^2-y^2 \quad -x^2-y^2.$$

The basic question one might ask about these is whether they're equivalent (under a linear change of basis). It turns out that if we work over $\mathbb{C}$ then all of these are equivalent, but over $\mathbb{R}$ they are not. Indeed, over $\mathbb{R}$, they are positive-definite, indefinite, and negative-definite. Notice that given

$$x^2+y^2 \quad x^2+3y^2$$

we can see that reducing mod 3 will degenerate the one on the right and cannot possibly be equivalent there. Hence they cannot be equivalent over $\mathbb{Z}$.

The mathematics I'm going to describe in this lecture began with the question as to whether these two methods shown above were the only ways of checking whether quadratic forms are equivalent. Let's be more precise. Let q, q' be positive definite quadratic forms over $\mathbb{Z}$. We say that q and q' are **in the same genus** if they are equivalent mod N for every N (from now on everything will be positive-definite and over $\mathbb{Z}$ and every form will involve the same number of variables). These two examples cannot be distinguished by the methods mentioned above. Are they equivalent? Not necessarily, but almost.

Let q be a quadratic form over $\mathbb{Z}$. Then for any commutative ring R we write $O_q(R)$ for the set of invertible $n \times n$ matrices preserving the form,

$$O_q(R) = \{A \in GL_n(R) \mid q = q \circ A\}.$$

In the case where $R = \mathbb{R}$ this is a familiar object, a compact Lie group. Inside here we have another group $O_q(\mathbb{Z}) \subset O_q(R)$, which is the set of matrices preserving a certain lattice. It is easy to check that $O_q(\mathbb{Z})$ is actually finite. We now define **the mass of** q

$$\mathsf{mass}(q) = \sum_{q \in \text{genus(q)}} \frac{1}{|O_q(\mathbb{Z})|}.$$

This is some sort of weighted or with-multiplicity count of genus-equivalence classes of quadratic forms.

There is a nice formula for the mass (see the title of this talk). I'm not going to state it for you in generality, since it can get kind of complicated. We'll state it in a special case, but we first need a definition. We say that a quadratic form q is **unimodular** if it is nondegenerate mod p for all p. This is a very strong condition – none of the examples above are unimodular. For instance notice that

$$x^2+y^2 \equiv (x+y)^2 \mod 2.$$

It turns out that the number of variables in a unimodular quadratic form has to be divisble by 8. Moreover it turns out that any two unimodular quadratic forms are in the same genus. The mass formula tells us

$$\sum \frac{1}{|O_q(\mathbb{Z})|} = \frac{\zeta(2)\zeta(4)\cdots\zeta(n-2)\zeta(n/2)}{\mathsf{vol}(S^1)\cdot\mathsf{vol}(S^2)\cdots\mathsf{vol}(S^{n-1})}.$$

where the sum is over the q unimodular in n variables.

Example 1. Let $n = 8$. Then the right hand side of the mass formula is

$$\frac{1}{2^{14}3^5 5^2 7},$$

which will be familiar as the rank of the E_8 lattice. Now the mass formula tells us that only one unimodular form in 8 variables.

In general things are not so nice.

Example 2. If $n = 32$ then it turns out that the right hand side is about 40 million. This implies that there exist millions and millions of unimodular quadratic forms in 32 variables.

We will now provide a different way of thinking about what this mass formula is saying, due to Tamagawa and Weil. Let q and q' be in the same genus. In other words, for each $N > 0$ there exists some matrix $A_N \in GL_n(\mathbb{Z}/N\mathbb{Z})$ such that

$$q = q' \circ A_N$$

for each N. One may as well assume that these A_N are compatible with each other, i.e. as N varies $A = \{A_N\}_{N>0} \in GL_n(\hat{\mathbb{Z}})$. Recall that $\hat{Z}$ is the profinite completion of the integers,

$$\hat{\mathbb{Z}} = \lim \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p \qquad \mathbb{Z}_p = \lim \mathbb{Z}/p^k\mathbb{Z}. \tag{1}$$

Hence q, q' are equivalent over the p-adics and thus over $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$ for all p. But of course they are also equivalent over $\mathbb{R}$ since they are positive-definite. We now invoke something non-trivial — the Hasse-Minkowski theorem. This theorem tells us that q and q' are equivalent over $\mathbb{Q}$ if we have equivalence over all the completions of $\mathbb{Q}$ i.e. the reals and the $\mathbb{Q}_p$. Hence $q = q' \circ B$ for some $B \in GL_n(\mathbb{Q})$. Now

$$q = q' \circ A = q \circ B^{-1} \circ A$$

whence $B^{-1} \circ A \in O_q(\mathbb{Q})\backslash O_q(\mathbb{A}^{\text{fin}})/O_q(\hat{Z})$ where

$$\mathbb{A}^{\text{fin}} = \hat{\mathbb{Z}} \otimes \mathbb{Q} \subset \prod_p \mathbb{Q}^p$$

is the ring of finite adeles, which contains both $\mathbb{Q}$ and $\hat{\mathbb{Z}}$. Notice that we are taking this double coset procedure because of the ambiguity in A and B^{-1}. To get the identity coset would mean that A and B are the same. But they are in different rings sitting inside of $\mathbb{A}^{\text{fin}}$. The intersection of $\mathbb{Q}$ and $\hat{Z}$ in $\mathbb{A}^{\text{fin}}$ is precisely $\mathbb{Z}$, as it turns out. Hence getting the identity coset means that q and q' are equivalent over $\mathbb{Z}$. We conclude that really we want to be counting the number of such double cosets.

There are now a few changes we would like to make. First off, let's take the ring of *all* adeles, not just the finite ones, $\mathbb{A} = \mathbb{A}^{\text{fin}} \times \mathbb{R}$. But to keep things finite now we will look at double cosets

$$O_q(\mathbb{Q})\backslash O_q(\mathbb{A})/O_q(\hat{Z} \times \mathbb{R}).$$

The reason we look at all adeles is because $\mathbb{A}$ is naturally a topological ring via the usual topology on $\mathbb{R}$ and the p-adic topologies. It turns out that $\mathbb{A}$ is locally compact. Moreover $\mathbb{Q}$ obtains the discrete induced topology. All of this is to say that $O_q(\mathbb{A})$ is now a locally compact group which contains a discrete subgroup $O_q(\mathbb{Q})$ and an open compact subgroup $O_q(\hat{Z} \times \mathbb{R})$. But the theory of locally compact groups tells us that we have a Haar measure (turns out here both left and right invariant) μ. Now

$$O_q(\mathbb{Q})\backslash O_q(\mathbb{A})$$

inherits a measure and is acted on from the right by $O_q(\hat{Z} \times \mathbb{R}$ which iteslf also inherits a measure. So here's what we might naively expect: the number of orbits is

$$X = \frac{\mu(O_q(\mathbb{Q})\backslash O_q(\mathbb{A}))}{\mu(O_q(\hat{Z} \times \mathbb{R})}.$$

But this is not true on the nose because the action is not free. In fact, it turns out that the number of orbits is precisely given by the mass of q.

Remark 3. The measure μ is not defined up to a scalar, which is fine for us since μ appears both in the numerator and the denominator.

Now define

$$SO_q(R) = \{A \in GL_n(R) \mid q \circ A = q, \det A = 1\}.$$

Then it turns out that

$$\frac{\mu(SO_q(\mathbb{Q})\backslash O_q(\mathbb{A}))}{\mu(SO_q(\hat{\times}\mathbb{R}))} = 2^k X.$$

Now notice that

$$SO_q(\mathbb{A}) = SO_q(\mathbb{R}) \times \prod_p^{\text{restr}} SO_q(\mathbb{Q}_p).$$

For $SO_q(\mathbb{R})$ write $V_\mathbb{R}$ for the line of left-invariant top-degree differential forms on $SO_q(\mathbb{R})$. Recall that $SO_q(\mathbb{R})$ is in fact an algebraic group over $\mathbb{R}$ that comes from $\mathbb{Q}$ (the equations defining it only use integers). Working over $\mathbb{Q}$ one finds a subspace $V_\mathbb{Q} \subset V_\mathbb{R}$ of algebraic differential forms. This is a bit better because it's now ambiguous only up to scaling by rationals. Next let's look at $V_{\mathbb{Q}_p}$, the invariant top-forms on the p-adic analytic Lie group $SO_q(\mathbb{Q}_p)$. As before inside this space we have a one-dimensional space over $\mathbb{Q}$ which is just $V_\mathbb{Q}$. Hence $V_\mathbb{Q}$ sits inside $V_\mathbb{R}$ and $V_{\mathbb{Q}_p}$. Now $\omega \in V_\mathbb{Q}$ determines measures $\mu_{\omega,\infty}$ and $\mu_{\omega,p}$ on $SO_q(\mathbb{R})$ and $SO_q(\mathbb{Q}_p)$.

Definition 4 (Tamagawa)**.** The Tamagawa measure is

$$\mu_{\text{Tam}} = \mu_{\omega,\infty} \times \prod_p \mu_{\omega,p}.$$

Two questions: convergence and well-definedness under choice of ω. For the latter suppose that we multiply ω by -5. The measure on $SO_q(\mathbb{R})$ is multiplied by 5. The measure on $SO_q(\mathbb{Q}_p)$ on the other hand shrinks by a factor of 5 (some argument here about p-adic measure of -5). Hence everything cancels out. So now consider the expression

$$\frac{\mu_{\text{Tam}}(SO_q(\mathbb{Q})\backslash SO(\mathbb{A}))}{\mu_{\text{Tam}}(SO_q(\hat{\times}\mathbb{R}))}$$

We obtain the Tamagawa-Weil version of the mass fomrula

$$\mu_{\text{Tam}}(SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})) = 2.$$

This is true for any q with at least 3 variables. Where does this number 2 come from? It comes from the fact that SO_q has a double cover Spin_q. It turns out that

$$\mu_{\text{Tam}}\left(\text{Spin}_q(\mathbb{Q})\backslash \text{Spin}_q(\mathbb{A})\right) = 1.$$

Motivated by all of this, Weil made the following conjecture.

Conjecture 5. *Let G be a semisimple simple connected algebraic group over $\mathbb{Q}$. Then*

$$\mu_{Tam}(G(\mathbb{Q})\backslash G(\mathbb{A})) = 1.$$

The proofs (for various generalities) were given by Langlands, Lai, and Kottwitz.

The purpose of these lecture series is as follows. The second lecture will be about the analog of Weil's conjecture works in the case of function fields. In this case we will be counting principal G-bundles over algebraic curves. We will reformulate this into a statement about cohomology of a certain stack. In the third lecture we will go back to characteristic zero and relate this story with factorization homology via nonabelian Poincaré duality. Finally, in the fourth lecture, we'll give notions of how these ideas gives a proof of Weil's conjecture for function fields.

Since we'll have an exercise session later:

Exercise 6. Let $\mathcal{H} = \{x + iy \mid y > 0\}$. This is equipped with a metric of constant curvature -1. The area form has a simple formula $dxdy/y^2$. This upper half plane is acted upon by $SL_2(\mathbb{Z})$. Compute the volume of the $\mathcal{H}/SL_2(\mathbb{Z})$ in two ways. The first is to just use calculus. The other way is to use Weil's conjecture. Hint: take $G = SL_2$.

2. June 11, 2018 – Kevin Costello

fill this in

Suppose now that $M = \Sigma \times \mathbb{R}$ for Σ a Riemann surface. Then the space of solutions to the equations of motion is

$$\begin{aligned}\mathsf{Flat}_G(\Sigma) &= \{\text{flat } G\text{-bundles on } \Sigma\}\\ &= \mathrm{Hom}(\pi(\Sigma), G)/G.\end{aligned}$$

where G is a complex semisimple group. The key point is that $\mathrm{Flat}_G(\Sigma)$ is a complex symplectic space (really a derived stack). Why is this? If we take a point $A \in \mathrm{Flat}_G(\Sigma)$ then

$$T_A\mathrm{Flat}_G(\Sigma) = \{\tilde{A} \mid F(A + \varepsilon\tilde{A}) = 0\}/\text{gauge}.$$

But

$$F(A + \varepsilon\tilde{A}) = d_A\tilde{A}$$

and the gauge transformations send $\tilde{A} \mapsto d_A X$ for $X \in \Omega^0(\Sigma, \mathfrak{g})$. In other words,

$$T_A\mathrm{Flat}_G(\Sigma) = H^1(\Sigma, \mathfrak{g}_A)$$

where $\mathfrak{g}_A$ is the adjoint bundle associated to flat bundle A. This a symplectic vector space: given $A', A'' \in T_A\mathrm{Flat}_G(\Sigma)$,

$$\omega(A', A'') = \int \langle A', A''\rangle_{\mathfrak{g}}$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is the Killing form. Why is it important that this moduli space is symplectic? This probably goes back to Hamilton. If we have a variational elliptic PDE on $X \times \mathbb{R}$ (for X a manifold) then germs on X to solutions to the equations of motion are always symplectic

Classical Chern-Simons theory on $\Sigma \times \mathbb{R}$ is equal to the classical mechanical system whose field is a map $f : \mathbb{R} \to \mathrm{Flat}_G(\Sigma)$ and whose action functional is

$$\int_{\mathbb{R}} f^*\theta$$

where $d\theta = \omega$. The operators of this system are functions $\mathcal{O}(\mathrm{Flat}_G\Sigma)$. They symplectic form gives rise to a Poisson bracket. The quantum Chern-Simons theory then should give rise to a deformation quantization $\mathcal{O}_\hbar(\mathrm{Flat}_G(\Sigma))$, i.e. $a, b \in \mathcal{O}_\hbar(\mathrm{Flat}_G(\Sigma))$,

$$\frac{1}{\hbar}[a, b] \xrightarrow{\hbar\to\infty} \{a, b\}.$$

Now we want to generalize Chern-Simons theory and obtain different symplectic manifolds (that are known to experts). Let's start with four dimensions. If $M^4 = \Sigma_1 \times \Sigma_2$ is the product of two Riemann surfaces and $\omega \in H^0(\Sigma_2, K_{\Sigma_2})$ (with poles allowed) then we consider $A \in (\Omega^1(M)/\omega\Omega^0(M)) \otimes \mathfrak{g}$ as our gauge fields. The action functional we take is

$$S(A) = \int_M \omega \mathsf{CS}(A).$$

Let's try to repeat what we did for Chern-Simons theory. This 4d Chern-Simons theory will be related to Yangians and quantum affine algebras $U_q(\hat{\mathfrak{g}})$. What are the equations of motion? We compute

$$\omega \wedge \mathsf{CS}(A + \varepsilon\tilde{A}) = \langle\tilde{A}, \omega \wedge F(A)\rangle$$

whence the equations of motion are $\omega \wedge F(A) = 0$. Now $F(A)$ has components in $\Omega^2(\Sigma_1)\hat{\otimes}\Omega^0(\Sigma_2)$ and $\Omega^1(\Sigma_1)\hat{\otimes}\Omega^0(\Sigma_2)$ (there are other components as well). The equations of motions are that these two components vanish. Let's try to understand what this means more geometrically — it's obviously some sort of flatness condition. These equations mean that for all $x \in \Sigma_1$, $A|_{x\times\Sigma_2}$ is a holomorphic bundle on Σ_2. If $x, y \in \Sigma_1$ the parallel transport over a path from x to y defines an isomorphism of holomorphic bundles. This is because the second component vanishes. Moreover for all $z \in \Sigma_2$ the bundle on $\Sigma_1 \times z$ is flat.

Let us in particular take $\Sigma_1 = \mathbb{R} \times S^1$ and Σ_2 an elliptic curve. What is the moduli of solutions to the equations of motion in this case? Since the cylinder is connected (but not simply-connected) the moduli space is the set of holmorphic bundles on E with an automorphism. This is actually a pretty famous hyperKähler manifold, the space of periodic monopoles or multiplicative Higgs bundles. Notice that it is important we are working with elliptic curves so that we have nonvanishing one-forms. One computes the tangent space to be

$$T_A\mathcal{M} = H^1_{\bar{d}}(E, \mathfrak{g}_A) \oplus H^0_{\bar{d}}(E, \mathfrak{g}_A).$$

coming from varying the connection and the isomorphism. The symplectic pairing is

$$\langle \phi, \psi \rangle = \int \omega \wedge \langle \phi, \psi \rangle_{\mathfrak{g}}$$

at a point where the automorphism is the identity (otherwise it doesn't split).

The next example is that of 5d Chern-Simons theory and is very similar to the previous example. Here we take $M^5 = \mathbb{R} \times X$ where X is a complex surface with a holomorphic volume form ω. Here our gauge fields are

$$A \in \Omega^1(\mathbb{R})\hat{\otimes}\Omega^0(X) \oplus \Omega^0(\mathbb{R})\hat{\otimes}\Omega^{0,1}(X).$$

If t, z_1, z_2 are coordinates then

$$A = A_t dt + a_{\bar{z}_1} d\bar{z}_1 + A_{\bar{z}_2} d\bar{z}_2$$

and

$$S(A) = \int_M \omega \wedge \mathsf{CS}(A).$$

We find that the equations of motion say that for all $t \in \mathbb{R}$ we have a holomorphic bundle on X and that the parallel transport on $\mathbb{R}$ is an isomorphism of holomorphic bundles. The symplectic manifold is $\mathsf{Bun}_G(X)$ the space of holomorphic bundles on X. Mukai showed, in the 90's, that because X has a holomorphic volume form, $\mathsf{Bun}_G(X)$ is symplectic.

So far we've seen a few examples of action functionals and how they lead to symplectic manifolds. Let's see how they relate to quantum groups. Here is a heuristic definition of a line operator in QFT: it is a function of fields that only depends on their behavior along a line.

Example 7. For example in Chern-Simons theory given a representation R of $\mathfrak{g}_A$ and a circle $S^1 \subset M$, we can define a Wilson line

$$\operatorname{tr}_R \mathsf{hol}_{S^1}(A) = \sum_n \int_{\Delta^n} \operatorname{tr}_R(A_{t_1} \wedge \cdots \wedge A_{t_n}).$$

We claim that to give a line operator in any of these theories is to give a representation of a quantum group.

3. June 11, 2018 – Damien Calaque

See slides

4. June 12, 2018 – Jacob Lurie

Let's recall the statement of Weil's conjecture. More generally this statement should be true for number fields. Recall that a number field is a finite extension of $\mathbb{Q}$.

Conjecture 8. *Let G be a semisimple, simply connected, algebraic group over a number field K. Then*
$$\mu_{Tam}(G(k)\backslash G(\mathbb{A})) = 1.$$

Here the adeles are the tensor of $\mathbb{A}$ over $\mathbb{Q}$ and K. Recall that a function field is a finite extension of $\mathbb{F}_p(t)$ for p prime. We state the same conjecture about function fields K.

Conjecture 9. *Let G be a semisimple, simply connected, algebraic group over a function field K. Then*
$$\mu_{Tam}(G(k)\backslash G(\mathbb{A})) = 1.$$

More generally we might think of $K(X)$, for X an algebraic curve over $\mathbb{F}_q$. Let's discuss how concepts in the context of $\mathbb{Q}$ carry over to those of function fields. First of all, the prime numbers p correspond to close points $x \in X$. Meanwhile $\mathbb{F}_p$ corresponds to $\kappa(x)$, the residue field, which is a finite extension of $\mathbb{F}_q$. Moreover $\mathbb{Z}_p$ corresponds to $\mathcal{O}_x$, the completed local ring $\mathcal{O}_x \simeq \kappa(x)[\![u]\!]$ (this isomorphism is not canonical). The field of p-adic rationals has as its counterpart the local field $\kappa_x \simeq \kappa(x)(\!(u)\!)$. It is sometimes also convenient to include $\mathbb{R}$, which we think of completion at a prime at ∞. There is nothing that corresponds to this on the function field side. Last time we had $\mathbb{A} = \mathbb{R} \times \prod_p^{\text{restr}} \mathbb{Q}_p$. For function fields we have
$$\mathbb{A} = \prod_{x \in X}^{\text{restr}} \kappa_x.$$
Just as before, $\mathbb{A}$ is a locally compact topological ring. Last time we were considering quadratic forms $q/\mathbb{Q}$, which determined a group SO_q. In the function field case we will just think of any semisimple algebraic group K. Last time we considered $SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})$. In this case we look at $G(K)\backslash G(\mathbb{A})$, which comes similarly with a Tamagawa number. Moreover we had a refinement: quadratic forms over $\mathbb{Z}$. In the function field setting we will similarly have a group scheme $G \to X$, which is affine smooth and has connected fibers (probably G is not great notation for this). There are however some points of bad reduction, where the fibers are not semisimple. This is analogous to the fact that a nondegenerate quadratic form over $\mathbb{Q}$ may be degenerate over finitely many primes. The analog of $SO_q(\hat{Z} \times \mathbb{R})$ is $G(\prod_{x\in X} \mathcal{O}_x)$, which is a compact open subgroup of $G(\mathbb{A})$. Last time we discussing a certain double quotient $SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})/SO_q(\hat{Z} \times \mathbb{R})$ which corresponded to quadratic forms in the genus of q. This will correspond to
$$G(K)\backslash G(A)/G(\prod_{x\in X} \mathcal{O}_x),$$

which corresponds to principal G-bundles $P \to X$. We now want a formulation of the mass formula.

$$\sum_{\text{G-bundles}} \frac{1}{|\operatorname{Aut}(P)|} = q^d \prod_{x \in X} \frac{|\kappa(x)|^{\dim G}}{G(\kappa(x))}.$$

Notice that the sum on the left is almost never finite — there is a convergence issue. So really the same statement is that the left converges and the right converges, and they converge to the same thing.

Recall the moduli stack of G-bundles $\operatorname{Bun}_G(X)$, which satisfies that maps $Y \to \operatorname{Bun}_G(X)$ correspond to principal G-bundles on $X \times Y$. The reason we need to treat this as a stack is so that we can really understand the automorphisms of principal bundles, which appears in our formula above. Our goal is to compute the number of points of $\operatorname{Bun}_G(X)$ (defined over $\mathbb{F}_q$). Let's look at something a little bit simpler first. Let Z be a projective variety over $\mathbb{F}_q$. A question that Weil was interested in was how big is $Z(\mathbb{F}_q)$? Notice that $Z(\mathbb{F}_q) \subset Z(\bar{\mathbb{F}}_q)$. Consider the Frobenius map $\phi : Z \to Z$, which in homogeneous coordinates (inside our projective space) is given by raising all coordinates to the qth power. Notice that this is indeed a map back to Z as the qth power acts as the identity on the coefficients of the polynomials defining Z. Notice that $Z(\mathbb{F}_q)$ is the precisely the fixed points of ϕ. Weil had the insight that one should be able to use some sort of fixed point formula to compute the size of this set.

Heuristic:

$$|Z(\mathbb{F}_q)| = \sum (-1)^i \operatorname{tr}(\phi \mid H^i(Z)) = \operatorname{tr}(\phi \mid H^*(Z)).$$

If we were working over $\mathbb{C}$ and ϕ had nondegenerate fixed points then this would just be a result of Lefschetz. This heuristic, after Grothendieck, is a theorem, known as the Grothendieck-Lefschetz theorem. The cohomology on the right is the étale or ℓ-adic cohomology.

We would like to apply this heuristic to our case $\operatorname{Bun}_G(X)$. Unfortunately this is no longer a projective variety. However all we have to do is replace H^* with compactly supported cohomology H^*_c. But what if we don't like compactly support cohomology and would like to stick with usual cohomology. Well suppose that Z is smooth of dimension d. In this case, the theory of étale cohomology has some version of Poincaré duality. Recall that this should be something like

$$H^i_c(Z) \simeq H^{2d-i}(Z)^*.$$

Hence we obtain

$$|Z(\mathbb{F}_q)| = q^d \operatorname{tr}(\phi^{-1} \mid H^*(Z)).$$

The q^d comes from the fact that Poincaré duality is not actually equivariant with respect to the Frobenius. Usually the pairing is given by pairing then integration. However the Frobenius is a map of degree q^d so carries the fundamental class to q^d times the fundamental class. Let's rewrite this a bit:

$$\frac{|Z(\mathbb{F}_q)|}{q^d} = \operatorname{tr}(\phi^{-1} \mid H^*(Z)).$$

Notice that the fraction on the left we expect to be close to 1 (number of points in $\mathbb{F}_q$ times the dimension of our variety). The higher cohomology groups are serving as the corrections to this not being exactly 1.

We now want to replace Z with $\mathrm{Bun}_G(X)$. This latter object is a smooth algebraic stack of dimension d. Some work needs to be done to show that the Lefschetz trace formula holds for this stack, but it turns out that it does.

$$\mathrm{tr}(\phi^{-1} \mid H^*(\mathrm{Bun}_G(X))) = \frac{1}{q^d}\sum \frac{1}{|\operatorname{Aut}(P)|}$$

This was proved by Kai Behrend in the case when G has good reduction (and can be generalized to our case). Now it turns out that one shows

$$\prod_{x\in X}\frac{|\kappa(x)|^{\dim G}}{|G(\kappa(x))|} = \prod_{x\in X}\mathrm{tr}(\phi^{-1} \mid H^*(\mathrm{Bun}_G(\{x\}))).$$

One now might expect some cohomological equality of these quantities, which is our new statement of Weil's conjecture.

Well there is some intuition that

$$\mathrm{Bun}_G(X) = \prod_{x\in X}^{\mathrm{cont}}\mathrm{Bun}_G(\{x\})$$

that bundles on X are bundles on points that vary continuously (algebraically) in some sense. Now if we knew some statement like this one might be able to use some sort of Kunneth formula,

$$H^*\mathrm{Bun}_G(X)) \cong \bigotimes_{x\in X}^{\mathrm{cont}} H^*(\mathrm{Bun}_G(\{x\})).$$

Now we might take the trace of the (inverse of the) Frobenius on each side,

$$\mathrm{tr}(\phi^{-1} \mid H^*\mathrm{Bun}_G(X)) = \prod_{x\in X}\mathrm{tr}(\phi^{-1} \mid H^*(\mathrm{Bun}_G(\{x\}))).$$

The rest of these lectures will deal with joint work with Dennis Gaitsgory that make precise this sketch proof. We will make these ideas more precise. Can we make sense of the Kunneth decomposition? What do we mean by this continuous tensor product? This will be made using factorization homology for algebraic curves over $\mathbb{F}_q$.

Next lecture we will transition to topology, and explain a formula that is very similar to the Kunneth decomposition. The finally in lecture 4 we will give a sketch about how this adapts to the algebraic geometry.

5. June 12, 2018 – Kevin Costello

Today we will discuss factorization algebras coming from Chern-Simons theories and the relation with quantum groups. Let's think about 3d Chern-Simons for now. Let M be a 3-manifold. Recall that the equations of motion cut out $\mathrm{Loc}_G(M)$. Here's a philosophy: every quantum field theory on M gives rise to a factorization algebra on M. An open subset U is sent to the complex $\mathrm{Obs}^q(U)$, which is a cochain complex over $\mathbb{R}[\![\hbar]\!]$. Modulo $\hbar$ we just obtain $\mathrm{Obs}^{\mathrm{cl}}(U)$, which is just the functions on the space of solutions to the equations of motion on U.

For Chern-Simons theory,

$$\mathrm{Obs}^{\mathrm{cl}}(U) = \mathcal{O}(\mathrm{Loc}_G(U)).$$

For example, if we take $M = \Sigma \times \mathbb{R}$ and $U = \Sigma \times (-\varepsilon, \varepsilon)$ then

$$\mathrm{Obs}^{\mathrm{cl}}(\Sigma \times (-\varepsilon, \varepsilon)) = \mathcal{O}(\mathrm{Loc}_G(\Sigma)).$$

We expect that $\mathrm{Obs}^q(\Sigma \times (-\varepsilon, \varepsilon))$ will be an E_1-algebra, a deformation quantization of $\mathrm{Loc}_G(\Sigma)$. Moreover, for the purposes of this lecture we will work perturbatively, i.e. we will replace $\mathrm{Loc}_G(M)$ by its formal completion $\hat{\mathrm{Loc}}_G(M)$ at the trivial bundle. The idea is that formal derived stacks is really much easier to work with. Roughly speaking, formal derived stacks are the same thing as dg Lie algebras (Jacob has an ICM address about this). In our case $\hat{\mathrm{Loc}}_G(M)$ just corresponds to $\Omega^*(M) \otimes \mathfrak{g}$ and functions on it correspond to $C^*(\Omega^*(M) \otimes \mathfrak{g})$.

We expect from 3d Chern-Simons theory that there is a factorization algebra on M sending

$$U \mapsto C^*_\hbar(\Omega^*(U) \otimes \mathfrak{g}).$$

Damien explained that factorization algebras under certain assumptions are the same as E_n-algebras. This assumption holds here: we will have a locally constant factorization algebra whence we obtain an E_3-algebra. This E_3-algebra deforms $C^*(\Omega^*(M) \otimes \mathfrak{g}) \simeq C^*(\mathfrak{g})$. We want to relate this E_3-algebra to quantum groups.

For 4d Chern-Simons theory, a quantization will give a factorization algebra on $\mathbb{R}^2 \times \mathbb{C}$ which sends

$$D_1 \times D_2 \to C^*(\Omega^*(D_1) \hat{\otimes} \Omega^{0,*}(D_2)) \simeq C^*(\mathrm{Hol}(D_2) \otimes \mathfrak{g})$$

If moreover we replace D_2 by a formal disc, we obtain $C^*(\mathfrak{g}[\![z]\!])$. In this case we expect an E_2-algebra, which quantizes $C^*(\mathfrak{g}[\![z]\!])$.

In the 5d Chern-Simons case we expect an E_1-algebra which quantizes $C^*(\mathfrak{g}[\![z_1, z_2]\!])$.

We now turn to line operators. For 3d Chern-Simons theory with an E_3 algebra of observables $C^\infty_\hbar(\mathfrak{g})$.

Definition 10. The category of line operators is $\mathrm{Perf} C^*_\hbar(\mathfrak{g})$.

If you're a physicist you might say that a line operator is what you get when you change the QFT over a line. This suggests that line operators give constructible factorization algebras. Line operators can end, so we also want pictures that look like rays. The underlying cochain complex of the module that we get associated to this ray is a module M over $C^\hbar_*(\mathfrak{g})$. What type of module is this (there are many in the homotopy algebra world). It turns out it is only a left-module

$$C^*_\hbar(\mathfrak{g}) \otimes M \to M$$

(for some reason that I missed). So physics suggests that line operators in a 3d TFT are an E_2 braided monoidal category. Jacob shows in Higher Algebra that perfect modules for an E_3-algebra is an E_2-category. Where does this E_2-structure come from? Given two different rays (in our three dimensional manifold), we can collide the lines (or as Damien said collapse the interval between the lines). The fact that there's two dimensions is giving us the E_2-structure.

There's another place where one finds braided monoidal categories: representations of $U_\hbar\mathfrak{g}$. Recall that $\mathsf{Rep}(U\mathfrak{g}) \simeq \mathsf{Perf}(C^*\mathfrak{g})$ sending $R \mapsto C^*(\mathfrak{g}, R)$.

Theorem 11 (Costello, Francis, Gwilliam)**.** *There is an equivalence of E_2-categories*

$$\mathsf{Perf}(C^*_\hbar(\mathfrak{g})) \to \mathsf{Rep}(U_\hbar(\mathfrak{g}))$$

possibly after a reparameterization $\hbar \mapsto \hbar + c\hbar^2 + \cdots$.

Let's now look at the 4d case. Quantizing 4d Chern-Simons theory gives rise to an E_2-algebra $C^*_\hbar(\mathfrak{g}[\![z]\!])$ is an E_1-category, which deforms representations of $U(\mathfrak{g}[\![z]\!])$. The Yangian $Y_\hbar(\mathfrak{g})$ is a Hopf algebra which deforms $U(\mathfrak{g}[\![z]\!])$.

Theorem 12 (Costello, Costello-Witten-Yamazaki)**.** *There is an equivalence of monoidal categories*

$$\mathsf{Perf}(C^*_\hbar(\mathfrak{g}[\![z]\!])) \xrightarrow{\sim} Y_\hbar(\mathfrak{g}) - \mathsf{mod}.$$

On both sides there is an extra structure that we haven't used. The Yangian has a universal R-matrix, and on the left we haven't used the extra complex direction.

More explicitly, $Y_\hbar(\mathfrak{g}) - \mathsf{mod}$ has extra structure. If λ is in the formal disc then there is a translation function T_λ on this category which quantizes $z \mapsto z + \lambda$. If λ is in the formal punctured disc, and $V, W \in Y_\hbar(\mathfrak{g})$-$\mathsf{mod}$, then there exists an isomorphism known as the R-matrix,

$$R(\lambda) : V \otimes T_\lambda W \xrightarrow{\sim} T_\lambda W \otimes V.$$

Let's see where this comes from in field theory. Given a factorization algebra on $\mathbb{R}^2 \times \mathbb{C}$ and a line defect at $z = 0$ and a line defect at $z = \lambda$, we can move them past each other without touching (we are in 4 dimensions).

Theorem 13. *These structures match each other.*

Now let's look at 5 dimensions. There is a technical caveat. Working on $\mathbb{R} \times \mathbb{C}^2$ we must have the $\mathbb{C}^2$ be noncommutative (and $\mathfrak{g} = \mathfrak{gl}_n$). In this case $C^*_\hbar(\mathfrak{gl}_n[\![z_1, z_2]\!])$ is an E_1-algebra

Theorem 14 (Costello, Yaping Yang, N Guang)**.** *There is an equivalence of categories*

$$\mathsf{Perf}(C^*_\hbar(\mathfrak{gl}_n[\![z_1, z_2]\!])) \xrightarrow{\sim} \mathsf{Rep}(U_\hbar \mathfrak{gl}_n[\![z_1, z_2]\!])$$

where on the right we have the deformed double current algebra.

If you like the ddca is a rational limit of the affine Yangian and plays a role in geometric representation theory. In this case one also should expect some extra structure, which should look something like the following. Denote $\mathcal{C} = \mathsf{Perf}(C^*_\hbar(\mathfrak{gl}_n[\![z]\!]))$, which is just a category. For each $V, W \in \mathcal{C}$ and λ in the formal disc we should get $T_\lambda : \mathcal{C} \to \mathcal{C}$. For all λ in the formal punctured disc we can tensor V and $T_\lambda W$. Hopefully one should be able to write this down more classically.

To summarize, 3d Chern-Simons yields an E_2-category, 4d Chern-Simons yields an E_1 chiral category, and 5d Chern-Simons theory is a chiral chiral category.

How to compute these things explicitly? One thing we have been using implicitly is the notion of Koszul duality. If V is a vector space, then the symmetric algebra S^*V is Koszul dual to the exterior algebra $\Lambda^* V^*$. If x_i is a basis of V and ε_i of V^* then $\mathbb{C}[x_i]$ is Koszul dual to $\mathbb{C}[\varepsilon^i]$ for $|\varepsilon^i| = 1$. One can make deformations to both sides. For instance $\mathbb{C}[x_i]$ can deform if we take

$$[x_i, x_j] = (\pi^k_{ij} x_k)\delta$$

for δ some parameter. This is dual to introducing a differential on the dual side

$$d\varepsilon^k = (\pi^k_{ij} \varepsilon_i \varepsilon_j)\delta.$$

The π_{ij}^k are structure constants of $\mathfrak{g}$. Then with a bit of care we conclude that $U\mathfrak{g}$ is Koszul dual to $C^*\mathfrak{g}$. We want to then quantize

$$\begin{array}{ccc} C^*(\mathfrak{g}[\![z]\!]) & \xrightarrow{KD} & U(\mathfrak{g}[\![z]\!]) \\ \downarrow & & \downarrow \\ C^*_\hbar(\mathfrak{g}[\![z]\!]) & \xrightarrow{KD} & Y(\mathfrak{g}) \end{array}$$

6. June 12, 2018 – Damien Calaque

Today we will discuss vertex models and E_n-algebras. We will start with two conjectures of Kontsevich.

The first will be a bit vague: any sensical definition of quantum field theory involves some infinite dimensional bundle of fields $\mathcal{F} \to X$.

Conjecture 15 (Conjecture A (Kontsevich)). *Given a quantum field theory on $\mathbb{R}^d$ which has translation and dilation symmetries, then translation invariant forms with values in $\mathcal{F}$ obtains an action of E_d. In other words we get an E_d-algebra in the category of complexes of $\mathbb{R}$-vector spaces.*

Remark 16. If A is an E_d-algebra in $\mathsf{Cpx}_{\mathbb{R}}$ then $A[d-1]$ is equipped with an L_∞-algebra structure. Presumably the higher brackets in this structure are related to renormalization and related combinatorial issues.

Remark 17. Since the statement of the conjecture there have been many contexts in which this conjecture has been turned into a theorem. See papers of Costello, Costello-Gwilliam, and in physics, Hollands, ... In most of these results we have similar statements for QFTs that are close (formally perturbed from) free theories. If you wish, there is a deformation theoretic interpretation: pick a QFT and look at the formal deformation problem of this theory. This deformation problem is governed by a dg Lie algebra and moreover if we consider the deformations along translation/dilation invariant theories then we get a deformation problem for E_n-algebras.

The next conjecture is a discretized version of the above which has not been published (but was given by Kontsevich in some talks). We will take, for clarity, dimension $d = 2$. In particular we consider a infinite square lattice in $\mathbb{R}^2$ that is oriented from left to right and bottom to top. Let H and V (for horizontal and vertical) be two vector spaces. We interpret

$$H = k\langle e_i \mid i \in I\rangle \qquad V = k\langle f_j \mid j \in J\rangle.$$

We have $R(e_i \otimes f_j) = R_{ij}^{kl} e_k \otimes f_l$. Here R_{ij}^{kl} is the probability to have a configuration where clockwise from top we have e_i, f_l, e_k, f_j. We now play the following game. We count the configurations in finite regions with these weights R_{ij}^{kl} and with prescribed boundary conditions. This is another way of saying that we are going to be doing "state sums."

We now define the space of boundary satates of a given region. Let U be bounded open convex region in $\mathbb{R}^2$. We write $\partial_+^h U$ for horizontal incoming boundary edges, i.e. horizontal edges e such that $e \cap U \neq \varnothing$ but the source $s(e) \notin U$. Similarly we have $\partial_+^v U$, $\partial_-^h U$, and $\partial_-^v U$. The space of boundary states is defined

$$W(U) = \mathrm{End}(H^{\otimes \partial_+^h U} \otimes V^{\otimes \partial_+^v U})$$

since we have a canonical bijection $\partial_+^{h,v} U \simeq \partial_-^{h,v} U$. The partition function, now is just an element of $W(U)$:

$$Z_U : k \to W(U).$$

given just by matrix multiplication at each lattice point in a certain order (left to right and bottom to top). We think of $k \to W(U)$ as coming from the inclusion $\varnothing \hookrightarrow U$. What happens if instead we are looking at an inclusion $U' \subset U$? One obtains a map

$$Z_{U',U} : W(U') \to W(U)$$

where we send an endomorphism A to A precomposed by the matrices R at each lattice point in U not in U'.

We now define

$$W(\mathbb{R}^2) = \operatorname*{colim}_U W(U),$$

which has a natural action of $\mathbb{Z}^2$. For example $W(U)^{(1,0)} \to W(U + (1,0))$. We now come to the second conjecture.

Conjecture 18 (Upside-down renormalization (Kontsevich))**.** *The complex giving us group homology valued in* $W(\mathbb{R}^2)$*,*

$$C_{-*}(\mathbb{Z}^2, W(\mathbb{R}^2))$$

is acted on by E_2*.*

Roughly speaking here are taking derived coinvariants.

Remark 19. Actually W can be turned into a (pre)factorization algebra.

Theorem 20 (Calaque-Lejay)**.** *Conjecture B is true.*

Consider the (2,1)-category of discretized disks $\mathbb{D}_2$. Objects here are finite disjoint unions of disks in $\mathbb{R}^2$. 1-morphisms are generated by two things, inclusions and translations (with some condition that I missed). There are 2-morphisms generated by recognizing that translating up then right is the same as translating right then up. On this category there is an E_2-monoidal structure. We use the fact (due to Dunn and Lurie) that a E_2-algebra is just an E_1-algebra in E_1-algebras. Indeed, consider the two partial orders on objects $A <_h B$ if A is to the left of B and $A <_v B$ if A is below B. Now we define

$$A \otimes_h B = A \sqcup (B + (m, 0))$$

where m is minimal such that $A <_h B + (m, 0)$. Similarly

$$A \otimes_v B = A \sqcup (B + (0, n)).$$

One checks that these are strict monoidal products. However they are not strictly compatible. A consequence is that $\mathrm{ho}(\mathbb{D}_2)$ is braided monoidal. The braiding is given by moving two blocks past each other (and then possibly retranslating).

Now let me give you a trick to construct E_n-algebras using E_n-monoidal functors. Take $\mathcal{C}, \mathcal{D}$ to be E_n-monoidal $(\infty, 1)$-categories. Given an E_n-monoidal functor $F : \mathcal{C} \to \mathcal{D}$ then the colimit of F is an E_n-algebra in $\mathcal{D}$. The colimit can be written as a left Kan extension along the functor $\mathcal{C} \to *$. It makes sense to do this operadically i.e. for everything E_n-monoidal.

Example 21. Take $\mathcal{C} = \mathbb{D}_2$ and let $\mathcal{D} = \mathsf{Cpx}_k$ and any E_2-monoidal functor $\mathbb{D}_2 \to \mathsf{Cpx}_k$. But the map $\mathbb{D}_2 \to *$ factors through the category which has one object and $\mathbb{Z}^2$ as automorphisms, i.e. $B\mathbb{Z}^2$. This functor forgets about the two-cell and sends paths to the pair of integers telling us the overall translation. The Kan extension along the functor $\mathbb{D}_2 \to B\mathbb{Z}^2$ yields complexes which will be the colimit over all U's of $F(U)$. Further Kan extension to the point will compute the coinvariants,

$$C_{-*}(\mathbb{Z}^2, \operatorname*{colim}_U F(U)).$$

So now we just need to choose the functor F appropriately. But W defines a braided monoidal functor

$$\mathbb{D}_2 \to \mathrm{ho}\mathbb{D}_2 \xrightarrow{W} \mathsf{Vect} \to \mathsf{Cpx}_k$$

Hence we see that

$$C_{-*}(\mathbb{Z}^2, W(\mathbb{R}^2))$$

has the structure of an E_2-algebra.

Let us conclude with a few remarks.

Remark 22. When $n \geq 3$ then the E_n-algebra structure turns out to be an E_∞-algebra structure. This is because the map from the homotopy category of $\mathbb{D}_2$ is symmetric monoidal and the colimit is taken over a symmetric monoidal functor. It would be nice to see explicit models (say from statistical mechanics) that are E_3 but not E_∞.

Remark 23. Assume that $H = V$ and $R \in GL(V^2)$ satisfies the Yang-Baxter equation. This tells us that we have certain commuting matrices. What does this tell us about the E_2-algebra structure?

Remark 24. There are other types of models where the interactions are not at the vertices but on the faces. Is there something interesting to say for this class of models?

7. June 13, 2018 – Lauren Bandklayder

Theorem 25 (Dold-Thom '58)**.** *For a based space $(X, *)$ and an abelian group (A, e) then there are isomorphisms*

$$\pi_*(\mathrm{Sym}(X; A)) \cong \tilde{H}_*(X; A).$$

The original proof roughly checks that $\pi_*(\mathrm{Sym}(-; A))$ satisfies the Eilenberg-Steenrod axioms. Our goal today is to outline a direct geometric proof.

Definition 26. For $(X, *)$ and (A, e) as above, define the infinite symmetric product of X with coefficients in A

$$\mathrm{Sym}(X; A) = \{(S, l) \mid * \in S \subset X \text{ finite}, l : (S - *) \to A\}/\sim$$

where the relation is given by declaring two configurations to be equivalent if they differ by a point marked by e.

The topology on this space is such that essentially points can either move around noninteracting, or they can collide, or they can get absorbed into the basepoint.

Two properties to remember about the infinite symmetric product is that (as a functor from based spaces to based spaces) it is homotopy invariant and preserves open embeddings.

Example 27. If I is a finite set then there is a homeomorphism $\mathrm{Sym}(I_+; A) \cong A^I$.

Example 28. For the sphere S^n Dold-Thom implies that

$$\mathrm{Sym}(S^n; A) \simeq K(A; n).$$

Let Mfld_n be the category of n-manifolds with open ends and Disk_n is the full subcategory with objects $\sqcup^k \mathbb{R}^n$. Consider now $\mathsf{Disk}_{*/M}$ be the full subcategory of Disk_M whose objects are embeddings $\sqcup^k \mathbb{R}^n \hookrightarrow M$ such that $* \subset \mathrm{im}(e)$.

Let's outline a three bullet point proof and then go back and elaborate.

(1) Identify
$$\mathrm{Sym}(M; A) \simeq \mathrm{hocolim}_{\mathsf{Disk}_{*/M}} \mathrm{Sym}(-; A);$$
(2) Move to the category $\mathsf{Ch}^{\geq 0}$;
(3) Show that the resulting chain complex is singular chains.

Theorem 29 (Dugger-Isaksen)**.** *For $\mathcal{U} = \{U_i\}_I$ a complete cover of X then there is an equivalence $X \simeq \mathrm{hocolim}\, U_i$. We say that a cover is complete if every finite intersection of elements of this cover admits a cover by $\{U_j\}_J$.*

We want to show now that $\{\mathrm{Sym}(U; A)\}_{U \in \mathsf{Disk}_{*/M}}$ is a complete cover of $\mathrm{Sym}(M; A)$. That it's a cover is clear: use Hausdorffness of manifolds to construct disjoint opens around each configuration of points. For completeness, I missed this argument.

Let's now move to chain complexes.

$$\begin{aligned} \mathrm{Sym}(M; A) &\simeq \mathrm{hocolim}_{\mathsf{Disk}_{*/M}} \mathrm{Sym}(-; A) \\ &\simeq \mathrm{hocolim}_{\mathsf{Disk}_{*/M}} \tilde{H}_0(-; A) \end{aligned}$$

Now $\tilde{H}_0(-; A) : \mathsf{Top} \to \mathsf{Top}$ factors through the Dold-Kan equivalence $\mathsf{Ch}^{\geq 0} \to \mathsf{sAb}$. We want to pass the homotopy colimit to this level of chain complexes. Of course the homotopy colimit passes through the geometric realization and it remains to check that it commutes with the forgetful functor. If the forgetful functor were a sifted colimit we would be done. It's not unfortunately.

However we can use (homotopy) finality of

$$\mathsf{Disk}_{*/M} \to \mathcal{D}\mathsf{isk}_{*/M}$$

where the right is a quasicategorical refinement (inverting isotopy equivalences). This result is proved by Ayala-Francis and Lurie. By finality it is enough to compute homotopy colimits over this quasicategory, and now the result follows.

Finally

$$\mathrm{hocolim}_{\mathsf{Disk}_{*/M}} \tilde{H}_0(-; A) \simeq C_*(M; A).$$

8. June 13, 2018 – Brian Williams

See slides

9. June 13, 2018 – Tian Lan

See slides

10. June 13, 2018 – Ricardo Campos

See slides

11. June 14, 2018 – Jacob Lurie

Let Σ be an algebraic curve over $\mathbb{C}$ and let G be a nice group scheme over Σ. Then we can talk about the moduli stack of algebraic G-bundles, $\mathrm{Bun}_G(\Sigma)$. We would like to understand the cohomology of this moduli stack. For simplicity let us take G to be constant, i.e. a complex semisimple Lie group. In this case we have an analytification $\mathrm{Bun}_G(\Sigma)^{\mathrm{an}}$, which we call the moduli stack of holomorphic G-bundles. Recall that holomorphic G-bundles can be identified with smooth G-bundles equipped with a $\bar{\partial}$-operator. The space of such $\bar{\partial}$-connections is contractible whence $\mathrm{Bun}_G(\Sigma)$ is homotopy equivalent this moduli stack of smooth G-bundles with $\bar{\partial}$-operators.

Now let me remind you that when we have a group G the classifying space is written $BG = EG/G$ where EG is some contractible space with a free action. Recall that BG classifies G-bundles. Once you've done the work in defining $\mathrm{Bun}_G(\Sigma)^{\mathrm{an}}$ it turns out that

$$\mathrm{Bun}_G(\Sigma)^{\mathrm{an}} \approx \mathrm{Maps}(\Sigma, BG).$$

Recall that π_0 of the right classifies isomorphism classes of G-bundles.

Suppose G were a discrete abelian group. In that case $\pi_0 \mathrm{Maps}(\Sigma, BG)$ is also known as $H^1(\Sigma, G)$. Now by Poincaré duality we should have

$$H^1(\Sigma, G) \cong H_1(\Sigma, G).$$

What we would like to talk about is the analog for this statement where G is no longer required to be discrete abelian.

Now recall that if M is a compact oriented d-manifold then Poincaré duality tells us

$$H^*(M; A) \simeq H_{d-*}(M; A)$$

What would happen if we applied this to the simplest d-manifold, $\mathbb{R}^d$? Well the compact-supports version of Poincaré duality tells us the following: on the right we have 0 except in degree 0 and A in degree 0. This tells us that the compactly supported cohomology of $\mathbb{R}^d$ is zero away from degree d where it is A.

Now let's suppose we had a nonabelian Poincaré duality for $\mathbb{R}^d$. Let's try to prove it for all manifolds. Let $\mathcal{U}(M)$ be the category of open subsets of M. Fix $U \in \mathcal{U}$ and notice (omitting A from the notation from now on) that $C_*(-)$ and $C_c^*(-)$ yield functors from $\mathcal{U}(M)$ to the category of chain complexes.

Proposition 30. *These functors $C_*(-)$ and $C_c^*(-)$ are homotopy cosheaves on M with values in chain complexes.*

Let's not get into the details here, but recall that the idea of a (co)sheaf is that the functor is determined on a bigger open by what it does to small opens inside of it. For instance say $U, V \subset M$. Then

$$\begin{array}{ccc} C_*(U \cap V) & \longrightarrow & C_*(U) \\ \downarrow & & \downarrow \\ C_*(V) & \longrightarrow & C_*(U \cap V) \end{array}$$

is homotopy pushout square. Even if you're not familiar with exactly what this means, morally the point is that this square yields a familiar Mayer-Vietoris long exact sequence. This is essentially the content of the proposition above.

If we accept that these two functors are homotopy cosheaves then we can prove that Poincaré duality holds for all manifolds. Let $\mathcal{U}_0(M)$ be the open disks in M. The proposition above implies that

$$\operatorname{hocolim}_{U\in\mathcal{U}_0(M)} C_*(U) \xrightarrow{\sim} C_*(M)$$

is a quasiisomorphism of chain complexes. Similarly

$$\operatorname{hocolim}_{U\in\mathcal{U}_0(M)} C^*_c(U) \xrightarrow{\sim} C^*_c(M).$$

Now we have

$$\begin{array}{ccc} \operatorname{hocolim}_{U\in\mathcal{U}_0(M)} C_*(U) & \longrightarrow & C_*(M) \\ \downarrow & & \downarrow \\ \operatorname{hocolim}_{U\in\mathcal{U}_0(M)} C^*_c(U) & \longrightarrow & C^*_c(M). \end{array}$$

with all arrows quasi-isomorphisms and the vertical quasiisomorphism can be made functorial (this is the data of an orientation.

Now let's look at the nonabelian case. Recall that $H^m(M,A)$ can be identified with the homotopy classes of maps from M to $K(A,m)$, where the latter is an Eilenberg-Maclane space. In other words, cohomology is about mapping into these Eilenberg-Maclane spaces. Nonabelian cohomology should be something where we replace A in $K(A,m)$ with a nonabliean group. Indeed if $m=1$ then $K(A,1)$ is actually just BA. Now let G be a nonabelian discrete group. We might define

$$H^1(M,G)=[M,BG].$$

We might also define it as isomorphism classes of G-bundles on X. If M is connected and we've chosen a basepoint then this is the same as group homomorphisms $\pi_1(M)\to G$ up to conjugacy in G. However this is not general enough for our applications where we need to work with nondiscrete groups.

Let us therefore introduce a much more general notion of nonabelian cohomology. Let X be any space. Then

$$H(M;X):=[M,X].$$

This is some absurdly general notion one can always define. Let's make a table of analogies. First our coefficient systems: we have an abelian group A and a degree m, which is replaced by X any space (which reduce to $X=K(A,m)$). Now the cohomology $H^m(M,A)$ is replaced by $[M,X]$. Similarly $H^m_c(M;A)$ corresponds to (after fixing a basepoint in X) $[M,X]_c$. This latter symbol is the homotopy classes of maps which carry everything to the basepoint away from a compact set in M (homotopies must satisfy the same thing). Now on the left we had $C^*_c(M;A)$ which is now replaced by the space $\operatorname{Maps}_c(M,X)$. But what about the analog of $C_*(M;A)$?

Let's try to apply our proof above on abelian Poincaré duality. The first part was some sort of local computation, $M=\mathbb{R}^d$. Consider the compactly supported maps $\operatorname{Maps}_c(\mathbb{R}^d,X)$. But we're doing homotopy theory now so we only care about the homotopy type of this space whence we identify

$$\operatorname*{Maps}_c(\mathbb{R}^d,X)\simeq \operatorname{Maps}((D^d,\partial D^d),(X,*))\simeq\Omega^d X.$$

This is the analog of the local computation we did earlier. Now we want to globalize. Consider the covariant functor $(U\in\mathcal{U}(M))\mapsto \operatorname{Maps}_c(U,X)$. Is this a homotopy cosheaf? It turns out that it's not! Suppose we have two open sets $U, V \subset M$. Then we have

$$\begin{array}{ccc} \mathrm{Maps}_c(U \cap V, X) & \longrightarrow & \mathrm{Maps}_c(U, X) \\ \downarrow & & \downarrow \\ \mathrm{Maps}_c(V, X) & \longrightarrow & \mathrm{Maps}_c(U \cup V, X) \end{array}$$

Is this a homotopy pushout? Well one consequence is that a compactly supported map defined on $U \cup V$ would have to be completely defined on either U or V. Why did it work in the abelian case and not here? Well here we have a diagram of topological spaces. The essential content was that we had a Mayer-Vietoris sequence

$$H_*U \oplus H_*V \to H_*(U \cup V) \to H_{*-1}(U \cap V)$$

The requirement imposed in this topological setting is much stronger than it was in the chain complex setting — the problem is that in the nonabelian case we can't add maps.

We can't add maps in general, but if U and V are disjoint then $\mathrm{Maps}_c(U \cup V, X) \simeq \mathrm{Maps}_c(U, X) \times \mathrm{Maps}_c(V, X)$. We might now attempt to salvage the above paragraph using this partially defined addition. Let's suppose we could. The canonical map

$$\mathrm{hocolim}_{U \in \mathcal{U}_0(M)} \underset{c}{\mathrm{Maps}}(U, X) \to \underset{c}{\mathrm{Maps}}(M, X)$$

would be a homotopy equivalence. Unfortunately, it's not generally a homotopy equivalence. Let's use this partial definition. Consider $\mathcal{U}_1(M)$ which is the category of finite disjoint unions of disks in M. Now instead we consider

$$\mathrm{hocolim}_{U \in \mathcal{U}_1(M)} \underset{c}{\mathrm{Maps}}(U, X) \to \underset{c}{\mathrm{Maps}}(M, x).$$

This is often a homotopy equivalence!

Theorem 31 (Nonabelian Poincaré duality)**.** *Let M be a manifold of dimension d and $(X, *)$ be $(d-1)$ connected. Then*

$$\mathrm{hocolim}_{U \in \mathcal{U}_1(M)} \underset{c}{\mathrm{Maps}}(U, X) \xrightarrow{\sim} \underset{c}{\mathrm{Maps}}(M, x).$$

is an equivalence.

Example 32. Let $X = K(A, d)$. In this case take π_* of both sides. If M is orientable then the homotopy colimit appeared in Lauren's talk yesterday and is the infinite symmetric product whose homotopy groups compute homology. On the right we have by representability the compactly supported cohomology.

Remark 33. The construction $(U \in \mathcal{U}_0(M)) \to \mathrm{Maps}_c(U, X)$ is the example of a factorization algebra on M taking values in topological spaces. Notice that $\mathrm{Maps}_c(U, X) \simeq \Omega^d X$ which is naturally an algebra over the E_d-operad. Say we have a disjoint union $U_1 \sqcup U_2$. Then we have maps

$$\prod_i \underset{c}{\mathrm{Maps}}(U_i, X) \simeq \underset{c}{\mathrm{Maps}}(U_1 \sqcup U_2, X) \to \underset{c}{\mathrm{Maps}}(U, X)$$

which are precisely giving this E_d-structure. Taking the homotopy colimit is computing the factorization homology of this primordial factorization algebra.

Why is this useful? Well $\text{Maps}_c(M, X)$ is something that is potentially quite complicated. This theorem is telling us that when X is sufficiently connected we can write this space as the homotopy colimit of a particular diagram. The category indexing this colimit has nothing to do with X and the image of the diagram has nothing to do with M, so one is effectively separating out the complicated interaction occuring in the mapping space. In the case when you're just computing something crude like an Euler characteristic it is often useful just to even know that this diagram exists.

Note that the hypothesis of connectivity is satisfied when $M = \Sigma$ a Riemann surface ($d = 2$) and $X = BG$ for G a complex semisimple Lie group. We were interested in a certain trace in the case where we were over $\mathbb{F}_q$. It turns out that one can ignore the fine details of the above in this case. Of course, there is the complication that we want to work over $\mathbb{F}_q$ and not $\mathbb{C}$. It is not at all obvious how we should pass from the above to the algebraic case. For instance, over a general field in the Zariski topology we don't really have things like open discs.

I want to close this lecture by formulating this theorem differently. Let M be a manifold. Then define $\text{Ran}(M)$ to be the set of nonempty finite subsets $S \subset M$. Let's describe a basis for this topology: write $\text{Ran}(U_1, \ldots, U_n)$ to be the set of nonempty finite subsets $S \subset M$, with $S \subset \cup U_i$ with each $S \cap U_i \neq \varnothing$ where the U_i are pairwise disjoint open sets. In fact we still get a basis if we take the U_i to be pairwise disjoint connected open sets. We call such open sets "special open sets" in $\text{Ran}(M)$. We define a functor $\mathcal{F}$ from the special open sets in $\text{Ran}(M)$ to Spaces sending

$$\text{Ran}(U_1, \ldots, U_n) \mapsto \underset{c}{\text{Maps}}(U_1 \cap \cdots \cap U_n, X).$$

Theorem 34 (Nonabelian Poincaré duality version 2)**.** *$\mathcal{F}$ is a homotopy cosheaf on* $\text{Ran}(M)$*.*

Assume that M is connected. Then

$$\text{hocolim}\, \mathcal{F}(\text{Ran}(U_1, \ldots, U_n)) \xrightarrow{\sim} \mathcal{F}(\text{Ran}(M))$$

where the homotopy colimt is over pairwise disjoint connected open sets. It is relatively easy to see that these two statements of the theorem are equivalent. The point, however, is that this second statement will be easier to apply in the algebraic geometric setting.

12. June 14, 2018 – Kevin Costello

Last time we were talking about Koszul duality and quantum groups. The classic instance of duality is

$$C^*(\mathfrak{g}[[z]]) \iff U(\mathfrak{g}[[z]]).$$

We claimed that the deformations coming from 4d Chern-Simons theory are related in the same way

$$C^*_\hbar(\mathfrak{g}[[z]]) \iff Y(\mathfrak{g}[[z]]) = U_\hbar(\mathfrak{g}[[z]]).$$

Similarly for the 5d theory,

$$C^*_\hbar(\mathfrak{gl}_n[[z_1, z_2]]) \iff U_\hbar(\mathfrak{gl}_n[[z_1, z_2]]).$$

Before we get to the commutation relations, etc. let's talk a bit about Koszul duality.

Let V be a vector space with basis x_i. Then we have Koszul duality between

$$S^*B \iff \Lambda^* V^*$$

i.e. $\mathbb{C}[x_i] \iff \mathbb{C}[\varepsilon^i]$ where ε^i is a basis for $V*$. One feature of Koszul duality is that Hochschild cohomology is isomorphic in a canonical way. This tells us that deformations of $\mathbb{C}[x_i]$ that preserve the maximal ideal generated by the x_i correspond to deformations of $\mathbb{C}[\varepsilon^i]$ as an A_∞-algebra. Recall that first order deformations of a commutative algebra to noncommutative are given by Poisson brackets.

Suppose we introduce a Poisson bracket $\{x_i, x_j\} = \pi_{ij}$. This is not allowed because it does not preserve the ideal. Something that would preserve the ideal would be $\{x_i, x_j\} = \pi^*_{ij} x_k$. This would correspond to introducing a differential on the other side $d\varepsilon^k = \pi^k_{ij}\varepsilon^i\varepsilon^j$. Next suppose we have $\{x_i, x_j\} = \pi^{kl}_{ij} x_k x_l$ corresponds to deforming the multiplication on the right, $m_2(\varepsilon^k, \varepsilon^l) = \pi^k_{ij}(\varepsilon_i, \varepsilon_j)$. Similarly $\{x_i, x_j\} = \pi^{klm}_{ij} x_k x_l x_m$ corresponds to $m_3(\varepsilon^k, \varepsilon^l, \varepsilon^m) = \pi^{klm}_{ij}\varepsilon^i\varepsilon^j$. Let's look at the very first Poisson bracket. Then π^k_{ij} are structure constants of a Lie algebra $\mathrm{Sym}^* \mathfrak{g}$. We are deforming $S^*\mathfrak{g} \to U\mathfrak{g}$. Dually $\Lambda^*\mathfrak{g}^* \to C^*\mathfrak{g}$.

Let's now look at what the commutators look like. Fix $\mathfrak{g} = \mathfrak{gl}_n$ and a basis E^i_j of elementary matrices. Then $Y_\hbar(\mathfrak{gl}_n)$ is generated by $E^i_j z^n$ with commutation relations

$$[E^i_j z^n, E^k_l z^m] = \delta^i_l E^k_j z^{n+m} - \delta^k_j E^i_l z^{n+m} + O(\hbar^2).$$

It turns out that $E^i_j, E^i_j z, z^n$ id generate the algebra. It suffices to check, then, what the commutation relations of these generators are. The formula is the following. The $[E^i_j, -]$ always acts by the adjoint representation. The interesting one is

$$[E^i_j z, E^k_l z] = \text{usual commutator} + \hbar^2(E^i_m E^m_l E^k_j - E^k_m E^m_j E^i_l).$$

Now what happens in what arises in the 5d Chern-Simons theory where we have $U(\mathfrak{gl}_n[[z_1, z_2]])$. Modulo $\hbar$, this is generated by $E^i_j z_1^n z_2^m$ which satisfiy

$$[E^i_j z_1^n z_2^m, E^k_l z_1^r z_2^s] = E^i_l \delta^k_j z_1^{n+1} z_2^{m+2} + O(\hbar).$$

Everything is quite similar to before — I only need to give you one new commutator

$$[E^i_j z_1, e^k_l z_2] = \text{usual commutator} + \hbar E^i_l E^k_j.$$

So what happens on the Koszul dual side? Well $C^*(\mathfrak{g}[[z]])$ is generated by linear functionals sending $X \in \mathfrak{gl}_n[[z]] \mapsto \mathrm{tr}(E^i_j \partial^n X)(0)$. In the Yangian the quantum correction took 2 generators and spit out something with 3 generators. Here on the Koszul dual side we're going to take 3 generators and spit out 2 generators, given by an m_3. In $C^*_\hbar(\mathfrak{g}[[z]])$ we should have

$$m_3(E^i_j, E^k_l, E^r_s) = \delta^i_s (E^k_j \partial)(E^r_l \partial) + \text{permutations}.$$

Similarly in $C^*_\hbar(\mathfrak{gl}_n[[z_1, z_2]])$ we must have a new m_2. The formula is

$$m_2(E^i_j, E^k_l) = (E^i_l \partial_1)(E^k_j \partial_2) + (E^k_j \partial_1)(E^i_l \partial_2)$$

where $E^i_j \partial_1^n \partial_2^m$ is a basis for dual of $\mathfrak{gl}_n[[z_1, z_2]]$. One expects to obtain these formulas by studying factorization algebras in the field theories we were discussing in the previous lectures. It turns out that in the 3d case there aren't any interesting deformations arising in this way (this can be done by looking at the degree 2 Hochschild cohomology).

How to compute? The 4d version is a bit tricky to compute with it turns out, so let's look at the 5d version. We'll use a Feynman diagram formula for the first order deformation of $C^*(\mathfrak{g}[[z_1, z_2]])$. To understand this we'll have to say a little bit about how Feynman diagrams work in this theory. The first thing we need to describe is the propogator. Fix a 2-form $P_0 \in \bar{\Omega}^*(\mathbb{R}\times\mathbb{C}^2)/\langle dz_1, dz_2\rangle$ (here we might be taking distributional coefficients) such that

$$dz_1 dz_2 dP_0 = \delta_0.$$

Let $c = E^i_j \otimes E^i_j \in \mathfrak{gl}_n \otimes \mathfrak{gl}_n$ be the quadratic Casimir. Let $\pi : (\mathbb{R}\times\mathbb{C}^2)^2 \to \mathbb{R}\times\mathbb{C}^2$ sending $(v, v') \mapsto (v - v')$. Write

$$P = \pi^* P_0 \otimes c \in \Omega^*(\mathbb{R}\times\mathbb{C}^2 \times \mathbb{R}\times\mathbb{C}^2)/\langle dz_i, dz'_i\rangle \otimes \mathfrak{gl}_n \otimes \mathfrak{gl}_n$$

Here are the Feynman rules for this theory. Given a graph, at external lines we place a field $\alpha \in \Omega^*(\mathbb{R}^\times\mathbb{C}^2)/\langle dz_1\rangle \otimes \mathfrak{gl_n}[i]$ and at vertices place $\int_{\mathbb{R}\times\mathbb{C}^2} \operatorname{tr}(\alpha_1\alpha_2\alpha_3) dz_1 dz_2$. On edges we place a propogator. These are the rules for when we're computing say a partition function. What about when we insert observables? We have certain observables

$$\alpha \mapsto \operatorname{tr}(E^i_j \partial_1^k \partial_2^l \alpha)(0).$$

These observables at the classical level are quasi-isomorphic to $C^*(\mathfrak{gl}_n[[z_1, z_2]])$ These are the one-cochains which generate. So what we need to understand how commutators deform when we quantize. Let $\mathcal{O}^i_j(\varepsilon, z_1, z_2)$ be the observable sending

$$\alpha \mapsto \operatorname{tr}(E^i_j \alpha)(\varepsilon, z_1, z_2).$$

We have observables $\mathcal{O}^i_j(\varepsilon, z_1, z_2)$ and $\mathcal{O}^k_l(0, 0, 0)$. What is their product in the factorization algebra? We need to compute

$$\lim_{\varepsilon\to 0^+} \mathcal{O}^i_j(\varepsilon, 0, 0)\mathcal{O}^k_l(0, 0, 0) - \lim_{\varepsilon\to 0^-} \mathcal{O}^i_j(\varepsilon, 0, 0)\mathcal{O}^k_l(0, 0, 0).$$

Here is the formula, to order $\hbar$, for

$$(\mathcal{O}^i_j(\varepsilon, 0, 0) \cdot \mathcal{O}^k_l(0, 0, 0))(\alpha)$$

that we represent diagrammatically. diagram here It turns out that only the following diagram contributes, diagram here What we need to show is that this diagram contributes precisely the deformation we saw above. There is a Lie algebra factor and an analytic factor coming from the Feynman rules. The analytic factor amounts to showing that

$$\int_{v,v'\in\mathbb{R}\times\mathbb{C}^2} P_0(v - (\varepsilon, 0, 0))\alpha(v)\alpha(v')P_0(v - v')P_0(v')dz_1 dz_2 dz'_1 dz'_2$$

looks like

$$\operatorname{sign}(\varepsilon)\partial_{z_1}\alpha\partial_{z_2}\alpha + \text{continuous}.$$

In the Yangian case the m_3 operation is computed by the diagram insert diagram. It's pretty tricky to get the factors of 2 right and this was computed by Costello, Witten, Yamazaki.

13. June 15, 2018 – Jacob Lurie

Recall that on Tuesday we translated Weil's conjecture for functional fields to a problem about computing the cohomology of the stack of principal G-bundles. Our goal is the following: let X be an algebraic curve and $G \to X$ be a group scheme and $\operatorname{Bun}_G(X)$ be the stack of G-bundles on X. We would like to use the topology/cohomology of this stack. Last time we discussed how to get ahold of the homotopy type of $\operatorname{Bun}_G(X)$, at least in the complex analytic setting.

Over $\mathbb{C}$, let $\mathrm{Ran}(X)$ be the set of nonempty finite subsets $S \subset X$. We defined a (homotopy) cosheaf $\mathcal{F}$ of spaces on $\mathrm{Ran}(X)$ such that

$$\mathcal{F}(\mathrm{Ran}(X)) = \mathrm{Maps}(X, BG) \approx \mathrm{Bun}_G(X).$$

Moreover we had a local statement that

$$\mathcal{F}(\mathrm{Ran}(U_1, \ldots, U_n)) = \mathrm{Maps}_c(U_1 \sqcup \cdots \sqcup U_n, BG) \approx \prod_{i=1}^{n} \Omega^2 BG.$$

How do we think about it? We are asking, roughly, for G-bundles on X equipped with a trivialization outside of the U_i. If we were to take a certain limit we would obtain G-bundles on X equipped with a trivialization outside n points.

Now over any field we introduce $\mathrm{Ran}_G(X)$, which classifies triples (P, S, γ) where $P \to X$ is a principal G-bundle, $S \subset X$ is a finite set, and γ is a trivialization of P on the complement of S. Recall that $\mathrm{Bun}_G(X)$ is not a variety but a stack. This means that I have to probe it by maps $Y \to \mathrm{Bun}_G(X)$ from a variety, which is defined to be the G-bundles on Y. So for $\mathrm{Ran}(X)$ we define the maps $Y \to \mathrm{Ran}(X)$ to associate to Y the set of nonempty finite subset S of Y. $\mathrm{Ran}_G(X)$ is defined similarly, and in fact maps to $\mathrm{Ran}(X)$ and $\mathrm{Bun}_G(X)$.

We now state an algebraic geometric version of NPD.

Theorem 35 (Nonabelian Poincaré duality)**.** *The map* $\mathrm{Ran}_G(X) \to Bun_G(X)$ *induces an isomorphism on étale cohomology.*

This result allows us to get the foot in the door for our proof, so let's discuss it briefly. Let us show that the fibers are contractible. Let's consider the fiber above the trivial bundle.

$$\begin{array}{ccc} \mathrm{Rat}(X, G) & \longrightarrow & \mathrm{Ran}_G(X) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Bun}_G(X) \end{array}$$

Here the fiber is the space of rational maps from X to G, which is reasonable because we are simply choosing another trivialization almost everywhere on X.

Example 36. Consider $G = GL_1 = \mathbb{G}_m$. This is not the type of group we're considering for Weil's conjecture. But of course nonabelian Poincaré duality holds in this case. We are trying to understand rational maps from our algebraic curve to $GL_1 = \mathbb{A} - \{0\} \subset \mathbb{A}$. Rational maps into $\mathbb{A}$ are pretty easy to understand, roughly just meromorphic functions $K(X)$, which is an infinite-dimensional affine space. This space oughta be contractible as it is affine. But every rational map from X into $\mathbb{A}^1 - \{0\}$ determines a rational map $X \to \mathbb{A}^1$ in the obvious way. However, this does not work if the entire map is zero. Hence $\mathrm{Rat}(X, GL_1) = K(X) - \{0\}$ but an infinite dimensional vector space with a point removed is contractible (just as in usual topology).

Now let's do a bit of a harder case. Another example, which again is not quite the kind of example that we're interested in, is the constant group scheme GL_n.

Example 37. We think of $G = GL_n \subset M_{n,n}$. Then $\mathrm{Rat}(X, M_{n,n}) = M_{n,n}(K(X))$. Now $\mathrm{Rat}(X, GL_n) \subset M_{n,n}(K(X))$ but now we're interested in those maps where the determinant does not vanish identically. How do we know that we get something contractible? It's hard to say when we're working with these infinite-dimensional spaces, so let's try to approximate with finite-dimensional spaces. If we fix a divisor $D \subset XS$ we can consider $M_{n,n}(\Gamma(X, \mathcal{O}(D))$ where the entries are global sections of $\mathcal{O}(D)$ (allowed to have poles only along D). And indeed by Riemann-Roch this space has approximately dimension $n^2 \deg(D)$ (we're really only interested in how this grows with the divisor D). Given such a matrix we can take the determinant to obtain a section $\Gamma(X, \mathcal{O}(nD))$. This is a nonlinear map between spaces of dimension roughly $n^2 \deg(D)$ to $n \deg(D)$. We are interested in taking our divisor to be bigger and bigger (to be the complement of a finite set). But in the limit the codimension is going to be larger and larger. The intuition now is that removing a very large codimension space will not change the contractibility.

Our goal is now to move to $\mathbb{F}_q$ and understand

$$\mathrm{tr}(\phi^{-1} \mid H^* \mathrm{Bun}_G(X)).$$

By virtue of what we just said this is the same as the cohomology of $\mathrm{Ran}_G(X)$. We want to compute the cohomology of this Ran space by some sort of Leray-Serre spectral sequence,

$$H^*(\mathrm{Ran}_G(X); \underline{\mathbb{Q}_\ell}) \simeq H^*(\mathrm{Ran}(X), R\pi_* \underline{Q_\ell}).$$

Let us denote

$$\mathcal{A} := R\pi_* \underline{Q_\ell}.$$

So really we wish to understand

$$\mathrm{tr}(\phi^{-1} \mid H^*(\mathrm{Ran}(X), \mathcal{A})).$$

We need to understand the fibers of the map π. Fix a point in the Ran space. Recall that this is a finite set of points of X, $S = \{x_1, \ldots, x_n\}$. The fiber should be the bundles equipped with trivializations away from these points. However we could take formal discs near our points and equip it with trivializations there. The trivializations won't agree on the punctured formal discs; they will disagree by the completed local ring with a coordinate inverted upto our choice of trivialization (which is the data of G evaluated at the completed local ring):

$$\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} \{S\} = \prod_{x \in S} G(K_x)/G(\mathcal{O}_x) =: \prod_{x \in S} \mathrm{Gr}_{G,x}.$$

This is what people call the affine Grassmannian. Notice that everything here is living as functors on commutative rings. One can show that this affine Grassmannian is a projective ind-scheme, i.e. it can be written as a direct limit of closed embeddings of projective schemes. It's kind of like a projective algebraic variety but has infinite dimension.

Now what can we say about $\mathcal{A}$? What if we were to take the stalk at S? At the risk of being a bit confusing write

$$S : * \to \mathrm{Ran}(X)$$

and consider $S^*\mathcal{A}$ which is the stalk, a chain complex. We find

$$H^*(S^*\mathcal{A}) = H^*(\prod_{x \in S} \mathrm{Gr}_{G,x}) \approx \otimes_{x \in S} H^*(\mathrm{Gr}_{G,X}).$$

This $\mathcal{A}$ is an example of a factorizable sheaf, which levels in the world of ℓ-adic sheaves. It has the feature that the stalk at a finite set can be written as a tensor product of things depending only on each point.

What we described last Tuesday was the Grothendieck-Lefschetz trace formula when we were in the context of a constant sheaf. We now need a slightly more general context. Let Y be a variety over $\mathbb{F}_q$ and let $\mathcal{F}$ be some kind of (ℓ-adic) sheaf on Y.

Theorem 38 (Grothendieck-Lefschetz trace formula)**.** *The trace*

$$\mathrm{tr}(\phi \mid H_c^*(Y, \mathcal{F})) = \sum_{x \in Y(\mathbb{F}_q)} \mathrm{tr}(\phi \mid x^* \mathcal{F})$$

One might now like to apply this to the Ran space. This isn't exactly right, however. Here there's a Frobenius instead of the inverse and moreover we're taking compactly supported cohomology. So we apply the Verdier dual of Grothendieck-Lefschetz

$$\mathrm{tr}(\phi^{-1} \mid H^*(Y, \mathcal{F})) = \sum_{x \in Y(\mathbb{F}_q)} \mathrm{tr}(\phi^{-1} \mid x^! \mathcal{F})$$

where $x^!$ is the costalk. Roughly speaking what we would like to do is apply this dual form to the case when $Y = \mathrm{Ran}(X)$.

Question: can we apply the dual of the Grothendieck-Lefschetz trace formula when $Y = \mathrm{Ran}(X)$ and $\mathcal{F} = \mathcal{A}$. If so, what are the costalks of $\mathcal{A}$?

As you might guess, the answer is no. The reason is that the costalks are all zero. So what goes wrong? This Ran space is not a variety over even an ind-scheme, but ignoring this for the moment we want to just emphasize that it is infinite dimensional. However we can write

$$\mathrm{Ran}(X) = \cup_n \mathrm{Ran}(X)_{\leq n},$$

which is the space of finite subsets of X with fewer than $n + 1$ elements. Really the Grothendieck-Lefschetz trace formula only tells you about sheaves living on finite-dimensional spaces. A little bit of wiggle room you might find is that if our sheaf is only supported on a finite-dimensional subspace, we could just apply the formula there. Unfortunately this isn't true because of the computation of $H^*(S^*\mathcal{A})$. Here's a workaround. Replace $\mathcal{A}$ by a "reduced" variant $\mathcal{A}_{\mathrm{red}}$. This variant has the property that

$$H^*(S^*\mathcal{A}_{\mathrm{red}}) = \otimes_{x \in S} H^*_{\mathrm{red}}(\mathrm{Gr}_{G,x}$$

and

$$H^*(\mathrm{Ran}(X), \mathcal{A}_{\mathrm{red}}) \simeq H^*_{\mathrm{red}}(\mathrm{Bun}_G(X)).$$

It's still supported in infinite dimensions, but if you're interested in a particular cohomological degree, the support is finite-dimensional. Here's teh first place that simply connectedness of G comes into play. The affine grassmannians are thus connected whence the cohomologies in the equations above only have support in degrees $\geq 2|S|$ as it turns out. As a result one is able to apply Grothendieck-Lefschetz.

We obtain:

$$\mathrm{tr}(\phi^{-1} \mid H^*(\mathrm{Ran}(X); \mathcal{A}_{\mathrm{red}})) = \sum_{S \subset X} \mathrm{tr}(\phi^{-1} \mid S^! \mathcal{A}_{\mathrm{red}})$$

where we're summing over nonempty finite subsets closed points. Now the left is

$$\mathrm{tr}(\phi^{-1} \mid H^*_{\mathrm{red}}(\mathrm{Bun}_G(X)))$$

and the right, using a factorizability property,

$$\sum_{S\neq\varnothing}\prod_{x\in S}\mathrm{tr}(\phi^{-1}\mid x^! A)$$

If we add 1 to both sides the reduced on the cohomology vanishes and the nonempty condition on the right vanishes. Now

$$\mathrm{tr}(\phi^{-1}\mid H^*(\mathrm{Bun}_G(X))) = \prod_{x\in X}\left(1+\mathrm{tr}(\phi^{-1}\mid x^!\mathcal{A}_{\mathrm{red}}\right).$$

It is this type of Euler product expansion that we have been after. Weil's conjecture is slightly different but hopefully you can appreciate already that we've used a local-to-global principle.

Let me close by saying how we analyze these costalks $x^!\mathcal{A}_{\mathrm{red}}$. To do this let us return to the setting of topology. This $\mathcal{A}_{\mathrm{red}}$ is an example of a factorization algebra over X. And at least over $\mathbb{C}$ this is a local system of nonunital E_2-algebras (these are coming from taking stalks and dualizing them). Indeed:

$$H^*(x^*\mathcal{A}_{\mathrm{Red}}) = H^*_{\mathrm{red}}(\mathrm{Gr}_{G,x}) = H^*_{\mathrm{red}}(\Omega^2 BG)$$

or more precisely

$$x^*\mathcal{A}_{\mathrm{red}} \approx C^*_{\mathrm{red}}(\Omega^2 BG).$$

In the theory of E_2-algebras there is a theory of Koszul duality, which is something we can apply in this particular case. Koszul duality for nonunital E_2-algebras corresponds, thinking of them as factorizable sheaves, to Verdier duality on $\mathrm{Ran}(X)$. We are interested in looking at costalks, which is Verdier dual to taking stalks. Hence we need to take the Koszul dual of $x^*\mathcal{A}_{\mathrm{red}}$. In the setting of topology Koszul duality tells us that

$$C_*^{\mathrm{red}}(\Omega^2 BG)\xrightarrow{KD} C^*_{\mathrm{red}}(BG).$$

Hence we should have

$$x^!\mathcal{A}_{\mathrm{Red}} = H^*_{\mathrm{red}}(BG_x).$$

So once we understand this Koszul duality in algebraic geometry we obtain

$$\prod_{x\in X}(1+\mathrm{tr}(\phi^{-1}\mid x^!\mathcal{A}_{\mathrm{red}})) = \prod_{x\in X}\mathrm{tr}(\phi^{-1}\mid H^*(BG_x)).$$

14. June 15, 2018 – Kevin Costello

15. June 18, 2018 – Ezra Getzler

I want to kick things off by telling you what a category of a fibrant objects is. The typical example is that of the fibrant objects in a closed model category less than a particular size.

Let $\mathcal{V}$ be a small category and $\mathcal{W}\subset\mathcal{V}$ be a subcategory of weak equivalences. Every isomorphism in $\mathcal{V}$ is required to be in $\mathcal{W}$, and for any diagram

$$X\xrightarrow{f} Y\xrightarrow{g} Z$$

if any two of the three morphisms (the third being the composition) are weak equivalences then the third is as well.

The typical example of weak equivalences is that of weak homotopy equivalences of topological spaces (here we are reflecting the isomorphisms in sets/groups/abelian groups). Dwyer and Kan showed that given such a subcategory of weak equivalence one can construct a simplicial localization where they are inverted — this is, roughly, what homotopy theory is about.

Let $\mathcal{V}_\bullet \to \mathcal{V}$ be a simplicial resolution of our category. The objects of $\mathcal{V}_n$ are just the objects of $\mathcal{V}$ for $n \geq 0$. Moreover $\mathcal{V}_n$ will be a free category for $n \geq 0$ (iteratively). This is a standard construction which you can look up.

Let's recall some simplicial basics. We take Δ to be the category of finite nonempty totally ordered sets. We use the finite ordinals $[n]$ to obtain a small category. The morphisms are just order-preserving functions. Notice that we may identify $[n] \mapsto \Delta^n$, where Δ^n is the convex hull of the unit vectors in $\mathbb{R}^{n+1}$. A covariant functor $\Delta \to \mathcal{V}$ is called a cosimplicial object in $X^\bullet$. On the other hand, a contravariant functor $\Delta^{\text{op}} \to \mathcal{V}$ is called a simplicial object $X_\bullet$.

Anyway, the vertices of $\mathcal{V}_n$ are words in the morphisms. For instance $\mathcal{V}_0$ is the free category generated by $\mathcal{V}$. Now we can consider the simplicial localization $\mathcal{V}_\bullet[\mathcal{W}^{-1}] = \mathcal{W}_\bullet^{-1}\mathcal{V}_\bullet$. Here we've taken a resolution of $\mathcal{W}$ as well. This is a simplicial category but all bets are off — it's very obscure what the properties are. The whole subject of categories of fibrant objects is to get our hands on this object.

One might ask whether every morphism of $\mathcal{V}$ that goes to an isomorphism of $\mathcal{W}^{-1}\mathcal{V}$ a weak equivalence? This is some sort of saturation property, which was studied by (Ezra forgets), and they found that there is a simple strengthening of the axioms, the 2-out-of-6 axiom, which yields the answer yes. The axiom says that if given

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

and gf and hg are in $\mathcal{W}$ then f, g, h are in $\mathcal{W}$. Check how this works in the case of spaces and weak equivalences.

Our goal now is to find a much more understandable realization of the simplicial localization. In particular we will obtain a fibrant resolution of categories! In fact it will be an ∞-category. To introduce categories of fibrant objects we need discuss fibrations. Verdier introduced, in SGA4, the notion of a carrable morphism, which is a morphism that can be pulled back (since we're in Canada it's probably ok to use this word). Fix $\mathcal{F} \subset \mathcal{V}$ a subcategory and assume that all pullbacks of morphisms of $\mathcal{F}$ exist and are in $\mathcal{F}$. Moreover we assume that there exists a terminal object e. We require also that all isomorphisms are fibrations. So far this structure is called a category of fibrations.

Now a category of fibrant objects is a category with weak equivalences and simultaneously a category of fibrations, satisfying some extra properties. We call morphisms which are both weak equivalences and fibrations trivial fibrations. These were invented by Ken Brown, who called them acyclic fibrations. The first axiom is that every object is fibrant, i.e. the unique map to the terminal object is a fibration. Next we ask that the pullback of a trivial fibration is again a trivial fibration. Finally we ask that every morphism may be factored into a weak equivalence followed by a fibration.

These axioms may seem a bit like an abstract painting where the logic is not so clear, but somehow this structure pops up all over math, and you can prove theorems about them!

What does the simplicial localization of a CFO look like? The vertices

$$\mathcal{V}[\mathcal{W}^{-1}](X, Y)_0.$$

will be "generalized morphisms". A morphism from $X \to Y$ is a diagram

$$X \overset{\sim}{\leftarrow} P \to Y.$$

where the first arrow is a trivial fibration. This has been known forever. Just a few years ago a student at Bonn, Szumilo extended this to a quasicategory (of frames).

Now recall that the nerve of a category is the simplicial set $N\mathcal{C}$ given

$$N_n\mathcal{C} = \mathrm{Fun}([n], \mathcal{C}),$$

which gives us a fully faithful embedding of small categories into simplicial sets. Now $N_\bullet \mathcal{V}_\bullet[\mathcal{W}^{-1}]$ will be a quasicategory whose zero simplices will be objects of $\mathcal{V}$ and whose one-simplices will be diagrams $X \leftarrow P \to Y$ as above. The two-simplices will look like fill in arrows

$$\begin{array}{ccc} & X_1 & \\ X_{01} & X_{012} & X_{12} \\ X_0 & X_{02} & X_2 \end{array}$$

where the acyclic fibrations are those where the face contains the initial vertex.

Recall we have a simplicial set Δ^n defined by

$$\Delta^n_k = \Delta([k], [n]).$$

Inside here we have the ith face

$$\partial_i \Delta^n \subset \Delta^n,$$

which has all the faces that don't include the vertex i. We obtain this way two important simplicial sets, the boundary

$$\partial \Delta^n = \cup_{0 \leq i \leq n} \partial_i \Delta^n$$

and the horn, for $0 \leq i \leq n$

$$\Lambda^n_i \Delta^n = \cup_{j \neq i} \partial_j \Delta^n.$$

Definition 39. A fibration of simplicial sets $f : X \to Y$ is a map f for which the map

$$X_n = \mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Delta^n_i, X) \times_{\mathrm{Hom}(\Lambda^n_i, Y)} Y_n.$$

is surjective for all $n < 0$ and $0 \leq i \leq n$. A Kan complex is a simplicial set for which $X \to e$ is a fibration.

Being a Kan complex is some sort of nonabelian analog of being a chain complex. It turns out that a category $\mathcal{C}$ is a groupoid if and only if $N\mathcal{C}$ is a Kan complex.

Example 40. Finite Kan complexes are an example of a CFO, where by finite we mean degreewise finite sets of simplices. Make this category small in your favorite way.

Lemma 41 (Brown's lemma)**.** *The weak equivalences in a category of fibrant objects factors into a section of a trivial fibration followed by a trivial fibration.*

Hence the trivial fibrations of Kan complexes for which

$$X_n \to \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} Y_n$$

is surjective, with $n \geq 0$. For $n = 0$ we find that $X_0 \to Y_0$ is surjective. Work out what you get for $n = 1$. The idea is that this allows us to access the weak equivalences combinatorially via the trivial fibrations which are often easier to work with.

16. June 18, 2018 – Chris Brav

Let me give a brief plan.

(1) DG categories as noncommutative spaces
(2) derived algebraic geometry, moduli space of objects
(3) orientations/calabi-yau structures on noncommutative spaces
(4) shifted symplectic structures on moduli spaces

Throughout we will fix a field k. For now it can be any field, but later it should be characteristic zero. Moreover everything will be implicitly derived.

Let's start with some motivation.

Example 42. Let X be a smooth variety (the smoothness is not crucial). On such a space we obtain a category $\mathsf{QCoh}(X)$, which is a differential graded category (there will be hom-complexes instead of hom-sets). The objects are complexes of $\mathcal{O}_X$-modules with quasicoherent cohomology, up to quasi-isomorphisms. The morphisms are $\mathrm{Hom}^*(F, G)$ (maps of various degrees) over k whose cohomology computes ext-groups. This is a convenient category to work in as most operations land us back in here.

There are examples of $X \ncong Y$ for which there is an equivalence of dg categories $\mathsf{QCoh}(X) \simeq \mathsf{QCoh}(Y)$. The most well-known example is due to Mukai: principally polarized abelian varieties and their duals. It is important that this equivalence does not preserve the tensor product. In other words, we're thinking of these categories as noncommutative spaces instead of commutative, where we can take tensor products. But the point is that you can't recover a commutative space from $\mathsf{QCoh}(X)$ but you can remember certain invariants.

For example, for certain dg categories one can compute Hochschild homology. There is a theorem of Hochschild-Kostant-Rosenberg for which

$$HH_*(\mathsf{QCoh}(X)) \simeq \Gamma(X, \oplus\Omega^p_X[p]) \simeq \Gamma(LX, \mathcal{O}_{LX})$$

(remember we are dropping all derived symbols). However it's hard to pick out certain degrees, can only really recover differences of degrees. This is because we forgot about $\otimes$. Another example we can pick up directly from this dg category is algebraic K-theory, though this is much finer than what we will be discussing.

The next example is more homotopical. Recall that dg categories can even be defined as modules for the Eilenberg-Maclane spectrum.

Example 43. Let X be a nice topological space. Consider $\mathsf{Loc}(X)$, which can be defined as

$$\mathsf{Loc}(X) = \mathrm{Fun}(X, \mathsf{Vect}_k),$$

where everything in sight is ∞-categorical. More concretely, if X is connected with basepoint the $\mathrm{Fun}(X, \mathsf{Vect}_k) \simeq C_*(\Omega X) - \mathsf{mod}$. There is also a tensor product here but we will forget it for our noncommutative purposes. Chris' favorite local system is the constant local system $k_x \in \mathsf{Loc}(X)$. We can ask what the hom-complex is

$$\mathrm{Hom}^*(k_X, k_X) = C^*(X, k_X).$$

This is an example of an invariant that is picked up. Another thing we could do is consider

$$HH_*(\mathsf{Loc}(X)) \simeq C_*(LX),$$

which is a theorem of Goodwillie-Jones. It is interesting to note that there is an S^1-action on each side and that this equivalence is S^1-equivariant.

The next is very general, which includes the above two in some sense.

Example 44. Let R be a dg algebra. Consider the category Mod_R of dg modules. Then $HH_*(\mathsf{Mod}_R)$ carries an S^1 action. There is no $\otimes$ of modules and no sort of $X \times X$, but HH_* plays the role of functions on the loop space with the loop rotation. Alternatively we can think of the de Rham complex with a funny grading..

A dg category $\mathcal{A}$ has some set (maybe collection) of objects, and has complex of morphisms

$$\mathcal{A}(x, y) = \mathrm{Hom}_A(x, y)$$

with a composition

$$\mathcal{A}(y, z) \otimes_k \mathcal{A}(x, y) \to \mathcal{A}(x, z)$$

that is a map of chain complexes and is associative and unital (the unit needs to be closed under the differential).

Really we want to work with big dg categories, so we will try to use colimits as much as possible, in stark contrast to Ezra's lecture. These big dg categories are going to be (co)complete, stable, and presentable. Presentable is a techincal condition that says there is a set of small objects and everything else can be built from sufficiently small filtered colimits. For example one might try to understand a vector space by understanding its poset of finite dimensional subspaces. So presentable is really a technical condition. Stable is more intuitive. Since we are (co)complete we have an initial and final object. Stable means:

- that the map $\varnothing \to *$ is an equivalence
- Consider the loop space

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

  and the suspension

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

  which are always adjoint. We require that these be inverse to each other.

In homological algebra stability is very easy: $\Sigma = [1]$ and $\Omega = [-1]$. This notation is independent of homological or cohomological grading. From now on dg categories will be big dg categories unless otherwise stated.

Now consider $\mathsf{dgCat}_{\mathrm{cont}}$, which will be the ∞-category of (big) dg categories with continuous (colimit-preserving) dg functors between them. Sometimes we will actually use a 2-categorical variant. $\mathsf{dgCat}_{\text{cont}}$ is a nice algebraic world to work in as it is symmetric monoidal and there is an internal hom Fun. The tensor product $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is similar to what we're used to in algebra. In particular such functors are in bijection with bilinear and continuous in each variable. The unit for this tensor product is just Vect_k. We say that a dg category $\mathcal{C}$ is dualizable if there exist $\mathcal{C}^\vee$ and pairings and copairings

$$\mathsf{Vect} \xrightarrow{\text{co}} \mathcal{C}^\vee \otimes \mathcal{C} \qquad \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\text{ev}} \mathsf{Vect}$$

such that

$$\mathcal{C} \xrightarrow{\text{id}_\mathcal{C} \otimes \text{co}} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\text{ev} \otimes \text{id}_\mathcal{C}} \mathcal{C} \sim \text{id}_\mathcal{C}$$

and similarly for $\mathcal{C}^\vee$.

Exercise 45. any big dg category $\mathcal{D}$ there is an equivalence

$$\mathcal{C}^\vee \otimes \mathcal{D} \sim \text{Fun}(\mathcal{C}, \mathcal{D}).$$

If $\mathcal{D} = \mathsf{Vect}$ then $\mathcal{C}^\vee \simeq \mathsf{Fun}(\mathcal{C}, \mathsf{Vect})$.

We'll skip the algebro-geometric example in the interest of time since it's similar to the topological example.

Example 46. Let X be a nice topological space. We claim that $\mathsf{Loc}(X)$ is self-dual. The essential fact that makes this work is a K unneth formula for local systems. Suppose we have X, Y nice topological spaces. There is a natural functor

$$\mathsf{Loc}(X) \otimes \mathsf{Loc}(Y) \to \mathsf{Loc}(X \times Y)$$

sending $E, F \mapsto E \boxtimes F = p_X^* E \otimes p^* F$, at least for small generators. Then extend by colimits. There is some lemma says that when you have a functor that induces isomorphisms on homs and everything is built from colimits then you get an equivalence overall. It's like checking isomorphisms of vector spaces by looking at a basis.

We now define the pairings and copairings. Note that $\mathsf{Loc}(*) = \mathsf{Vect}$. Now the copairing is given

$$\text{co}_X : \mathsf{Loc}(*) \xrightarrow{p^*} \mathsf{Loc}(X) \xrightarrow{\Delta^!} \mathsf{Loc}(X \times X) \simeq \mathsf{Loc}(X) \otimes \mathsf{Loc}(X)$$

where $\Delta^!$ is roughly an induction of representations. The evaluation or pairing goes the other way,

$$\text{ev}_X : \mathsf{Loc}(X) \otimes \mathsf{Loc}(X) \simeq \mathsf{Loc}(X \times X) \xrightarrow{\Delta^*} \mathsf{Loc}(X) \xrightarrow{p_!} \mathsf{Loc}(*),$$

where Δ^* is roughly a restriction of representations.

Let's now turn to Hochschild homology of dg categories. Suppose $\mathcal{C}$ is dualizable. Then we define

$$HH_*(\mathcal{C}) := \text{tr}(\text{id}_C).$$

Let's unpack what this means. Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor of $\mathcal{C}$ (recall that it preserves colimits). We write

$$\text{tr}(F) : \mathsf{Vect} \xrightarrow{\text{co}} \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\text{id}_C \otimes F} \mathcal{C}^\vee \otimes \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\text{ev}} \mathsf{Vect}.$$

Check that for vector spaces (choose bases) this recovers the familiar trace. This is a functor: since Vect is built by colimits from a one-dimensional space it is enough to know what happens to k. Hence we really mean $\text{tr}(F)(k)$.

Let's see what this means in our two examples. Consider $\mathcal{C} = \mathsf{Loc}(X)$. We obtain

$$HH_*(\mathsf{Loc}(X)) := \mathsf{Vect} \xrightarrow{p^*} \mathsf{Loc}(X) \xrightarrow{\Delta_!} \mathsf{Loc}(X \times X) \xrightarrow{\Delta^*} \mathsf{Loc}(X) \xrightarrow{p^!} \mathsf{Vect}.$$

To see what this means consider

$$\begin{array}{ccccc} LX & \longrightarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & X \times X & & \\ \downarrow & & & & \\ * & & & & \end{array}$$

but local systems satisfy base change, i.e. $\Delta^*\Delta_! = \pi_!\pi^*$. So we have

$$p_!\pi_!\pi^*p^*k \simeq p_!\pi_!k_{LX} \simeq (p_{LX})_!(k_{LX}) \simeq C_*(LX)$$

though the last equivalence is not so clear yet. Now going the other way we find

$$HH_*(\mathsf{Loc}(X)).$$

It doesn't look very nice with respect to S^1 since we've broken the symmetry in two, but that can be fixed.

Now consider $\mathcal{C} = \mathsf{Mod}_R$. Then $\mathcal{C}^\vee = \mathsf{Mod}_{R^{\mathrm{op}}}$. The pairing is the relative tensor product

$$\mathrm{ev} : \mathsf{Mod}_R \otimes \mathsf{Mod}_{R^{\mathrm{op}}} \to \mathsf{Vect}$$

sending $M, N \mapsto M \otimes_R N$. For the copairing,

$$\mathrm{co} : \mathsf{Vect} \to \mathsf{Mod}_{R^{\mathrm{op}}} \otimes \mathsf{Mod}_R \simeq \mathsf{Mod}_{R^{\mathrm{op}} \otimes R}$$

sending $k \mapsto R$. Identifying $\mathsf{Mod}_{R \otimes R^{\mathrm{op}}} \simeq \mathsf{Mod}_R \otimes \mathsf{Mod}_{R^{\mathrm{op}}}$ we obtain

$$(M \otimes_k N) \simeq (M \otimes_k N) \otimes_{R^{\mathrm{op}} \otimes R} \bar{R}.$$

Composing, we find that

$$HH_*(\mathsf{Mod}_R) = R \otimes_{R^{\mathrm{op}} \otimes R} R,$$

where of course everything is fully derived.

17. June 18, 2018 – Claudia Scheimbauer

There are two motivating notions for dualizability. The first is that of an adjoint pair of functors L, R between two categories $\mathcal{C}, \mathcal{D}$. The unit and counit yield that $L \implies LRL \implies L$ and $R \implies RLR \implies R$ are the identity. The second is that of a vector space, which has a dual V^* if and only if we have evaluation and coevaluation maps satisfying $V \to V \otimes V^* \otimes V \to V$ and $V^* \to V^* \otimes V \otimes V^* \to V^*$ are identities. Notice the similarity between these two examples.

In the first example we are working in the bicategory Cat whose objects are categories, morphisms are functors, and 2-morphisms are natural transformations. On the other hand we could just as well take any bicategory $\mathcal{B}$ and have our categories $\mathcal{C}$ and $\mathcal{D}$ be instead just objects in $\mathcal{B}$ and the adjoint pair just 1-morphisms in $\mathcal{B}$ satisfying the relevant properties. Recall for instance Damien's example in the cobordism category.

In the second example we are working in Vect, a symmetric monoidal category where we have objects vector spaces, morphisms linear maps, and then suggestively let's notice that we have a tensor product $\otimes$. Instead we could just take any symmetric monoidal category $\mathcal{C}$ and take two objects in here V and V^* satisfying the same properties.

The goal for today is to discuss a generalization of the two examples above. Today will be purely algebraic, with the geometric intuition presented next time, about TFTs. Recall from last week the cobordism hypothesis (sketched by Lurie, in progress Ayala-Francis) which said that the evaluation

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}}} \mathcal{C}$$

factors through $\mathcal{C}^{\mathrm{nd}}$, which is the subcategory of n-dualizable objects and the map $\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{Fr}}, \mathcal{C}) \to \mathcal{C}^{\mathrm{nd}}$ is an equivalence of underlying ∞-groupoids. The details here are not so important here but we just want to highlight the importance of n-dualizability. For instance having a n-dualizable object in a symmetric monoidal ∞-category automatically yields, by the cobordism hypothesis, a TFT.

Let's now look at the definition of n-dualizable. As a warm-up, let's talk about algebras. If we go back to our second example above, recall that V a vector space has a dual if and only if V is finite-dimensional. What about other targets? Consider Alg_k, the bicategory having objects algebras over k, morphisms being bimodules ${}_AM_B$, and 2-morphisms being maps of bimodules. Here the symmetric monoidal structure is given by $\otimes_k$. The composition of bimodules is given

$${}_BN_C, {}_AM_B \mapsto M \otimes_B N.$$

Algebraically, to study this bicategory we might try to study modules. More speficially we build $\mathsf{Alg}_k \to \mathsf{Cat}_k$ sending $A \mapsto \mathsf{Mod}_A$ and ${}_AM_B \mapsto \mathsf{Mod}_A \xrightarrow{-\otimes_A M} \mathsf{Mod}_B$. Notice that if the bimodule is invertible then one obtains an equivalence of module-categories.

We claim that every algebra A is dualizable. Let's start the proof — you can finish it. The dual of A is the opposite algebra A^{op}. We take

$$\mathrm{ev}_A = {}_{A^{\mathrm{op}}\otimes A}A_k, \qquad \mathrm{coev}_A = {}_kA_{A\otimes A^{\mathrm{op}}}$$

and then one needs to check that the appropriate compositions yield the identity.

Consider now the category of pointed vector spaces $v \in V$ where the morphisms are linear maps preserving the pointing. Then we claim that only $1 \in k$ is dualizable.

Now what about adjoints? Let's go back to Alg_k. A 1-morphism ${}_AM_B$ in Alg_k has a left adjoint if and only if M is finitely presented and projective over A. Replacing left with right just switches A to B. This is an exercise, with the hint to use the dual basis lemma.

Let's now pass to n-categories. In this setting we have objects, 1-morphisms, 2-morphisms, and so on up to n-morphisms. If we pass to (∞, n)-categories we have even higher morphisms, but they are all required to be invertible. Now if $k \leq n$ then we define $\tau_{\leq k}\mathcal{C}$ by discarding noninvertible morphisms for $\ell > k$. Let's start with $\mathcal{C}$ a symmetric monoidal (∞, n)-category. Truncate and take the homotopy category $h_1\tau_{\leq 1}\mathcal{C}$, which is symmetric monoidal. If $\mathcal{C}$ is an (∞, n)-category then we obtain $h_2(\tau_{\leq 2}\mathcal{C})$ a homotopy bicategory. By truncating in this way we pass to settings where we can actually talk about adjoints.

From now on we will set $\mathcal{C}$ to be a symmetric monoidal (∞, n)-category.

Definition 47. An object $x \in \mathcal{C}$ is dualizable if its image in $h_1(\mathcal{C})$ is dualizable.

If f is a k-morphism in $\mathcal{C}$ (for $1 < k \leq n-1$) it might be a map between maps α, β that are maps between a and b (which may or may not be objects). Then

$h_2\tau_{\le 2}\operatorname{Hom}_C(a,b)$ is a 2-category and we say that f has a left/right adjoint if the image of f here has a left/right adjoint.

Definition 48. We say that $\mathcal{C}$ is n-dualizable if every object has a dual and every k-morphism, for $1 \le k \le n-1$ has a left and right adjoint. In this case we say that any object $x \in \mathcal{C}$ is n-dualizable.

Example 49. The category Bord_n with or without framing is n-dualizable.

Now if $\mathcal{D}$ is an arbitrary symmetric monoidal (∞, n)-category then we have $\mathcal{D}^{\mathrm{nd}} \to \mathcal{D}$ the subcategory of n-dualizable objects and k-morphisms with adjoints.

Let's look at some examples of 2-dualizable objects in Alg_k. We saw that every object is (1-)dualizable. Moreover we saw that ${}_AM_B$ has a left-adjoint if and only if it is finitely presented and projective over A. To test whether an object of A is 2-dualizable we need to check that $\mathrm{ev}_A = {}_{A^{\mathrm{op}}\otimes A}A_k$ has left and right adjoints as well as that $\mathrm{coev}_A = {}_kA_{A\otimes A^{\mathrm{op}}}$ has left and right adjoints. The existence of right adjoints requires finitely presentable and projective over k and the existence of left adjoints requires separable over k.

Recall that (Calaque-Scheimbauer) using factorization algebras we obtain an (∞, n)-category where the objects are E_n-algebras, maps are bimodules are E_{n-1}-algebras, 2-morphisms are bimodules of bimodules in E_{n-2} algebras, etc. So for $n = 2$ we might give a definition of objects being locally constant factorization algebras on a disk, maps are constructible factorization algebras on a disk with a line passing through, 2-morphisms are constructible factorization algebras on a disk with a line passing through with a point on the line, etc.

Theorem 50 ((Gwilliam-Scheimbauer))**.** *This category Alg_n^{fact} is n-dualizable.*

Recall that our motivation was to invoke the cobordism hypothesis: every A determines a fully extended TFT

$$\mathsf{Bord}_n^{\mathrm{fr}} \to \mathsf{Alg}_n.$$

This map is explicitly given by factorization homology, as outlined by Damien.

Theorem 51 ((Gwilliam-Scheimbauer))**.** *Only* 1 *is $(n+1)$-dualizable.*

This is not so great because we would actually like to have more than just n-dualizability. This means we "need to get rid of points".

18. June 18, 2018 – Chris Brav (exercise section)

We will provide various exercises of different tastes.

1. Let X be a smooth scheme. Compute $HH_*(\mathsf{QCoh}(X))$ via diagonal and via base change. Use

$$\mathrm{co} : \mathsf{Vect} \xrightarrow{\Delta_* p^*} \mathsf{QCoh}(X \times X)$$
$$\mathrm{ev} : \mathsf{QCoh}(X \times X) \xrightarrow{p_* \Delta^*} \mathsf{Vect}$$

This is halfway to HKR.

2. Recall looping delooping: for X connected, $X \sim B\Omega X$. We claimed that $\mathsf{Loc}(X) \simeq C_*(\Omega X) - \mathsf{Mod}$. Now consider $\iota : * \to X$. We can induct $\iota_! k$ First, what is $\operatorname{End}(\iota_! k_x)$? Second, show that the induced $\iota_! k$ is a compact generator for $\mathsf{Loc}(X)$. Third, take $X = BS^1$. Describe $C_*(\Omega BS^1)$ and modules over it. To make things easier assume that the ground field we are taking chains over is $\mathbb{Q}$ (hint: formality).

3. More abstractly, let $\mathcal{C}$ be a (big) dg category. We say that $x \in \mathcal{C}$ is compact if $\mathcal{C}(x, -) : \mathcal{C} \to \mathsf{Vect}$ is continuous. We say that $x \in \mathcal{C}$ is a compact generator if x is compact and the functor $\mathcal{C}(x, -)$ has trivial kernel. Then the basic result/fact of Morita theory is that $\mathcal{C} \simeq \mathsf{Mod} - \mathrm{End}(x)$.

Fact: for $\mathcal{C} = \mathsf{Mod}(R)$, $M \in \mathcal{C}$ is compact if and only if it's a retract of a finitely built module (i.e. repeated cofibers of (shifted) direct sums of R).

Let's return to the algebro-geometric setting. Consider a polynomial map $F : \mathbb{A}^n \to \mathbb{A}^m$ with $F = (f_1, \dots, f_m)$ with $f_i \in k[x_1, \dots, x_n]$ (say nonconstant). Consider the fiber $F^{-1}(0)$. Classically this is the spectrum of the polynomial ring

$$\mathrm{Spec}\, \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_n)}.$$

Often this sequence is not regular (especially if m is very big). Instead we should take some sort of derived or homotopy fiber:

$$\begin{array}{ccc} F^{-1}(0) & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}^m \end{array}$$

We should take $\mathcal{O}$ whence we take Spec of the derived tensor product

$$\mathrm{Spec}(k \otimes_{k[t_1, \dots, t_m]} k[x_1, \dots, x_n]).$$

To compute this derived fiber we resolve k as a commutative dg algebra over $k[t_1, \dots, t_m]$. There is a standard way to do this called the Koszul resolution. What we'll get after the tensor product is the free commutative algebra

$$k[x_1, \dots, x_n, y_1, \dots, y_m]$$

where $|y_i| = -1$ and $dy_i = f_i$. You can check that this has, in degree zero, the cohomology the usual quotient. Play around and notice that you may have cohomology in nonzero degrees.

Let's say some more about the HKR theorem. C.f. Loday for the commutative case. Let's do one example where you see most of what's going on. So remember the claim was that

$$HH_*(\mathsf{QCoh}(X)) \simeq \Gamma(X, \oplus \Omega^p[p]).$$

Let's consider the proof for $X = \mathbb{A}^n$ (in general use free resolutions). There would be an intermediate step

$$HH_*(\mathsf{QCoh}(X)) \simeq \Gamma(X \times X, \Delta_* \mathcal{O}_X \otimes \Delta_* \mathcal{O}_X) \simeq \Gamma(X, \Delta^* \Delta_* \mathcal{O}_X)$$

and recall that everything is derived so you need to resolve the sheaf first. Koszul resolution of $\Delta_* \mathcal{O}_{\mathbb{A}^n}$ on $\mathbb{A}^n \times \mathbb{A}^n$ with coordinates x, x'. Take $f_i = x_i - x'_i$.

Let's back up a bit and describe the lower-shriek operator. Recall that $\mathsf{Loc}(X) = \mathrm{Fun}^\infty(X, \mathsf{Vect})$. Given a map $f : X \to Y$ there is an obvious restriction functor $\mathsf{Loc}(Y) \to \mathsf{Loc}(X)$. One definition of $f_!$ is that it is the left adjoint of f^*. This is a good definition if you can show that it exists. Say we choose $x \in X$. Then we have a map of chains on based loop spaces $C_*(\Omega X) \to C_*(\Omega Y)$. If X, Y are $K(\pi, 1)$'s then this is really just a map on π_1's and on the group algebras. But for any map of dg algebras $R \to S$ we can restrict and induce representations. The induction is given $- \otimes_R S$ and the restriction is given $\mathrm{Hom}_S(S, -)$.

Let's sketch the solution to the topological exercise above. Notice that

$$\mathrm{End}(\iota_! k) \simeq \mathrm{Hom}(\iota_! k, \iota_! k) \simeq \mathrm{Hom}(k, i^* \iota_! k)$$

But using base change we obtain

$$\mathrm{End}(\iota_! k) \simeq \mathrm{Hom}(k, \pi_! \pi^* k) \simeq C_*(\Omega_x X; k)$$

where π is the map from $\Omega_x X \to \{x\}$. This last equivalence you want to think of as coming from the chain level Eilenberg-Steenrod axioms. Local systems on a point are completely determined by what happens at the point as spaces are just homotopy colimits of points.

Now for the last part of the problem notice that $C_*(\Omega BS^1) \simeq C_*(S^1)$ is a dg algebra. In characteristic zero one shows that any dg algebra is quasi-isomorphic to its homology as an algebra. In our case we have (homologically) $\mathbb{Q}[\varepsilon]/\varepsilon^2$ where $|\varepsilon| = 1$. Let R be a homologically graded dga whose homology is $\mathbb{Q}[\varepsilon]$ with $\varepsilon^2 = 0$ and $d = 0$. Then it is quasi-isomorphic to $\mathbb{Q}[\varepsilon]$ as a dga. What is a dg module over this algebra? These are called mixed complexes, and it is one way of discussing complexes with an S^1-action. For instance Hochschild homology is a mixed complex under b and B (the latter corresponding to ε). More explicitly given a $\mathbb{Q}[\varepsilon]$-module E_*, it is a complex under a differential b together with a square-zero grading-increasing differential B. Notice that Leibniz tells us that, since $d\varepsilon = 0$

$$b(Be) = b(B)e - B(be) = -B(be).$$

We conclude that $bB + Bb = 0$. Notice that we should not think of this as a double complex! Quasi-isomorphism between these mixed complexes does not care about B! It is just a quasi-isomorphism of the underlying complexes.

You could instead consider modules over cochains over S^1. This would be a very different category: you'd get instead unipotent local systems.

We have $\mathsf{Loc}(BS^1) \simeq \mathsf{Mod}k[\varepsilon]$ which is just mixed complexes. We have maps $p_!, p_* \mathsf{Loc}(BS^1) \to \mathsf{Vect}$ and $p^* : \mathsf{Vect} \to \mathsf{Loc}(BS^1)$. Notice that

$$p_! E \simeq k \otimes_{k[\varepsilon]} E, \qquad p_*(E) \simeq \mathrm{Hom}_{k[\varepsilon]}(k, E)$$

which we think of as coinvariants and invariants of S^1 respectively. In the cyclic homology literature these are known as cyclic chains $HC_*(E)$ and negative cyclic chains $HC_*^-(E)$.

What about periodic cyclic homology? Note that endomorphisms of the trivial module k, $\mathrm{End}_{k[\varepsilon]}(k)$ acts on $HC^-(E)$. Here's a fun exercise (never read a book on cyclic homology just do this exercise). Resolve k as a $k[\varepsilon]$-module. There's only one way to do this unless you're really crazy. Using this resolution you will find the standard complexes for cyclic and negative cyclic chains and you will be able compute this endomorphism algebra.

Under the equivalence between $k[\varepsilon]$-modules and local systems on BS^1 k and k_{BS^1} are identified. Hence their endomorphisms are equivalent. But $\mathrm{End}(k_{BS^1}) \simeq C^*(BS^1, k) = k[u]$, which has cohomology of one-dimension in each degree and in fact turns out to be formal in characteristic zero. Here $|u| = 2$ (cohomological). Now $k[u]$ acts on $\mathrm{End}_{k[\varepsilon]}(k)$ and one obtains periodic cyclic homology

$$HP(E) = HC^-(E)[u^{-1}].$$

Why is this something you might be interested in? Well consider the HKR theorem. For X a smooth affine variety we have

$$(HH_*(\mathsf{QCoh}(X), b, B) \simeq (\oplus\Omega^p_X[p], 0, d_{dR}).$$

If X were singular you'd have the cotangent complex instead. Let's isolate the component p, $HH_*(X)(p) = \Omega^p_X[p]$. Now $HC^-_*(p) = (0 \to \cdots \to \Omega^p_X \to \cdots \to \Omega^d_X)[2p]$ is the pth level of the Hodge filtration. Think of $HC^- \to HP$ giving a Hodge filtration for de Rham cohomology where we think of HP as the noncommutative de Rham cohomology.

Here are some recommended references:

- Gaitsgory has a short note on dg categories
- Gaitsgory-Rozenblyum
- for more on local systems there's Lurie's DAG XIII on rational homotopy theory
- Lurie's notes on surgery theory have some nice examples

19. June 19, 2018 – Ezra Getzler

Let's start with a calculation: Brown's lemma. There are two different ways of stating it. The first that all morphisms can be factorized, or alternatively, the diagonal maps $Y \to Y \times Y$ can be factorized. Recall that since everything is fibrant we can consider the pullback $Y \to * \leftarrow Y$, which is denoted $Y \times Y$. Any factorization

$$Y \xrightarrow{\sim} PY \twoheadrightarrow Y \times Y.$$

we call the path space. Since $PY \to Y \times Y$ is a fibration, the two projections $PY \to Y$ are trivial fibrations. Thus for any X,

$$X \times_Y PY \overset{\twoheadrightarrow}{\sim} X$$

has a section $s : X \to X \times_Y PY$ whence since PY has a map to Y, we obtain a factorization of any map $X \to Y$ through $X \times_Y PY$. In other words, every map factors into a section of a trivial fibration followed by a fibration. The map is a weak equivalence if and only if the fibration is a trivial fibration.

Now given any functor $F : \mathcal{V} \to \mathcal{C}$ a functor from a category of fibrant objects to a category with weak equivalences, if it takes trivial fibrations to weak equivalences then it takes weak equivalences to weak equivalences, i.e. it is a homotopy functor. The theme here is that weak equivalences are often hard to get a grip on, whence we work with trivial fibrations instead. We will explore this theme in the context of Lie groupoids.

Recall from last time that Kan complexes of bounded size form a category of fibrant objects. We defined the fibrations and trivial fibrations last time. It is then a bit of a story to check that you actually have a category of fibrant objects. For instance you should check that every trivial fibration or isomorphism is a fibration. What about something like 2-out-of-3? Suppose we have a pair of fibrations.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

If f is a trivial fibration and the composite is a trivial fibration then it is easy to show that g is. It is much harder to do the same if we swap f and g in that statement (see Ezra's paper). For 2-out-of-6 in a category of fibrant objects, it was shown by Blumberg-Mandell in a slightly different setting, which still works, is equivalent to

2-out-of-3 together with the axiom that the retract of a weak equivalence is a weak equivalence.

Our approach to Lie groupoids now, will be to completely imitate Kan complexes. In particular, take $\mathcal{V}$ to be our (small) category of "spaces", which we assume to have finite limits and we assume to be equipped with a subcategory of "covers". These covers must satisfy that: every isomorphism is a cover, pullbacks of covers are covers, and a cancellation axiom:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

if f and gf are covers then g is a cover. This last axiom is not generally taken classically, but if you check the Stacks project you'll find that all of them satisfy the cancellation property. Notice that the subcanonicalness is not as important in the derived case.

Example 52. Let $\mathcal{V}$ be the category of finite sets with the subcategory of covers being the surjective functions. This will recover the finite Kan complexes.

Example 53. The example we want to focus on is that of (higher) Lie groupoids. We can't consider manifolds since they don't have finite limits (well ok one could work with C^∞-schemes, etc.) so maybe consider $\mathcal{V}$ to be complex analytic spaces where covers are surjective submersions. There is no particular reason to work finite-dimensionally — instead one might take $\mathcal{V}$ to be a Banach analytic spaces (originally due to Douady).

Fix $k \in \{0, 1, \ldots, \infty\}$. Define $\mathsf{s}\mathcal{V}$ to be the category of simplicial spaces. This will actually be way too general for our purposes.

Definition 54. We say that $f : X_\bullet \to Y_\bullet$ is a fibration of simplicial spaces if for every $n \geq 0$ and $0 \leq i \leq n$, the map

$$X_n \to \mathrm{Hom}(\Lambda^n_i, X) \times_{\mathrm{Hom}(\Lambda^n_i, Y)} Y_n$$

is a cover. A trivial fibration is a map for which ($n \geq 0$)

$$X_n \to \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} Y_n.$$

Now the same proofs in the case of Kan complexes will go through, replacing everywhere "surjective" with "cover.

Now we turn to an idea of Duskin, from the 70's, which he called k-dimensional hypergroupoids. But after working in the field for 20 years, Ezra has decided that there's no better definition of a k-groupoid...

Definition 55. A simplicial space $X_\bullet$ is a k-groupoid if

$$X_n \to \mathrm{Hom}(\Lambda^n_i, X)$$

is a cover for $n > 0$ with $0 \leq i \leq n$ and an isomorphism if $n > k$.

Notice that for $n = 1$ we are roughly generalizing that source and target maps in a Lie groupoid are submersions.

Theorem 56 (Behrend-Getzler). *The category of k-groupoids forms a category of fibrant objects.*

The basic strategy is the to phrase things so that everything is phrased in terms of covers.

Notice that in general a k-groupoid is $(k+1)$-coskeletal (assuming that retracts of covers are covers, which is very likely an innocuous condition). This means that

$$X_n \cong \mathrm{Hom}(\mathrm{sk}_{k+1}\Delta^n, X).$$

For $k = 1$ we obtain just Lie groupoids. The basic story, then, is that X_0 will be the space of objects of our groupoids and X_1 will be the space of morphisms. Now notice that $\mathrm{Hom}(\Lambda^2_1, X)$ is the space of composable morphisms and we ask that X_2 be isomorphic to this space. Similarly for Λ^2_0 and Λ^2_2, which gives us left and right inverses.

Let A^* be a dg algebra (really this should be a category). Assume that $A^i = 0$ if $i \leq -k$. From this data we will produce a Lie k-groupoid. We think of this as the classifying space of the (homotopy) invertible elements in A. Define $E(n) = N_\bullet[\![n]\!]$, i.e. the nerve of the groupoid where we take the category $[n]$ and invert all morphisms. It's a rather stupid groupoid as it has one object and everything is isomorphic in one way; in that sense it's a presentation of the trivial groupoid. The terminology $E(n)$ is due to Rezk and Joyal calls $J = E(1)$ as it is an interval object.

Given $\mu \in A^1$ we have a map $A^1 \to A^2$ sending

$$\mu \mapsto d\mu + \mu^2.$$

This Maurer-Cartan map/equation describes perturbations of the differential of A. Notice that if we define $d_\mu = d + \mathrm{ad}(\mu)$ i.e. $d_\mu x = dx + \mu x \mp x\mu$, then $d_\mu^2 = 0$ if $d\mu + \mu^2 = 0$. This is of course the condition that a connection be flat but in this context probably goes back to Kodaira and Spencer. Now we define

$$\mathbb{N}_n A = \mathsf{MC}(C^*(E(n)) \otimes A^*)$$

where MC is the Maurer-Cartan locus and C^* is actually normalized cochains.

Theorem 57. *The nerve $\mathbb{N}_n A$ is a k-groupoid.*

The hard part of the proof is showing that the source and target maps are covers, i.e.

$$\mathsf{MC}(C^*(E(1) \otimes A^*)) \to \mathsf{MC}(A^*)$$

This is related to the theory of complete Segal spaces somehow.

What if A is an algebra? Then a little calculation shows that for $n = 0$ we just get a point (the unique Maurer-Cartan element). For $n = 1$ you'll find that you get the invertible elements $GL(A)$ of A. In other words we reproduce the theory of the Lie group of invertible elements of A. As an extreme case suppose A has a vanishing product. Then we are working just with chain complexes and one produces the Eilenberg-Maclane space of $\tau_{\leq 1}A[1]$.

There's another construction $N_\bullet A$ where we use the usual simplex, not the fat simplex. One obtains $\mathbb{N}_n A = \mathrm{Hom}(E(n), N_\bullet A)$.

20. June 19, 2018 – Chris Brav

Today I want to talk about something a bit more geometric, an introduction to derived algebraic geometry, in particular how to deal with differential forms. The point of derived algebraic geometry is to deal with degenerate situations, particularly intersections. Yesterday we saw an example where we took a homotopy pullback to obtain a free loop space. In classical algebraic geometry, however, there is no a priori notion of homotopy — identifying two things sets them equal. To take a derived fiber product, for instance, we need to 'resolve' rings of functions. For simplicity we will work in characteristic zero.

Functions in derived algebraic geometry form commutative dg algebras. In the affine case we assume that our functions are bounded above at zero:

$$\cdots \to A^{-2} \to A^{-1} \to A^0 \to 0.$$

We denote the category of such $\mathsf{cdga}_k^{\leq 0}$.

Example 58. Recall the Koszul cdga from yesterday. Let $F : \mathbb{A}^n \to \mathbb{A}^m$ be a map given by $f_1, \dots, f_m$. We obtained a cdga $k[x_1, \dots, x_n, y_1, \dots, y_m]$ with $|y_i| = -1$ and $dy_i = f_i$.

Example 59. The following is a nonexample. Let X be a smooth affine variety. Then its de Rham complex

$$\mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^d_X$$

is not an example! Indeed, it is functions on some sort of stack instead of an affine.

Very formally, the category of derived affine schemes is given

$$\mathsf{Aff} = (\mathsf{cdga}^{\leq 0})^{\mathrm{op}}$$

where a cdga A corresponds to $\operatorname{Spec} A$. Let $U = \operatorname{Spec} A$. Then we define

$$\mathsf{QCoh}(U) = \mathsf{Mod}_A,$$

the dg category of dg-modules over A. Because A is connective, truncation $\tau^{\leq k} M$ is again an A-module (for $k \geq 1$). This is good because we get a t-structure.

Now given a cdga A the cohomology in degree zero $H^0(A)$ is a usual commutative algebra with a map $A \to H^0(A)$ of cdgas. Dually, we obtain a map

$$\operatorname{Spec} H^0(A) \hookrightarrow \operatorname{Spec} A,$$

which is some kind of infinitesimal thickening in a way that can be made precise.

What is the derived analog of derivations? Classically a derivation is a k-linear map $\delta : A \to M$ satisfying the Leibniz rule. If A is a cdga and M is a dg module, we use the same definition except we use graded Leibniz. The usual K ahler differentials Ω^1_A is the universal module that receives a derivation from A, $A \xrightarrow{d_{\mathsf{dR}}} \Omega^1_A$ such that $\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega^1_A, M)$. We can do the same corepresentability in the derived setting:

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_A(\Omega^1_A, M),$$

where Ω^1_A is now a complex. **Warning**: Ω^1 does not respect quasi-isomorphisms of cdgas. It does respect quasi-isomorphisms for (at least) quasi-free cdgas (graded free after forgetting differentials). The basic example is the Koszul complex from before. Claim: this is one way to compute the cotangent complex $\mathbb{L}_A$ in characteristic zero.

Example 60. Let's do the Koszul cdga A. Over A,

$$\Omega^1_A = \langle d_{dR} x_1, \dots, d_{dR} x_n, d_{dR} y_1, \dots, d_{dR} y_n \rangle$$

where $|dx_i| = 0$ and $|dy_i| = -1$ with differential $d = d_\Omega$

$$d_\Omega(d_{dR} y_i) = d_{dR}(d_A y_i) = d_{dR} f_i = \sum \frac{\partial f_i}{\partial x_j} d_{dR} x_j$$

where first equality follows since the universal derivation is a map of complexes. So K ahler differentials for the Koszul cdga detect the rank of the Jacobian dF. Notice that if F is not a submersion then there is cohomology in degree -1 of Ω^1_A.

To pass to the nonaffine setting we will use the functor of points. In this derived setting, to get a good Yoneda lemma, has to be valued in Spaces or homotopy types or ∞-groupoids (all of these are equivalent in all models for higher categories). We will just call them Spaces here. Thus the most general object in derived geometry is an ∞-functor from cdgas to spaces.

Definition 61. A (derived) prestack X is an ∞-functor

$$\mathsf{Aff}^{\mathrm{op}} = \mathsf{cdga}_k^{\leq 0} \xrightarrow{X} \mathsf{Spaces}.$$

The ∞-category of all such we denote by PrStk.

This is doing two things at once — generalizing our affines to derived affines and generalizing stacks to higher stacks (we are valued in homotopy types instead of just groupoids).

Among these presheaves there are those that are representable:

$$(\operatorname{Spec} A)(B) = \operatorname{Maps}(A, B).$$

This is just the Yoneda embedding

$$\mathsf{Aff} \hookrightarrow \mathsf{PrStk}.$$

What's a nonaffine example? Let's suppose that G is a group in PrStk, e.g. something like GL_n. Then we have the usual simplicial diagram associated to G. Taking the geometric realization (homotopy colimit) we obtain, by definition, the stack BG.

Notice that PrStk has an internal hom, right adjoint to the product $\times$. Consider the following example. If $K \in \mathsf{Spaces}$ then we obtain a constant prestack $A \mapsto K$ which we call just K. Hence $\mathsf{Spaces} \hookrightarrow \mathsf{PrStk}$ is a fully faithful embedding. In particular, consider

$$LX := \mathsf{PrStk}(S^1, X).$$

Writing the circle as a homotopy pushout

$$\begin{array}{ccc} S^0 & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & S^1 \end{array}$$

we obtain

$$LX \simeq \mathsf{PrStk}(*, X) \times_{\mathsf{PrStk}(S^0, X)} \mathsf{PrStk}(*, X) \simeq X \times_{X \times X} X.$$

This is an object that is not meaningful in classical algebraic geometry but is meaningful here. For instance the underived pullback would just be X again.

Now we pass to quasi-coherent sheaves. Let X be a (derived) prestack. By definition we know how to map affines into it. Whatever a qc sheaf is, we should be able to pull it back along such a map. And if I had factorization of the map I should get a commutative diagram. We just take this as the definition.

Definition 62. The category $\mathsf{QCoh}(X)$ is the category of quasi-coherent sheaves on affines mapping to X that are compatible under pullbacks. Formally

$$\mathsf{QCoh}(X) = \lim_{(\mathsf{Aff}/X)^{\mathrm{op}}} \mathsf{QCoh}(U).$$

where the limit is computed in $\mathsf{dgCat}_{\mathrm{cont}}$ (or alternatively what Jacob Lurie calls Pr^L, presentable ∞-categories and in fact even just in Cat_∞).

Likewise $\mathsf{Perf}(X) = \lim \mathsf{Perf}(U)$.

Remark 63 (Comparison to quasi-coherent sheaves on schemes). Traditionally quasi-coherent sheaves on schemes are defined by gluing in the Zariski topology or the faithfully flat topology (fppf). To compare, you just have to check that our construction satisfies descent.

Example 64. Here's an example of a prestack that is really not a scheme. Let X_{dR} be the de Rham stack of X, given

$$X_{\mathsf{dR}}(A) = X(H^0(A)/\text{nilradical}).$$

In other words, X_{dR} identifies nearby points. For instance there is a map $X \to X_{\mathsf{dR}}$ that identifies infinitesimally nearby points (in the form of maps $\operatorname{Spec} A, \operatorname{Spec} A'$ into X that differ only by nilpotents).

Now we define

$$\mathscr{D}{-}\mathsf{Mod} := \mathsf{QCoh}(X_{\mathsf{dR}}).$$

If X is smooth then this is the usual notion of a $\mathscr{D}$-module. This is also the same as the definition when you have singularities. This idea actually goes back to Grothendieck probably, c.f. "Dix exposes de cohomologie".

Notice that each prestack has a structure sheaf (coming from the structure sheaves of affines mapping in). Then it turns out that

$$\Gamma(X_{\mathsf{dR}}, \mathcal{O}_{X_{\mathsf{dR}}}) \simeq H^*_{\mathsf{dR}}(X).$$

21. June 20, 2018 – Laura Wells

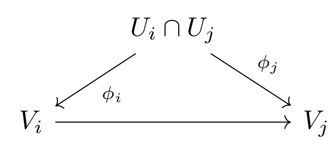
Recall that there are two different ways of talking about sheaves: one, we might have sheaves on a specific manifold, or two, we might have a sheaf on the site of all manifolds of dimension n. There is analogous notion for factorization algebras. So far people have mostly been discussin factorization algebras on a fixed manifolds. We'll describe the second approach.

Our plan is to first define these $\mathcal{G}$-factorization algebras and then relate them to equivariant factorization algebras. We will then sketch a proof of the fact that these are equivalent.

Why might we be interested? Well the latter approach gives a nice categorical interpretation of the descent condition. Moreover Scheimbauer has shown (in her thesis) that locally constant factorization algebras yield fully extended TFTs. More generally Dwyer-Stolz-Teichner are showing (work in progress) that $\mathcal{G}$-factorization algebras yield twisted $\mathcal{G}$-field theories.

To start off I should tell you the site of manifolds that we're working with. Fix M an n-manifold and a Lie group G which acts smoothly $G \times M \to M$.

Definition 65. We define the symmetric monoidal category $\mathcal{G}\mathsf{Man}$ whose objects are manifolds X equipped with a $\mathcal{G}$-structure. This means that X has a maximal atlas $\{U_i, \phi_i)$ where $\phi : U_i \xrightarrow{\sim} V_i \subset M$ and moreover we have $\{g_{ij} \in G\}$ such that



commutes and satisfying $g_{jk}g_{ij} = g_{ik}$. The morphisms in this category are maps $f : X \to Y$ together with $\{f_{i'i} \in G\}$ for all pairs of charts with

$$\begin{array}{ccc} U_i & \xrightarrow{f} & U'_i \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i'}} \\ M & \xrightarrow{f_{i'i}} & M \end{array}$$

satisfying $f_{j'j}g_{ji} = g_{j'i'}f_{i'i}$. The monoidal structure is given by disjoint union.

Example 66. One can define a Euclidean structure, where $M = \mathbb{R}^n$ and $G = \mathbb{R}^n \rtimes O(n)$. More interestingly we might define a Euclidean spin structure, where $M = \mathbb{R}^n$ and $G = \mathbb{R}^n \rtimes Spin(n)$, or a conformal Euclidean structure where $M = \mathbb{R}^n$ and $G = \mathbb{R}^n \rtimes (SO(n) \times \mathbb{R}_+)$.

Definition 67. A $\mathcal{G}$-factorization algebra is a lax symmetric monoidal functor

$$\hat{F} : \mathcal{G}\mathsf{Man} \to \mathsf{Ch}$$

satisfying

(1) multiplicative axiom: $X_1, \ldots, X_n \in \mathcal{G}\mathsf{Man}$,
$$\hat{F}(X_1) \otimes \cdots \otimes \hat{F}(X_n) \xrightarrow{\sim} \hat{F}(X_1 \sqcup \cdots \sqcup X_n)$$
(2) descent axiom: for any Weiss cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $X \in \mathcal{G}\mathsf{Man}$, there is a weak equivalence
$$\text{hocolim}\left(\oplus\hat{F}(U_{\alpha_0}) \to \oplus\hat{F}(U_{\alpha_0} \times U_{\alpha_1} \to \cdots\right) \xrightarrow{\sim} \hat{F}(X)$$

Recall that a Weiss cover of $X \in \mathcal{G}\mathsf{Man}$ is $\{f_\alpha : U_i \hookrightarrow X\} = \mathcal{U}$ such that for any finite $S \subset X$ there exists some f_α such that $f_\alpha(U_\alpha)$. Notice that this defines a Grothendieck topology on $\mathcal{G}\mathsf{Man}$.

Let's recall the previous notion of a factorization algebra and extend it to the G-equivariant case.

Definition 68. A G-equivariant factorization algebra on M (connected) is

$$F : \mathsf{Open}(M) \to \mathsf{Ch}$$

with structure maps satisfying associativity and commutativity satisfying multiplicativity and descent. Moreover for each $g \in G$ and $U \subset M$ we requite equivalences $\sigma_g^U : F(U) \xrightarrow{\sim} F(gU)$ such that

(1) $\sigma_1 = \text{id}$;
(2) $\sigma_{gh} = \sigma_g \circ \sigma_h$;
(3) we have a commutative diagram

$$\begin{array}{ccc} F(U_1) \otimes \cdots \otimes F(U_n) & \longrightarrow & F(gU_1) \otimes \cdots F(gU_n) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(gV) \end{array}$$

Notice that now instead of only looking at maps induced from $U \subset V$ we can now look at maps coming from the fact that U can be G-translated to be in V.

The following theorem is work in progress.

Theorem 69 (Wells)**.** *G-equivariant factorization algebras on M are equivalent (as dg categories) to $\mathcal{G}$-factorization algebras.*

To prove this it is useful to pass through an intermediary category.

Definition 70. Define Disj_M to be the subcategory of $\mathcal{G}\mathsf{Man}$ consisting of finite disjoint unions of connected opens in M. This inherits the symmetric monoidal product of disjoint union.

This is similar to $\mathsf{Disk}_{/M}$ from Lauren's talk, but here we are not requiring the opens to be disks. Moreover we are not remembering that they be embedded in M (this way we can have things like $M \sqcup M$.

Proposition 71. *G-equivariant factorization algebras on M (à la Costello-Gwilliam) are equivalent to symmetric monoidal functors $\mathsf{Disj}_M \to \mathsf{Ch}$ which satisfy multiplicativity and Weiss descent.*

Why does this work? Given a Weiss cover of a disjoint union one can decompose it to Weiss covers for each open, from which the conditions will follow. I don't want to go into the details here, and instead I want to sketch a proof of the theorem.

Proof sketch. To go from $\mathcal{G}$-equivariant factorization algebras just consider

$$\begin{array}{ccc} \mathsf{Disj}_M & & \\ \downarrow & \searrow^{F:=\hat{F}|_{\mathsf{Disj}_M}} & \\ \mathcal{G}\mathsf{Man} & \xrightarrow{\hat{F}} & \mathsf{Ch} \end{array}$$

The other direction is more interesting. We left Kan extend

$$\begin{array}{ccc} \mathsf{Disj}_M & \xrightarrow{F} & \mathsf{Ch} \\ \downarrow & \nearrow_{\iota_! F =: \hat{F}} & \\ \mathcal{G}\mathsf{Man} & & \end{array}$$

Let's check multiplicativity of this extension. For $X_1, X_2 \in \mathcal{G}\mathsf{Man}$ we wish to show that we have a weak equivalence $\hat{F}(X_1)\otimes\hat{F}(X_2) \xrightarrow{\sim} \hat{F}(X_1 \sqcup X_2)$ (say $X := X_1 \sqcup X_2$). Well we have an explicit formula

$$\hat{F}(X) = i_! F(X) = \operatorname{colim}\left(i_{/X} \to \mathsf{Disj}_M \xrightarrow{F} \mathsf{Ch}\right)$$

where we send $(U \to X) \mapsto U \mapsto F(U)$. But notice that $i_{/X} \simeq i_{/X_1} \times i_{/X_2}$. Now we use a Fubini-type property of colimits over product categories.

What about descent. We want to show that for each Weiss cover $\mathcal{U}$ of X we can write $\hat{F}(X)$ as a homotopy colimit over a certain simplicial diagram. Define $\hat{\mathcal{U}}$ to be the full subcategory of $\mathcal{G}\mathsf{Man}_{/X}$ of objects subordinate to $\mathcal{U}$ in the sense that

$$\begin{array}{ccc} Y & & X \\ & \searrow \quad \nearrow & \\ & U_i & \end{array}$$

Now consider the composition

$$\hat{\mathcal{U}} \xrightarrow{j_U} \mathcal{G}\mathsf{Man}_{/X} \to \mathcal{G}\mathsf{Man} \xrightarrow{\hat{F}} \mathsf{Ch}$$

where we call the composition of the last two arrows $\hat{F}$.

But now $\hat{F}$ satisfies descen with respect to a Weiss cover $\mathcal{U}$ of X if and only if

$$\operatorname{colim}(\hat{\mathcal{U}} \xrightarrow{(j_{\mathcal{U}})^* \hat{F}} \mathsf{Ch}) \xrightarrow{\sim} \hat{F}(X).$$

Hence $\hat{F}$ satisfies descent for any $Y \in \mathcal{G}\mathsf{Man}_{/X}$ with respect to $\mathcal{U}|_Y$ (the pullback Weiss cover),

$$(j_{\mathcal{U}})_!(j_{\mathcal{U}})^*\hat{F}$$

i.e. $\hat{F}$ satisfies descent with respect to $j_{\mathcal{U}}$. This will imply descent for $\hat{F}$. □

22. June 20, 2018 – Pelle Steffens

We have a long term goal of understanding the moduli of flat G-connections as a derived smooth Artin stack (and maybe extend this to other solutions of elliptic PDEs). Today we will just try to describe what derived smooth geometry is. Recall that in derived geometry in general we enlarge our category of spaces by add certain limits via passing from algebras of functions to graded algebras of functions and then adding colimits by passing to ∞-groupoid valued sheaves.

The first part consists of passing from manifolds to certain derived smooth affines; this is the main subject of today's talk. Recall that in manifolds there are certain good pullbacks that we can construct, of submersions. Consider however a smooth function $\mathbb{R} \to \mathbb{R}$ whose preimage is the Cantor set. This is of course not a manifold. Sometimes on the other hand we take an intersection which is a manifold but of the wrong dimension. We want to work with objects for which we can define virtual dimension, etc. that is correct/as expected.

In what follows all notions are ∞-categorical.

Definition 72 (Lurie, DAG V)**.** Let T be an ∞-category. An admissibility structure on T is the data of:

- a subcategory T^{ad} containing all objects of T
- a topology on T such that every covering sieve on $X \in T$ contains a covering sieve generated by a collection $\{U_i \to X\}$ satisfying
  - for all $f : U \to X$ admissible and for all $G : Y \to X$, the pullback $Y \times_X U$ exists and the map $Y \times_X U \to Y$ is admissible
  - if $f = gh$ then if f and g are admissible then h is too
  - admissible are stable under retracts

Definition 73 (Lurie, DAG V)**.** A pair (T, T^{ad}) with T an ∞-category with finite products and T^{ad} an admissibility structure is called a pregeometry. A pair $(\mathcal{G}, \mathcal{G}^{ad})$ as above but with $\mathcal{G}$ containing all finite limits is called a geometry.

Example 74. Define $T^{diff} = N(\mathsf{Man})$. The admissibility structure is $(T^{diff})^{ad}$ consisting of open embeddings. We could also have chosen surjective submersions (maybe not says Ezra?). This would yield the smooth topology instead of the étale topology. For the purposes of adding limits they should both work the same. A collection $\{U_i \to X\}$ generates a covering sieve if and only if U_i cover X.

Example 75. Another example is Lawvere theories. We take $T = \{\mathbb{R}^0, \mathbb{R}^1, \dots\}$ Cartesian spaces with the trivial admissibility structure where only equivalences are admissible.

A small variant of this would be to take T Cartesian spaces where the morphisms are polynomial maps.

Example 76. If you wish to work analytically take $T^{an} = N(\mathsf{Cplx})$ and the admissibility structure to be the open immersions. See for instance Mauro Porta's thesis and papers.

Example 77. Consider $T_k^{et} = N(\mathsf{CAlg})^{\mathrm{op}}$ with admissible maps the étale maps. This will recover derived algebraic geometry.

Now if T is a pregeometry and $\mathcal{C}$ an ∞-category with finite limits, we denote $\mathrm{Fun}^{ad}(T, \mathcal{C})$ to denote functors preserving pullbacks along admissible maps.

Definition 78. Let T be a pregeometry. Then the derived geometry associated to T is a geometry $\mathcal{G}$ equipped with a map $\phi : T \to \mathcal{G}$ with $\phi \in \mathrm{Fun}^{ad}(T, \mathcal{G})$ such that for any ∞-category $\mathcal{C}$ with finite limits such that

$$\mathrm{Fun}^{lex}(\mathcal{G}, \mathcal{C}) \to \mathrm{Fun}^{ad}(T, \mathcal{C})$$

is an equivalence (here lex means left exact).

Of course $\mathcal{G}$ always exists by adding certain limits but we would like a less abstract description that we can actually do geometry with.

So our goal is now to find $\mathcal{G}$ for various pregeometries.

Example 79. The category $\mathsf{cdga}^{\leq 0}$ is the derived geometry for the example of commutative algebras.

What about the differentiable case?

Example 80. In the differentiable case we take $\mathcal{G}^{diff} = \mathsf{dSmAff}_{fp}$.

Definition 81. A simplicial C^∞-ring is a functor $F : \mathsf{CartSp} \to \mathsf{Spaces}$ such that F preserves products. The underlying algebra we think of as $F(\mathbb{R}^1)$. There is an underlying simplicial commutative algebra coming from the functor from cartesian spaces with polynomial maps to cartesian spaces. This sends $A \mapsto A^{alg}$.

Theorem 82 (Steffens)**.** *There is a functor $C^\infty : T^{diff} \to \mathsf{s}C^\infty\mathsf{Rings}$ factoring through $\mathsf{s}C^\infty\mathsf{Rings}_{fp}$. Then*

$$C^\infty : T^{diff} \to (\mathsf{s}C^\infty\mathsf{Rings}_{fp})^{op}$$

is the derived geometry for T^{diff}.

23. June 21, 2018 – Ezra Getzler

Last time we discussed briefly the Maurer-Cartan locus. We'll talk more about this today. Last time we used it for a dg algebra, but actually it's defined for a dg Lie algebra L^*. It's going to only know about L in degrees one and two:

$$\mathsf{MC}(L) = \{\mu \in L^1 \mid d\mu + \frac{1}{2}[\mu, \mu] = 0\}.$$

Recall that a dg algebra gives rise to a dg Lie algebra via the graded bracket $[x, y] = xy - (-1)^{|x||y|}yx$. In this case the Maurer-Cartan equation becomes $d\mu + \mu^2 = 0$. Hence this is a generalization from last time.

The key idea is that all standard deformation problems can be reformulated as a Maurer-Cartan locus. Suppose for instance we have an algebra $\mathcal{A}$ and we are interested in deforming its product. Then Gerstenhaber introduced the dg Lie algebra of Hochschild cochains $C^*(\mathcal{A}, \mathcal{A})$:

$$C^k(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes k}, \mathcal{A}).$$

The bracket is

$$[-,-] : C^i \times C^j \to C^{i+j-1}$$

so really we should take $L^* = C^{*+1}$. It turns out that the Maurer-Cartan equation says that

$$a_1 *_\mu a_2 = a_1 a_2 + \mu(a_1, a_2)$$

is an associative product. We think of this as a deformation of the usual product.

The one in the back of our minds last time was that of a compact complex manifold X. Let $E \to X$ be a holomorphic vector bundle and consider the Dolbeault resolution

$$A^{0,*}(X, \mathrm{End}(E)).$$

In this case the Maurer-Cartan elements are deformations of the vector bundle. This comes about because the element μ is a (0,1)-form whence it yields a deformation of the $\bar\partial$ operator (this requires some analysis to show that one can in fact find a local holomorphic frame). This example can be generalized in many ways. Suppose for instance E^* is a complex of vector bundles. Then

$$\mu \in \oplus_{q=0}^{d=\dim X} A^{0,q}(X, \mathrm{End}(E^*)^{1-q}).$$

Here End is the complex linear endomorphisms. In general now μ allows deformations into a twisted complex of vector bundles. This is related to the fact that the best notion of a perfect complex on a compact complex manifold is a twisted complex.

We now want to derive the Maurer-Cartan locus. There are two basic ways of realizing derived geometry. One is to use dg objects and the other is to use cosimplicial objects. The local story of the dg case goes back at least to 1958 due to Tate (the first paper in derived geometry). The cosimplicial approach dates back to the 60's to Quillen (and Michael Barr) showed that the two approaches are equivalent in characteristic zero. Let's see how the derived Maurer-Cartan locus can be expressed in these two frameworks.

We start with the dg realization. Let L be a dgla. Let M be a manifold and $V^* = \oplus_{i=-\infty}^{1} V^i$ be a graded vector bundle on M. Define for the pair $\mathcal{M} = (M, V^*)$,

$$\mathcal{O}(\mathcal{M}) = \left(\Gamma(M, \mathrm{Sym}(V^*)), \delta_{\mathcal{M}}\right).$$

This is a dg manifold or scheme. Tate's theorem was that given an affine variety one can can find such an object something something. What we will do is take

$$\mathcal{O}(\mathcal{MC}(L)) = C^*_{\mathsf{CE}}(\sigma_{\geq 1} L^*).$$

In this case $M = L^1$ an affine space, and $V^i = L^1 \times L^{1-i}$. We define the classical locus

$$\pi^0(\mathcal{M}) = \mathrm{Spec}(H^0(\mathcal{O}(\mathcal{M}))).$$

Lemma 83. *The classical locus of the derived Maurer-Cartan locus is naturally isomorphic to the Maurer-Cartan locus:*

$$\pi^0(\mathcal{MC}(L)) \cong MC(L).$$

The other approach is via cosimplicial manifolds (just as simplicial models stackiness). Recall that sufficiently fibrant cosimplicial objects represent homotopy limits. A cosimplicial manifold recall is a functor $M^\bullet : \Delta \to \mathsf{Mfld}$. The main thing to remember for the cosimplicial manifold is that it stands in for the equalizer $\pi_0(M^\bullet)$ of the coface maps $d^0, d^1 : M^0 \to M^1$. This is a prototypical example of a sifted limit. Hence a cosimplicial manifold is a generalization of equations. Hence we might look for a cosimplicial manifold whose corresponding π_0 precisely cuts out the Maurer-Cartan locus. In fact, there is a simpler construction.

Recall that the Dold-Kan correspondence is a correspondence between chain complexes and simplicial abelian groups. For instance C_* corresponds to $K(C_*)$ where

$$K(C_*)_n = Z_0(\mathrm{Hom}(C_*(\Delta^n), C_*)) = Z_0(C^(\Delta^n) \otimes C_*).$$

Notice that this is very close to the construction from last time. Kaledin remarked to Ezra that this works for any abelian category. So apply it to the opposite category of abelian groups: this yields cochain complexes and cosimplicial abelian groups $K(C^*)$. The derived MC locus will be very similar to this, except a nonabelian version.

We're not interested in just any cosimplicial manifold. We want some sort of fibrancy condition. Say we have a simplicial algebra $\mathcal{O}(X^\bullet)$. There is a well-known Reedy cofibrancy condition on this simplicial algebra, which translated back to manifolds yields the following. Define $M^n(X^\bullet)$ to be the matching space

$$M^n(X^\bullet) = \mathrm{eq}\left((X^{n-1})^n \to (X^{n-2})^{\binom{n}{2}}\right)$$

there should be two codegeneracy maps here. The fibrancy condition is as follows: for all $n \geq 0$ the map $M^n \to M^n(X^\bullet)$ is a submersion. This map is roughly all the codegeneracies assembled together. The best book on descent theory according to Ezra is Bousfield-Kan where you read about matching spaces, etc. Let's look at the first few conditions. For $n = 0$ this is saying that $X^0 \to *$ is a submersion, i.e. that X^0 is a manifold. The next condition is that $X^1 \to X^0$ is a submersion. For $n = 2$ this is saying that $X^2 \to X^1 \times_{X^0} X^1$ is a submersion.

Ezra, years ago, wrote down an explicit formula for the derived MC locus using these combinatorics (a bit more elementary than Chevalley-Eilenberg). It goes as follows. Let Λ^n be the exterior algebra generated by the vertices of Δ^n, i.e. we have generators $\{e_1, \ldots, e_n\}$ where $\deg e_i = -1$ and $\delta e_i = 1$ for all $0 \leq i \leq n$. The cosimplicial Maurer-Cartan locus of a dgla L^* is

$$MC^n(L^*) = MC(\Lambda^n \otimes L^*).$$

This yields a cosimplicial scheme. This is more or less combinatorially checked that it is fibrant. It turns out that the two maps $MC^0(L) \to MC^1(L)$ are just the inclusion $L^1 \to L^1 \times L^2$ and the MC equation $x \mapsto (x, \delta x + [x,x]/2)$.

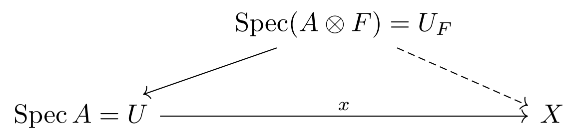
Next we are interesting in discussing derived stacks. Here we've just been writing charts, and we need to assemble these into an atlas. This is going to be in a sequel to Kai and Ezra's paper. Let $\mathcal{V}$ be a category of fibrant objects with a subcategory of covers. The covers will sit between the subcategories of trivial fibrations and fibrations of $\mathcal{V}$. The additional axiom we need, recall from last time, is that if we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ then if f and gf are covers than g is a cover.

24. June 21, 2018 – Chris Brav

Let A be a cdga and M a dg module. some things that I missed

More generally suppose we have a map $B \to A$ of cdgas and a map $B \to A \oplus M \to A$ commuting with the map $B \to A$. Write $f : U \to V$ for the map of affine spaces corresponding to the map $B \to A$. This is the same data of a derivation $B \to f_*M$, which is the same as a B-linear map $\Omega^1_B \to f_*M$, which is the same as an A-linear map $f^*\Omega^1_B \to M$. We now want to globalize this.

Given a prestack $X : \mathsf{Aff}^{\mathrm{op}} = \mathsf{cdga}_k^{\leq 0} \to \mathsf{Spaces}$ and a point $\operatorname{Spec} A = U \xrightarrow{x} X$ and $F \in \mathsf{QCoh}(U)^{\leq 0}$, we look at extensions



Define

$$\mathsf{QCoh}(U)^{\leq 0} \to \mathsf{Spaces}$$

sending $F \mapsto \mathrm{Maps}_{U/}(U_f, X) = \mathrm{Maps}(U_F, X) \times_{\mathrm{Maps}(U,X)} \{x\}$ (this last expression of course being homotopy fiber product).

Definition 84. Say that X has a cotangent space at x if this functor (above) is corepresented by $T_x^*(X) \in \mathsf{QCoh}(U)^-$ (bounded above).

Note that if $X = \operatorname{Spec} B$ then $T_x^*(X) = f^*\Omega_B^1$.

Next we consider how cotangent spaces pullback. Suppose we have $f : U \to V$ and insert diagram here. In other words, we have a precomposition

$$\underset{V/}{\mathrm{Maps}}(V_{f_*F}, X) \to \underset{U/}{\mathrm{Maps}}(U_F, X).$$

Now suppose we have cotangent spaces at x and y. Then

$$\underset{U}{\mathrm{Maps}}(f^*T_y(X), F) \simeq \underset{\mathsf{QCoh}(V)^-}{\mathrm{Maps}}(T_y(X), f_*F) \to \underset{\mathsf{QCoh}(U)}{\mathrm{Maps}}(T_x^*(X), F).$$

You can check that this is natural in f. Now (co)Yoneda yields a map

$$T_x^*(X) \to f^*T_y(X).$$

Definition 85. If the prestack X has all cotangent spaces and this (co)Yoneda map is always an isomorphism then this compatible system of quasicoherent sheaves defines a quasicoherent sheaf on X which we write $T^*(X) \in \mathsf{QCoh}(X)$. We call this the cotangent complex of X.

Remark 86. $T^*(X)$ exists for Artin stacks. See for instance HAG II or Gaitsgory-Rozenblyum.

Definition 87. Let X have a cotangent complex $T^*(X)$. Then a p-form of degree n is a section

$$\alpha : \mathcal{O}_X \to \Lambda^p T^*(X)[n].$$

This definition is fine but it is difficult to describe when they are de Rham closed. We need a quasi-isomorphism invariant notion of closedness. We solve this issue by introducing higher homotopies. There is an inexpensive and an expensive way to do this. The inexpensive way (due to Pantev-Toen-Vaquié-Vezzosi) to do it is easy to define and is good enough for Artin stacks. The expensive way (due to Gaitsgory-Rozenblyum) is via the Hodge filtration on $H_{dR}^*(X)$. Let's talk about the inexpensive way. First define de Rham closed forms on affines.

$$\alpha \in \Omega_U^p[n], \qquad d_{dR}\alpha \in \Omega^{p+1}n + 1].$$

Let's do an example. Recall the Koszul complex. We have $\mathsf{DR}(A) = \mathrm{Sym}_A(\Omega_A^1[-1])$ so we had $d_{dR}x, d_{dR}y$ generators which have degree 1 and 0 respectively. Suppose we have an expression $xd_{dR}y$. Then

$$d_{dR}(xd_{dR}) = d_{dR}x \wedge d_{dR}y$$

whence the shift.

Now given $\alpha \in \Omega^p_U[n]$, we obtain $d_{dR}\alpha \in \Omega^{p+1}[n+1]$. We do not ask for this to be zero. We ask for something in

$$A^{p,cl}(A) := \big((\Omega^p)^n \times (\Omega^{p+1})^{n-1} \times (\Omega^{p+2})^{n-2} \times \cdots, d_\Omega + d_{dR}\big)$$

which means that we ask:

$$\begin{aligned} d_{dR}\alpha_0 + d\alpha_1 &= 0 \\ d_{dR}\alpha_1 + d\alpha_2 &= 0 \\ \vdots &= \vdots \end{aligned}$$

This is, roughly, being de Rham closed up to homotopy. Denote $\Omega^p = \mathcal{A}^p(A)$.

Definition 88. We define the p-forms of degree n to be $\mathcal{A}^p(-,n) : \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Spaces}$ sending $\operatorname{Spec} A \mapsto |\Omega^p[n]|$. Here $|-|$ means to truncate above zero and apply Dold-Kan. Similarly we have the close forms $\mathcal{A}^{p,cl}(-,n) : \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Spaces}$ sending $\operatorname{Spec} A \mapsto |\Omega^p[n] \times \Omega^{p+1}[n-1] \times \cdots|$.

Remark 89. These prestacks are in fact étale stacks as the cotangent complex satisfies étale descent.

This was all just for an affine space. On a general prestack X, we would like to define a (closed) form as a compatible (under pullback) system of closed forms on $\mathsf{Aff}_{/X}$. A concise way of saying this is that the space of p-forms on X of degree n is

$$\begin{aligned} \mathcal{A}^p(X,n) &= \operatorname{Maps}(X, \mathcal{A}^p(-,n)) \\ &= \operatorname{Maps}(\operatorname*{colim}_{\mathsf{Aff}_{/X}} U, \mathcal{A}^p(-,n)) \\ &= \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} \operatorname{Maps}(U, \mathcal{A}^p(-,n)) \\ &\simeq \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} \mathcal{A}^p(U,n). \end{aligned}$$

We do the same for the space of closed p-forms of degree n:

$$\begin{aligned} \mathcal{A}^{p,cl}(X,n) &= \operatorname{Maps}(X, \mathcal{A}^{p,cl}(-n) \\ &\simeq \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} \mathcal{A}^{p,cl}(U). \end{aligned}$$

Remark 90. For X a (higher, locally) Artin stack, PTVV check that

$$\operatorname{Maps}(\mathcal{O}_X, \Lambda^p T^*(X)[n]) \simeq \mathcal{A}^p(X,n).$$

In general the left hand side might be a better definition, though the right hand side has easier to define notions of closedness (no need for Hodge filtrations).

Remark 91. There is a map $\mathcal{A}^{p,cl}(X,n) \to \mathcal{A}^p(X,n)$ that forgets all but the first form.

Let's now go on a bit of a digression about Hochschild-Kostant-Rosenberg (see for instance Loday chapter 5). Let $U = \operatorname{Spec} A$ and consider $(HH_*(U), b, B)$. Just to be careful let's assume that A is almost finite type, which means that $H^0(A)/k$ is finitely generated as a k-algebra and $H^i(A)/H^0(A)$ is finitely generated as a module. One statement of HKR is that as mixed complexes we have

$$(HH_*(U), b, B) \simeq (\operatorname{Sym}_A(\Omega^1_A[1]), d, d_{dR}).$$

Junwu actually wrote down the formula for this map; one checks that it is a map of mixed complexes. Notice here that there is a +1 shift unlike the de Rham complex! So there's a bit of a grading shift going on. We can write/get via HKR, (more precisely there are a few ways of getting at this weight via say Adams operations or S^1-actions, etc.).

$$HH_*(U)(p) \simeq \Omega_U^p[p]$$

and

$$HC^-(U)(p) \simeq \Omega_U^p[p] \times \Omega_U^{p+1}[p+1] \times \cdots$$

so we obtain a way of computing these prestacks of forms and closed forms using Hochschild and cyclic homology. More precisely, we have isomorphisms of prestacks

$$\mathcal{A}^p(-,n) \simeq |HH_*(-)(p)[n-p]|$$

and

$$\mathcal{A}^{p,cl}(-,n) \simeq |HC^-(-)(p)[n-p]|.$$

25. June 21, 2018 – Kai Behrend

The main results of this talk are joint work with H.-Y. Liao and P. Xu.

Problem: The category of C^∞-manifolds lacks fiber products, for instance intersections or fibers or zero loci, etc. We would like to add these fiber products in a way such that things remain smooth and controllable. In other words we would like to construct the ∞-geometry associated to the pregeometry of smooth manifolds with étale maps (recall that étale means local diffeomorphism). What we will end up will be some sort of dg manifolds with étale maps. The explicit model for such things will be as a category of fibrant objects. Notice that dg manifolds contain deformation theory in the sense that they know about the homotopy theory of dg Lie algebras $L = L^{\geq 1}$.

We will start out by examining this case of deformation theory. Let $L = L^1 \oplus L^2 \oplus \cdots$ a dg Lie algebra that is finite-dimensional (the whole algebra). We construct a bundle as follows. Let $M = L^1$ and consider a graded vector bundle over M: $\mathcal{L}^2 = L^2_M, \mathcal{L}^3 = L^3_m, \ldots$ (these are trivial). We have a section $F : M \to \mathcal{L}^2$ coming from the MC equation map $x \mapsto dx + [x,x]/2$. Notice that we obtain vector bundle homomorphisms

$$\mathcal{L}^2 \xrightarrow{d\mu} \mathcal{L}^3 \to \mathcal{L}^4 \to \cdots$$

In the fiber over $\mu \in M$ we have $d_\mu = d + [\mu, -]$. Hence we have $[-,-] : \Lambda^{\mathcal{L}} \to \mathcal{L}$ a bracket which is constant along fibers. One checks that $(M, \mathcal{L}^{\geq 2}, F, d\mu, [-,-])$ is a bundle of curved dg Lie algebras. In other words:

- $d_\mu F = 0$,
- $d\mu^2 = [F, -]$,
- d_μ is a derivation over $[-,-]$ and the bracket satisfies Jacobi.

If $L = L^{\geq 0}$ then one might find a Lie group G such that $\mathsf{Lie}(G) = L^0$. Let G act on this whole structure $(M, \mathcal{L}^{\geq 2}, F, d_\mu, [-,-])$. We now pass to a quotient $(M/G, \ldots)$ a bundle of curved dg Lie algebras on M/G.

Example 92. Let K be a finite simplicial complex and let $L = C^*(K; M_{n\times n})$. This becomes a dg Lie algebra under cup product and bracket of matrices. If we take $C^{\geq 1}(K, M_{n\times n})$ we obtain a derived moduli space of flat connected on the trivial bundle of rank n on K.

Definition 93. A dg manifold is a triple (M, L, λ) where M is a smooth manifold, L is $L = L^2 \oplus \cdots \oplus L^{n+1}$ (here n is called the amplitude) a graded vector bundle over M, and $\lambda = (\lambda_k)_{k\geq 0}$ is a sequence of operations

$$\lambda_k : \Lambda^k L \to L[2-k]$$

a homomorphism of vector bundles. We require that for all $p \in M$, $(L|_p, \lambda|_p)$ is a curved L_∞ algebra, i.e. $\lambda \circ \lambda = 0$.

If we look at $k = 0$ we obtain $\lambda_0 = \mathcal{O}_M \to L[2]$ so just a section of L^2. This is called the "curvature". If we look at $k = 1$ we obtain $\lambda_1 : L^2 \to L^3 \to L^4 \to \cdots$ a twisted differential. For $k = 2$ we obtain $\Lambda^2 L \to L$ a bracket. For $k \geq 3$ we obtain higher brackets.

Example 94. Suppose $n = 0$. Then we obtain just a manifold M. For $n = 1$ we obtain $M \xrightarrow{\lambda_0 = F} L^2$ just a section of a vector bundle. If $n = 2$ then we have $M \to L_2 \to L_3$ where the first map is the section/curvature and $\lambda_1(F) = 0$. To see the brackets we need $n = 3$ in which case $dF = 0$ and $d^2 = [F, -]$.

Remark 95. Let $\mathcal{M} = (M, L, \lambda)$ be a dg manifold. Then $\mathcal{O}(\mathcal{M}) = (\Gamma(M, \mathrm{Sym}(L[1])^\vee))$. Here λ corresponds to a derivation Q on $\mathcal{O}(\mathcal{M})$ satisfying $Q^2 = 0$.

Definition 96. A morphism $(f, \phi) : (M, L, \lambda) \to (N, L', \lambda')$ is a smooth map $f : M \to N$ and $\phi = (\phi_k)_{k\geq 1}$ where $\phi_k : \Lambda^k L \to f^* L'[1-k]$ such that for all $p \in M$ with $q = f(p)$,

$$\phi_k|_p : \Lambda^k L|_p \to L'|_q[1-k]$$

is a morphism of curved L_∞-algebras (this is roughly a morphism respecting the bracket structures).

See for example Getzler's MC homotopical perturbation theory for all of this written out.

Now let $\mathcal{M} = (M, L, \lambda)$ be a dg manifold. Consider $\pi^0(M) := Z(F) \subset M$ closed. This is the classical Maurer-Cartan locus of $\mathcal{M}$.

Remark 97. It is an exercise to check that $\pi^0(\mathcal{M}) = \mathrm{Maps}(*, \mathcal{M})$

Take $p \in \pi^0(\mathcal{M})$ a classical point. We define the tangent complex of $\mathcal{M}$ at p to be

$$T\mathcal{M}|_p = TM|_p \xrightarrow{dF} L^2|_p \to L^3|_p \to \cdots.$$

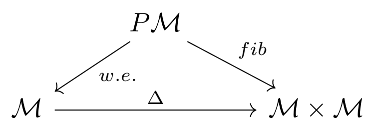
This is a complex of vector spaces since we are at a classical point ($d_\mu^2 = 0$).

Definition 98. We say that a morphism $f : \mathcal{M} \to \mathcal{N}$ is étale at $p \in \pi^0(\mathcal{M})$ if $T\mathcal{M}|_p \to T\mathcal{N}|_{f(p)}$ is a quasi-isomorphism.

We say that $f : \mathcal{M} \to \mathcal{N}$ is a fibration if the underlying map $f : M \to N$ is a submersion and $\phi_1 : L \to f^* L'$ is a epimorphism of graded vector bundles (i.e. surjective everywhere). A map is a weak equivalence if it is étale and a bijection on classical loci.

For instance the map $\varnothing \to \mathcal{M}$ is a weak equivalence if and only if $\pi^0(\mathcal{M}) = \varnothing$. The restriction to an open neighborhood of the classical locus in $\mathcal{M}$ includes into $\mathcal{M}$ and this inclusion is a weak equivalence.

One checks that we obtain a category of weak equivalences (also the 2-out-of-6 property, probably). To make this ∞-category tractable and to make the fiber products/hom-spaces more explicit, we introduce fibrations. Indeed we now claim that this category of dg manifolds is in fact a category of fibrant objects. The axioms are relatively easy to check straight from the definition, except for one. The factorization axiom is harder to check, but it turns out it suffices to check the diagonal $\mathcal{M} \xrightarrow{\Delta} \mathcal{M} \times \mathcal{M}$.



Let's explain the existence of such a path space. As a warm-up let's consider M as a manifold. Define in this case $PM = \mathrm{Maps}(I, M)$, which is of course infinite-dimensional. Notice that the map $M \to PM$ in this case is not a weak equivalence. Consider TPM the tangent bundle of the path space. Notice that if $a : I \to M$ is a path then $TPM|_a = \Gamma(I, a^*TM)$ and we obtain naturally a section a' whence a map of dg manifolds $PM \to TPM$. Notice that M near the constant path $M \to PM$ yields a bijection $M \xrightarrow{const} \pi^0(PM \to TPM)$. As an exercise check that the map $M \to PM$ is étale. As a result the map is a weak equivalence. Of course $PM \to M \times M$ is a fibration so we have solved our problem, up to the fact that we're using an infinite-dimensional model for the path space PM. We will cut down to finite dimensions as follows: choose a connection ∇ on TM and notice that a path $[0, 1] \xrightarrow{a} M$ is geodesic if a' is covariant constant as a section of a^*TM with respect to the connection $a^*\nabla$. Now define P_gM to be the geodesic paths. Now we can argue by working in small neighborhoods of M (can do this due to classical locus argument) and using the exponential map.

26. June 22, 2018 – Ezra Getzler

Let $\mathcal{V}$ be a category of spaces. If you remember, we're not going to assume it has all finite limits. Instead we are going to carefully keep track of which limits we do have. Recall that we have

$$\text{isomorphisms} \subset \text{trivial fibrations} \subset \text{covers} \subset \text{fibrations}.$$

Remark that we will want covers to be preserved by forming retracts. Our basic examples will be categories of derived spaces of some sort.

Consider, for instance, open subsets of the derived Maurer-Cartan loci of dg Lie algebras. In this context fibrations are given by submersions (surjections of dg Lie algebras). Recall that the classical locus

$$\pi^0(\mathcal{MC}(L)) = MC(L).$$

Fix $\mu \in MC(L)$ and consider $L_\mu = (L, d_\mu)$ where $d_\mu = d + \mathrm{ad}(\mu)$. One finds that $T_\mu\mathcal{MC}(L) = L_\mu[1]$. What are trivial fibrations in this context? They are fibrations that induce isomorphisms on π^0 and quasi-isomorphism of tangent complexes. Covers are similar: they are fibrations which are surjective submersions on π^0 and moreover quasi-isomorphisms of tangent complexes in non-zero degrees.

Denote by $\mathsf{s}\mathcal{V}$ the simplicial objects of $\mathcal{V}$. This again has the structure of a category of fibrant objects. We are going to be considering a map $f : X_\bullet \to Y_\bullet$. We say that f is a fibration if it is Reedy fibrant, i.e. the map

$$X_n \to \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y} Y_n$$

is a fibration in $\mathcal{V}$ for all $n \geq 0$, and moreover that

$$X_n \to \mathrm{Hom}(\Lambda^n_i, X) \times_{\mathrm{Hom}(\Lambda^n_i, Y)} Y_n$$

is a cover in $\mathcal{V}$ (here $n > 0$ and $0 \leq i \leq n$). To get a category of fibrant objects we'd better cut ourselves down to only working with objects that are fibrant. An ∞-groupoid is by definition a fibrant object. Notice that the classical locus is an ∞-groupoid. k-groupoids will be ∞-groupoids that additionally satisfy

$$X_n \to \mathrm{Hom}(\Lambda^n_i, X)$$

for $n > k$ and $0 \leq i \leq n$ is a trivial fibration (it is already a fibration so really the content lies in the fact that it's a weak equivalence). The maps between k-groupoids are all simplicial morphisms. Now it remains to tell you what the trivial fibrations of k-groupoids are. They are also known as hypercovers: we require

$$X_n \to \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y} Y_n$$

is a cover for all $n \geq 0$ (we already knew it was a fibration). With this data we find that k-groupoids forma a category of fibrant objects. We won't prove this, but we point out that if $X_\bullet$ is a k-groupoid it is crucial to understand that $\mathrm{Hom}_\bullet(\Delta^\bullet, X)$ is again a k-groupoid and that the map to $\mathrm{Hom}_\bullet(\Lambda^n_i, X)$ is a cover of k-groupoids.

The 2-out-of-6 axiom is required to show that the inclusion of k-groupoids into ∞-groupoids induces a full faithful embedding of the corresponding simplicial localizations (i.e. k-stacks).

Recall on Tuesday we discussed the nerve of a dg algebra

$$\mathbb{N}_\bullet A = MC(C^*(E[n]) \otimes A).$$

We asserted that if $A^i = 0$ for $i \leq k$ then this is a k-groupoid. We think of this as the classical Maurer-Cartan locus. The derived refinement of this result is

$$[n] \mapsto \mathcal{MC}(C^*(E[n]) \otimes A).$$

We don't have time to talk about differential forms on Perf so let's instead look just at perfect complexes of amplitude zero.

$$N_k GL_N = (GL_N)^k = \mathrm{Fun}([k] \to GL_N).$$

Let's check Reedy fibrancy:

$$(GL_N)^2 = N_2 GL_N \to \mathrm{Hom}(\partial\Delta^2, N_\bullet GL_N) \cong (GL_N)^3.$$

This is what Kapranov realized: there's no way this could be submersion.

Now notice that we have a de Rham complex of our derived stack,

$$\Omega^*(X_\bullet) = \prod_{n=0}^{\infty} \Omega^{*-n}(X_n)$$

with the de Rham differential plus the differential induced by the face maps The key theorem is that a hypercover gives a quasi-isomorphism of de Rham complexes. Consider now

$$H^*(\Omega^*(N_\bullet GL_N)) = \mathbb{C}[c_1, \ldots, c_N]$$

the Chern classes. The existence of the first Chern class tells you that there is a determinant map. In particular,

$$c_1 \in \Omega^2(N_0 GL_N) \times \Omega^1(N_1 GL_N) \times \Omega^0(N_2 GL_N) \times \Omega^{-1}(N_3 GL_N) \times \cdots$$

As far as the classical locus is concerned there are no negative degree differential forms. The two-form component is zero since it's forms on a point. In Ω^1 we have $\mathrm{tr}(g^{-1}dg)$ the Maurer-Cartan element. The second Chern class leads to the Polykov-Wigman formula that has also been studied by Gawedski.

27. June 22, 2018 – Chris Brav

We specialize now to compactly generated dg categories $\mathcal{C}$. Actually what I want to say will make sense formally for dualizable, but I don't know any interesting examples. What does compactly generated mean? It means that

$$\mathrm{Ind}\ \mathcal{C}^c \xrightarrow{\sim} \mathcal{C}$$

where $\mathcal{C}^c$ are the compact objects. Consider, now, the functor Perf, given

$$X \mapsto \mathsf{Perf}(X) = \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} \mathsf{Perf}(U)$$

This yields a dg category. There is a functor the other way $\mathsf{dgcat}^{\mathrm{op}} \to \mathsf{PrStk}$ which is a right adjoint denote it $\mathcal{A} \mapsto \mathcal{M}_{\mathcal{A}}$. It is defined by

$$\begin{aligned}\underset{\mathsf{PrStk}}{\mathrm{Maps}}(U, \mathcal{M}_{\mathcal{A}}) &\simeq \underset{\mathsf{dgcat}^{\mathrm{op}}}{\mathrm{Maps}}(\mathsf{Perf}(U), \mathcal{A}) \\ &\simeq \underset{\mathsf{dgcat}}{\mathrm{Maps}}(\mathcal{A}, \mathsf{Perf}(U)) \\ &\simeq \underset{\mathsf{dgcat}_{\mathrm{cont}}}{\overset{r}{\mathrm{Maps}}}(\mathrm{Ind}\ \mathcal{A}, \mathsf{QCoh}(U)).\end{aligned}$$

The definition of this moduli space is due to Toen and Vaquié. Now there was a bunch of stuff I didn't follow at all.

For example consider $\mathcal{C} = \mathsf{Vect}$. Notice that $\mathcal{M}_{\mathcal{C}} = \mathsf{Perf}$ and its k-points are perfect k-modules. Now we need to justify why we call it moduli of objects, instead of moduli of certain special functors.

Definition 99. We say that $\mathcal{C}$ a (big) dg category is smooth if

(1) it is dualizable
(2) there is a left adjoint to the evaluation functor $\mathcal{C}^\vee \otimes \mathcal{C} \to \mathsf{Vect}$. This is equivalent to the coevaluation $\mathsf{Vect} \to \mathcal{C} \otimes \mathcal{C}^\vee$ to have a continuous right adjoint

In fact we will require more, that $\mathcal{C} \simeq \mathsf{Mod}_R$ for some dg algebra R that is finitely presented. This is a techincal thing that Toen and Vaquié use to get nice charts on the moduli of objects. This has something to do internally in dg categories where they are the compact objects (or something like this?). From now on such dg categories are called finite type.

Theorem 100 (To en-Vaquié)**.** *For $\mathcal{C}$ a dg category of finite type, the moduli space $\mathcal{M}_{\mathcal{C}}$ is locally Artin and locally of finite presentation (basically, the cotangent complex is perfect). Moreover, for $E \in \mathcal{M}_{\mathcal{C}}(k)$*

$$T_E\mathcal{M}_{\mathcal{C}} \simeq \mathrm{End}^*_{\mathcal{C}}(E)[1].$$

Let's make sense of this. First, for $\mathcal{C}$ smooth, every continuous adjunction between $\mathcal{C}$ and $\mathsf{QCoh}(U)$ is corepresented by $F \in \mathcal{C} \otimes \mathsf{QCoh}(U)$,

$$f \simeq \underline{\mathrm{Hom}}_U(F, -).$$

One checks this using $f^\vee$, evaluation, coevaluation, and left evaluation. So $\mathcal{M}_\mathcal{C}$ parameterizes certain compact objects of $\mathcal{C}$. For instance if $\mathcal{C} = \mathsf{QCoh}(X)$ for X a finite-type scheme/variety then we obtain the perfect complexes on X with compact support. If $\mathcal{C} = \mathsf{Loc}(X)$ for X some topological space, we obtain local systems of finite rank.

What are differential forms on $\mathcal{M}_\mathcal{C}$? Recall that Hochschild homology was supposed to give us noncommutative differential forms. Moreover there is S^1-functoriality for HH_* with respect to functors with continuous right adjoint. In particular given such a functor $f : \mathcal{C} \to \mathcal{D}$ we obtain $HH_*\mathcal{C} \to HH_*\mathcal{D}$. Apply to $\mathcal{C} \xrightarrow{\varepsilon} \mathsf{Ind}(\mathsf{Perf}(\mathcal{C}))$. We get a corresponding map on Hochschild homologies,

$$HH_*\mathcal{C} \to HH_*\mathsf{Ind}(\mathsf{Perf}(\mathcal{M}_\mathcal{C})) \to \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} HH_*(U)$$

and taking S^1-invariants

$$HC^-\mathcal{C} \to HC^-\mathsf{Ind}(\mathsf{Perf}(\mathcal{M}_\mathcal{C})) \to \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} HC^-(U) \to \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} HC^-_w(U)(p).$$

By the HKR theorem on an affine we have

$$HC^-_w(U)(p) \simeq \Omega^p[p] \times \Omega^{p+1}_U[p+1] \times \cdots$$

with the differential $d + d_{dR}$. This is exactly the complex we used to define closed forms. Getting rid of the p,

$$|HC^-_w(U)[n-p]| \simeq \mathcal{A}^{p,cl}(U,n).$$

Now given any negative cyclic chain for our category $\mathcal{C}$, say of degree d,

$$\alpha : k[d] \to HC^-(\mathcal{C}),$$

we obtain a map for each p

$$k[d] \to HC^-(\mathcal{C}) \to \lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} HC^-_w(U)(p)$$

and truncating and taking the underlying space we get an element of the space $|\lim_{(\mathsf{Aff}_{/X})^{\mathrm{op}}} HC^-_w(U)(p)[-d]|$, i.e. a closed p form of degree $p-d$ on the moduli space $\mathcal{M}_\mathcal{C}$.

What we will be most interested in is degree $p = 2$.

Definition 101. Given a locally Artin stack X locally of finite presentation, an n-shifted symplectic structure is

$$\omega \in \mathcal{A}^{2,cl}(X,n)$$

such that under the forgetful map

$$\mathcal{A}^{2,cl}(X,n) \to \mathcal{A}^2(X,n) \simeq |\operatorname{Hom}(\mathcal{O}, \Lambda^2 T^*(X)[n])|$$

say sending $\omega \mapsto \omega_0$, we get an isomorphism

$$T(X) \xrightarrow{\sim} T^*(X)[n].$$

We expect such shifted symplectic structures whenever tangent spaces are isomorphic to themselves under a shift. This usually comes about from Poincaré or Serre duality. From above, for instance,

$$\operatorname{End}(E)[1] \xrightarrow{\sim} (\operatorname{End}(E)[1])^*[n].$$

What we want to do is provide conditions on the category $\mathcal{C}$ which naturally produce this (this will be the notion of a noncommutative orientation or Calabi-Yau structure on a category).

Definition 102. Let $\mathcal{C}$ be smooth. Denote by $\mathrm{id}^! : \mathrm{ev}^L(k) \in \mathcal{C}^\vee \otimes \mathcal{C} \simeq \mathrm{End}(\mathcal{C})$ (this is kind of an inverse Serre functor for those who know what that means).

Let's compute Hochschild homology. Recall

$$HH_*(\mathcal{C}) \simeq \mathrm{tr}(\mathrm{id}_\mathcal{C}) \simeq \mathrm{ev} \circ \tau \circ \mathrm{coev}(k).$$

Notice that

$$HH_*(\mathcal{C}) \simeq \mathrm{Hom}_k(k, HH_*(\mathcal{C})) \simeq \mathrm{Hom}_k(k, \mathrm{ev} \circ \tau \circ \mathrm{coev}(k)) \overset{ev^L}{\simeq} \mathrm{Hom}_{\mathrm{End}\,\mathcal{C}}(\mathrm{id}^!, \mathrm{id}).$$

In the following definition we are "trivializing the anticanonical bundle".

Definition 103 (Kontsevich, Ginzburg)**.** A nc orientation or Calabi-Yau structure of dimension d on a smooth dg category $\mathcal{C}$ is a negative cyclic chain of degree d

$$k[d] \to HC^-(\mathcal{C})$$

such that

$$\begin{array}{ccc} k[d] & \xrightarrow{\alpha} & HC^-(\mathcal{C}) \\ & \searrow^{[\alpha]} & \downarrow \\ & & HH_*(\mathcal{C}) \simeq \mathrm{Hom}(\mathrm{id}^!, \mathrm{id}) \end{array}$$

such that $[\alpha] : \mathrm{id}^! \simeq \mathrm{id}[-d]$ is an equivalence.

Theorem 104 (Brav-Dyckerhoff)**.** *If $\mathcal{C}$ is a dg category of finite type with nc orientation*

$$\alpha : k[d] \to HC^-(\mathcal{C})$$

then the induced closed 2-form of degree $2-d$ on $\mathcal{M}_\mathcal{C}$ is nondegenerate, i.e. $\mathcal{M}_\mathcal{C}$ is shifted symplectic of degree $2-d$.

After T oen-Vaquié what remains is to check nondegeneracy. This follows from a few things:

- $T(\mathcal{M}_\mathcal{C})[-1] = (\mathrm{End})(\mathcal{E}) \simeq \Phi(\mathrm{id}^!)$
- $T^*(\mathcal{M}_\mathcal{C})[1] \simeq \bar{\Phi}(\mathrm{id})$
- the map $T(\mathcal{M}) \to T^*(\mathcal{M})[2-d]$ induced by the two-form is the same as (up to a shift) the map $\Phi[\alpha] : \Phi(\mathrm{id}^!) \xrightarrow{\sim} \Phi(\mathrm{id})[-d]$

To understand this last map one needs to understand HKR not via formulas, but instead via deformation theory and the geometry of the loop space.

One can then ask how to get Lagrangians in these moduli spaces. There exists a notion of a relative orientation structure on a functor $\mathcal{C} \to \mathcal{D}$ such that the induced map on moduli $\mathcal{M}_\mathcal{D} \to \mathcal{M}_\mathcal{C}$ gives a Lagrangian (e.g. local systems and local systems on boundaries). We also get a map $HC(\mathcal{C}) \to \Gamma(\mathcal{M}_\mathcal{C}, \mathcal{O})$. The left is shifted Lie via a string bracket (see Chas-Sullivan for local systems, Brav-Rozenblyum have been studying this case) and the right is shift Poisson, and the map intertwines these structures.

Remark 105. Shifted symplectic geometry appears in the Q-manifold literature. It was imported to algebraic geometry by PTVV. In the latter they construct lots of examples on mapping stacks. We pick up some of their examples, miss some, but have some that they don't have. This is a kind of noncommutative AKSZ.

This is the beginning of interesting stuff, so we'll stop here.

28. June 22, 2018 – Kai Behrend

Recall that the category of dg manifolds consists of triples (M, L, λ) where $L = L^2 \oplus \cdots \oplus L^{n+1}$ is a graded vector bundle and operations $\lambda_k : \Lambda^k L \to L[2-k]$ for $k \geq 0$. These operations together defined for us a bundle of curved L_∞-algebras. A map of dg manifolds recall was a map of manifolds together with a morphism of graded vector bundles $L \to f^*L'$ that is a bundle of morphisms of curved L_∞-algebras. Recall that the classical locus of $\mathcal{M}$ is the topological space $Z(\lambda_0) \subset M$. One could put more structure on this, such as a C^∞-ring, etc. but we don't want to do that here. Given a point $p \in \pi^0(\mathcal{M})$ we had a complex of vector spaces $T\mathcal{M}|_p$. Finally we said that $\mathcal{M} \to \mathcal{N}$ is étale at $p \in \pi^0(\mathcal{M})$ if $T\mathcal{M}|_p \to T\mathcal{N}|_{f(p)}$ is a quasi-isomorphism. The weak equivalences of dg manifolds are those maps which are étal and induce bijections on π^0. This is most probably equivalent to $\mathcal{O}(\mathcal{N}) \to \mathcal{O}(\mathcal{M})$ being a quasi-isomorphism. Fibrations are defined to have the underlying map of manifolds to be submersions and the map of bundes ϕ to be an epimorphisms.

To construct factorizations of maps (we want a category of fibrant objects) we needed to construct path spaces. Last time we sketched the case of $\mathcal{M} = M$ where we chose a connection on TM and defined the geodesic path space $P_g M$, the sufficiently short geodesic paths in M. The factorization of the diagonal map $M \to M \times M$ is then given by embedding M as the constant path in $P_g M$ and then evaluating at $t = 0$ and $t = 1$ (roughly speaking this is just constructing a tubular neighborhood of the diagonal in $M \times M$). To give $P_g M$ the structure of a dg manifold we need a graded vector bundle over it: consider $(TM)_{\text{const}}$ the bundle over $P_g M$ with fiber over $a \in P_g M$ the covariant constant sections in $\Gamma(I, a^*TM) \cong TM|_{a(0)}$ (in degree zero). Notice that this bundle has a canonical section which is the derivative of the path. One then checks that the map $\mathcal{M} \to P_g M$ is étale and a weak equivalence. The evaluation maps are easily checked to be fibrations.

Example 106. Suppose $X, Y \subset M$ are submanifolds. Then the derived intersection of X and Y in M is the homotopy fiber product $X \times^h_M Y$. some argument that I didn't understand using the path space. Roughly look at

$$X \to M \leftarrow PM \to M \leftarrow Y$$

and take three appropriate pullbacks But the point is that we end up with $P_g(X, Y)$, the space of sufficiently small geodesic paths in M starting on X and ending at Y. This is the base – what's our vector bundle? It will be TM_{const} with section $a \mapsto a'$. Notice that

$$\dim X^h \times_M Y = \dim X + \dim Y - \dim M.$$

So far this was all for just a manifold M. What about the general case $\mathcal{M}$. We want a factorization $\mathcal{M} \to P\mathcal{M} \to \mathcal{M} \times \mathcal{M}$. The general case is formally very similar to the baby case – we use connections to cut down the path space to something finite-dimensional. The underlying manifolds of the derived path space will be the same as the construction above. Now take $a \in P_g M$ and consider as the fiber:

$$TM|_{a(0)} = (TM)_{\text{const}} dt \oplus \Gamma(I, a^* L \otimes \Omega_I^*) =: \tilde{L}$$

where here $\Omega_I^* = \mathcal{O}_I \oplus \Omega_I^1$ and the first term is in degree 2. The operations are given as follows:

$$\mu = D + a^*\lambda + a^*(\nabla\lambda)dt$$

where D is the canonical curvature on $(TM)_{\text{const}}$ and $a^*\lambda$ is base extended from $\mathcal{O}_I$ to $\mathcal{O} \oplus \Omega^1$ and here

$$\nabla\lambda_k : T_M \otimes \Lambda^k L \to L[2-k].$$

We have one more operation, which is

$$\delta = a^*\nabla : a^*L \to a^*L dt.$$

Now $(\tilde{L}, \delta)$ is a complex of vector spaces and $(\tilde{L}, \delta + \mu)$ is a curved L_∞-algebra.

Now we use homotopical perturbation lemma due to Fukaya (for a good reference see the paper of Getzler mentioned yesterday). Given

$$\tilde{L} \to \tilde{L}[-1], \qquad \eta^2 = 0, \eta\delta\eta = \eta,$$

then $\delta\eta, \eta\delta$ are orthogonal idempotents on $\tilde{L}$ and we obtain a splitting

$$\tilde{L} = \text{im}(\delta\eta) \oplus \text{im}(\eta\delta) + (\ker(\delta\eta) \cap \ker(\eta\delta).$$

Call the intersection of the kernels H. H includes into $\tilde{L}$ and we have a projection $\pi : \tilde{L} \to H$, $\pi : 1 - [\delta, \eta]$. Hence H and L are homotopy equivalent complexes. The theorem now is that there exists a curved L_∞ structure on (H, δ) and a morphism of curved L_∞-algebras

$$\phi : (H, \delta + \theta) \to (\tilde{L}, \delta + \mu)$$

(in fact an equivalence). ϕ is the unique solution to the fixed point equation

$$\phi = \iota - \eta\lambda \circ \phi.$$

Checking that this structure comes through together to get a L_∞ structure is straightforward.

What is η? We send $\Gamma([0,1], a^*L)dt \to \Gamma([0,1], a^*L)$ sending $\alpha(t)dt \mapsto \int_0^t \alpha(u)du - \int_0^1 \alpha(u)du$. So the idea is to work with H instead of $\tilde{L}$, because it is a finite dimensional quasi-isomorphic complex that we can actually work with. Of course the cost is that we had to transfer structures.

Example 107. Suppose we have a homomorphism of complexes of vector bundles over M. This is an example of a map of dg manifolds. Fiorenza and Manetti examined this case in detail and explicitly solved this recursion: one finds that the fiber is the shifted mapping cone $C_\phi[-1]$ with no higher operations. If L, L' have bracket then C_ϕ already has higher brackets.