Seeing Planetary Motion Through Quantum Mechanics

Semiclassical phase space distributions and scattering theory of the harmonic oscillator and hydrogen atom

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To my teachers

Joseph Tanzosh

Steve Zelditch

Jared Wunsch

Honorable Mentions

Classical Mechanics

$$F = ma = mx''$$

- $F = -\nabla V \implies H := \frac{1}{2}m|x'|^2 + V(x)$ is conserved
- Hooke's Law: $F = -kx, \quad V = \frac{k}{2}|x|^2$
- Newton's ULoG:

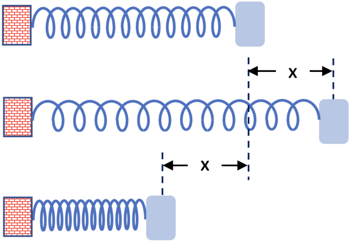
$$F = -\frac{GmMx}{|x|^3}, \quad V = -\frac{GmM}{|x|}$$

Assume $k = G = m = M = 1$ and write $\xi := x'$:

$$F = \xi'$$

Hooke's Law/Simple Harmonic Oscillation

$$-x = F = x'', \quad V = \frac{1}{2}|x|^2, \quad H^{\text{spring}}(x,\xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|x|^2$$

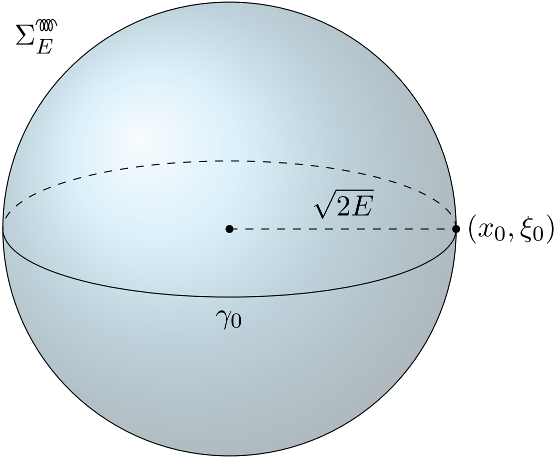


- Solutions: $\cos t$ and $\sin t$.

$$x(t) = x_0 \cos t + \xi_0 \sin t,$$
$$\xi(t) = -x_0 \sin t + \xi_0 \cos t$$

<u>Hamilton's Reform</u>: $\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix}$

- In the plane spanned by (x_0, ξ_0) and $(\xi_0, -x_0)$
- Great circle on $\frac{1}{2}|x|^2 + \frac{1}{2}|\xi|^2 = E$, the energy surface Σ_E^{spring}

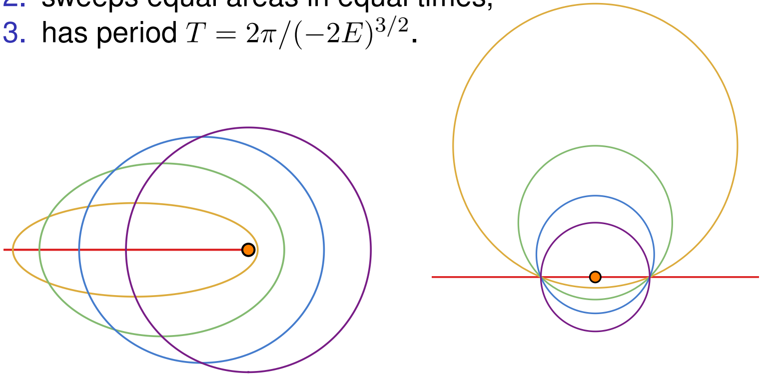

Σ_E
$\sqrt{2E}$
(x_0, ξ_0)
γ_0

Newton's ULoG and Kepler's Laws

$$F = -\frac{x}{|x|^3}, \quad V = -\frac{1}{|x|}, \quad H^{\odot}(x,\xi) = \frac{|\xi|^2}{2} - \frac{1}{|x|}$$

$E < 0$: Any trajectory of energy E

1. is a planar ellipse with the origin at one focus,
2. sweeps equal areas in equal times,
3. has period $T = 2\pi/(-2E)^{3/2}$.



(a) Position graphs (b) Momentum graphs

Quantum Mechanics

Classical Mechanics		Quantum Mechanics
(x, ξ)	$\rightsquigarrow$	$\psi : \mathbb{R}^3 \to \mathbb{C}$
x	$\rightsquigarrow$	$\lvert\psi\rvert^2 dx$
ξ	$\rightsquigarrow$	$\lvert\mathcal{F}_\hbar[\psi]\rvert^2 d\xi$
$(x, \xi) \mapsto x$	$\rightsquigarrow$	$\psi \mapsto x\psi$
$(x, \xi) \mapsto \xi$	$\rightsquigarrow$	$\psi \mapsto \hbar D[\psi]$
$(ax + b\xi)^j$	$\rightsquigarrow$	$(ax + b\hbar D)^j[\psi]$
$H(x, \xi) = \frac{1}{2}\lvert\xi\rvert^2 + V(x)$	$\rightsquigarrow$	$\widehat{H}_\hbar[\psi] := -\frac{\hbar^2}{2}\Delta\psi + V(x)\psi$
$a(x, \xi)$	$\rightsquigarrow$	$\mathrm{Op}_\hbar(a)[\psi]$
E	$\rightsquigarrow$	E such that $\widehat{H}_\hbar\psi = E\psi$
average x	$\rightsquigarrow$	$\langle \mathrm{Op}_\hbar(x)\psi, \psi\rangle$
average $a(x, \xi)$	$\rightsquigarrow$	$\langle \mathrm{Op}_\hbar(a)\psi, \psi\rangle$

Quantum Mechanics Cont.

Classical Mechanics		Quantum Mechanics
x	$\rightsquigarrow$	$\lvert\psi\rvert^2 dx$
ξ	$\rightsquigarrow$	$\lvert\mathcal{F}_\hbar[\psi]\rvert^2 d\xi$
average $a(x,\xi)$	$\rightsquigarrow$	$\langle \mathrm{Op}_\hbar(a)\psi, \psi\rangle$

Uncertainty Principle:

$$\|(x-a)\psi\|_2^2 \|(\hbar D - b)\psi\|_2^2 \geq \hbar^2/4.$$

Caution to the wind: Want $W_\psi(x,\xi)$ such that

$$\langle \mathrm{Op}_\hbar(a)\psi, \psi\rangle = \int_{\mathbb{R}^6} a(x,\xi) W_\psi(x,\xi)\,dx\,d\xi$$

$$W_\psi(x,\xi) = \frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3} \psi(x+\tfrac{1}{2}v)\overline{\psi(x-\tfrac{1}{2}v)}e^{-\frac{i}{\hbar}v\cdot\xi}\,dv$$

Quantum Mechanics Cont.

$$\langle \mathrm{Op}_\hbar(a)\psi, \psi\rangle = \int_{\mathbb{R}^6} a(x,\xi) W_\psi(x,\xi)\,dx\,d\xi$$

$$W_\psi(x,\xi) = \frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3} \psi(x+\tfrac{1}{2}v)\overline{\psi(x-\tfrac{1}{2}v)}e^{-\frac{i}{\hbar}v\cdot\xi}\,dv$$

Key Properties:

- $\int_{\mathbb{R}^3} W_\psi(x,\xi)\,d\xi = |\psi|^2, \quad \int_{\mathbb{R}^3} W_\psi(x,\xi)\,dx = |\mathcal{F}_\hbar[\psi]|^2$
- $\int_{\mathbb{R}^6} W_\psi(x,\xi)\,dx\,d\xi = 1$
- W_ψ can be negative

Hudson's Theorem: $W_\psi \geq 0$ if and only if ψ has minimal uncertainty

Send $\hbar \to 0$

Bohr's Correspondence Principle: As $\hbar \to 0$, quantum "looks" classical.

Definition 1 (Semiclassical Measure).

A measure μ is a semiclassical measure of a sequence $\psi_\hbar$ if $W_{\psi_\hbar} \rightharpoonup \mu$. That is,

$$\langle \mathrm{Op}_\hbar(a)\psi_\hbar, \psi_\hbar \rangle = \int_{\mathbb{R}^6} a(x,\xi) W_{\psi_\hbar}(x,\xi)\, dx d\xi \xrightarrow{\hbar \to 0} \int_{\mathbb{R}^6} a(x,\xi)\, d\mu$$

for any $a \in C_c^\infty(\mathbb{R}^6)$.

$$\widehat{H}_\hbar \psi_\hbar = E\psi_\hbar \implies \mu \text{ is inv. under Ham. flow of } H \text{ on } \Sigma_E$$

Understanding the backwards direction is notoriously hard!

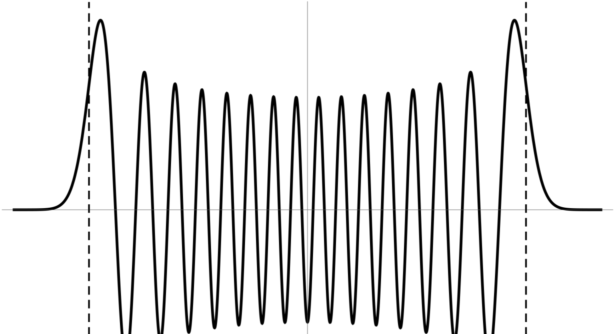
Quantum Simple Harmonic Oscillator

$$H^{\text{ꝏ}}(x,\xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|x|^2 \rightsquigarrow \widehat{H}_\hbar^{\text{ꝏ}} = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}|x|^2$$

- $\widehat{H}_\hbar^{\text{ꝏ}}\psi = E\psi \implies E = \hbar(N+\frac{3}{2}), N = 0,1,2,\dots$
- The $\hbar(N+\frac{3}{2})$-eigenspace has a basis of $\sim N^2$ functions

$$h_\alpha(x) = c_\alpha p_\alpha(x) e^{-\frac{1}{2\hbar}|x|^2}$$

- Well-known $W_{h_\alpha}(x,\xi) = c_\alpha e^{-\frac{1}{\hbar}(x^2+\xi^2)} L_\alpha(\frac{2}{\hbar}(x^2+\xi^2))$.



Theorem A (Arnaiz 2018, Studnia 2019, Arnaiz-Macià 2020).

For any $E > 0$ and any Radon probability measure μ on $\Sigma_E^{\mathfrak{m}}$ invariant under Ham. flow $H^{\mathfrak{m}}$, there exists $u_\hbar$ such that $\widehat{H}_\hbar^{\mathfrak{m}} u_\hbar = E u_\hbar$ where $u_\hbar \rightharpoonup \mu$ as $\hbar \to 0, N \to \infty, \hbar(N + \frac{3}{2}) = E$.

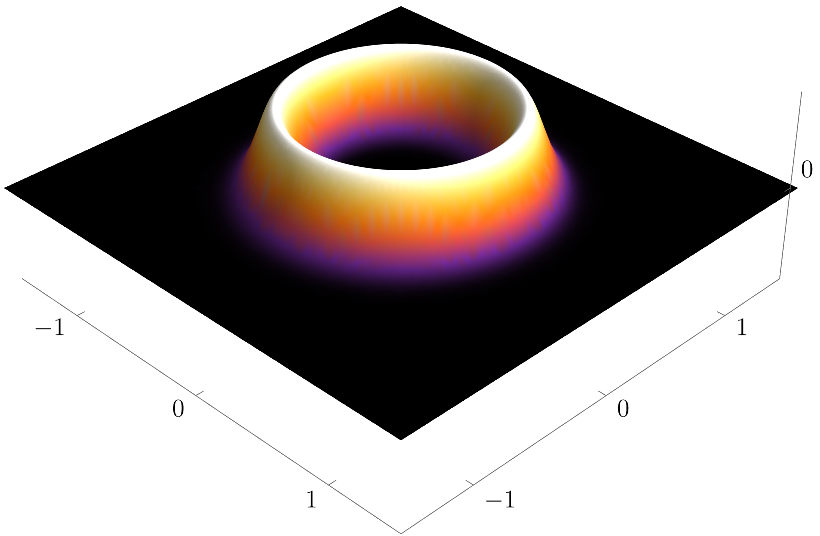
Definition 2.

For any Ham. orbit γ, define

$$\varphi_{\hbar,N}^{\gamma} := \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{i}{\hbar} tE} e^{-\frac{i}{\hbar} t \widehat{H}_\hbar^{\mathfrak{m}}} [\varphi_\hbar^{(x_0,\xi_0)}] dt$$

where $(x_0, \xi_0) \in \gamma$.

- $x_0 = \sqrt{2E} e_1, \xi_0 = 0 \implies \varphi_{\hbar,N}^{\gamma} = h_N(x_1)$
- $x_0 = \sqrt{E} e_1, \xi_0 = \sqrt{E} e_2 \implies \varphi_{\hbar,N}^{\gamma} = c_{\hbar,N}(x_1 + i x_2)^N e^{-\frac{1}{2\hbar}|x|^2}$



−1
0
1
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For any $E > 0$ and any Radon probability measure μ on $\Sigma_E^{\text{ꝏ}}$ invariant under Ham. flow $H^{\text{ꝏ}}$, there exists $u_\hbar$ such that $\widehat{H}_\hbar^{\text{ꝏ}} u_\hbar = E u_\hbar$ where $u_\hbar \rightharpoonup \mu$ as $\hbar \to 0, N \to \infty, \hbar(N + \frac{3}{2}) = E$.

We know $W_{\varphi_\hbar^\gamma} \rightharpoonup \delta_\gamma$, but what about finer asymptotics? Pointwise? Scaling near γ?

Theorem 1 (Pointwise Asymptotics, L. 2023).

Fix $E = 1/2$ *and* γ *a H.O. in plane* P_γ *Then*

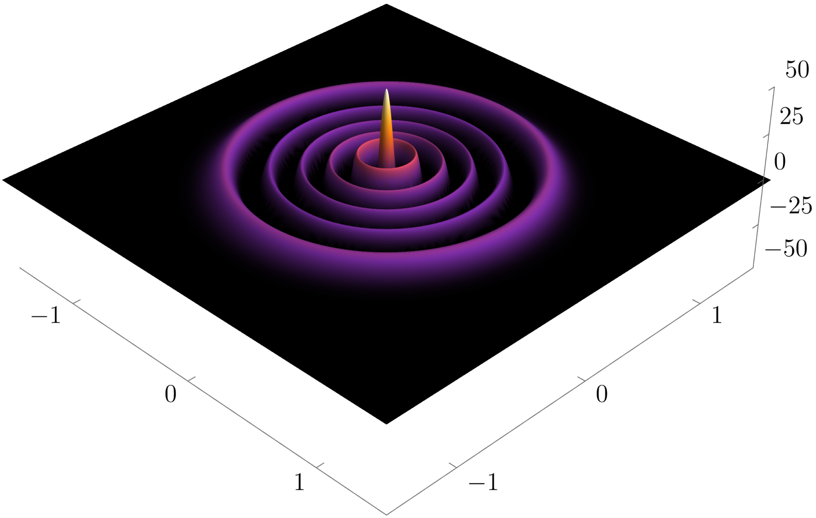
$$\hbar^d W_{\varphi^\gamma_{\hbar,N}}(0,0) = (-1)^N/\pi^d.$$

Define the notation $I := |\operatorname{proj}_{P_\gamma}(x,\xi)|^2, I^\perp := |\operatorname{proj}_{P_\gamma^\perp}(x,\xi)|^2.$
For $\varepsilon \le I \le 1-\varepsilon$*,*

$$\hbar^{d-\frac{1}{2}} W_{\varphi^\gamma_{\hbar,N}}(x,\xi) \sim A(I) e^{-\frac{1}{\hbar}I^\perp} \cos(\theta(I)).$$

For $I > 1$*,*

$$\hbar^{d-\frac{1}{2}} W^\hbar_{\varphi^\gamma_{\hbar,N}}(x,\xi) \sim C e^{-\frac{1}{\hbar}H^{\text{ꝏ}}}$$



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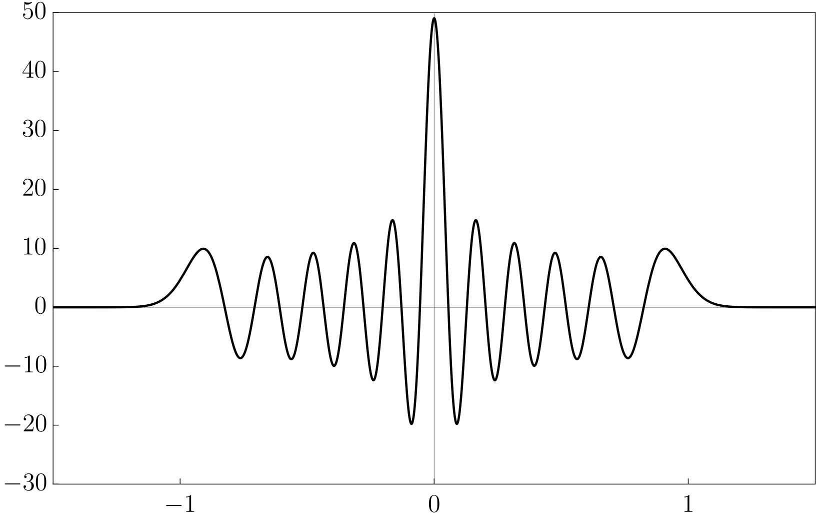
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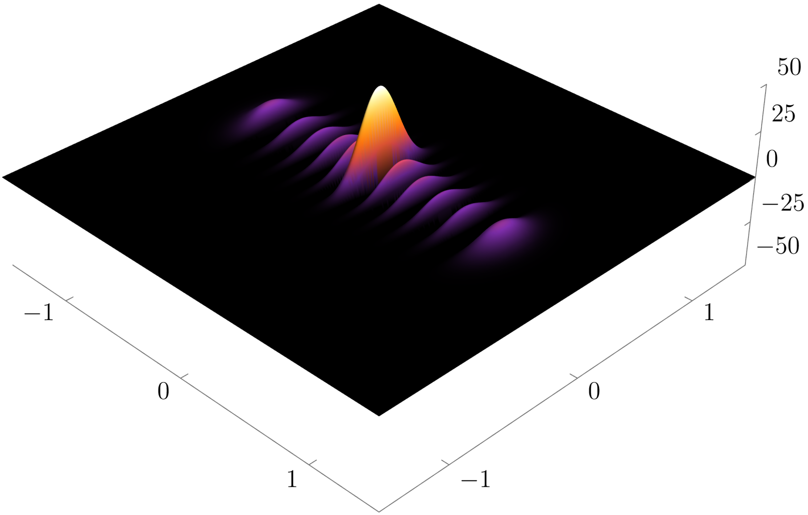

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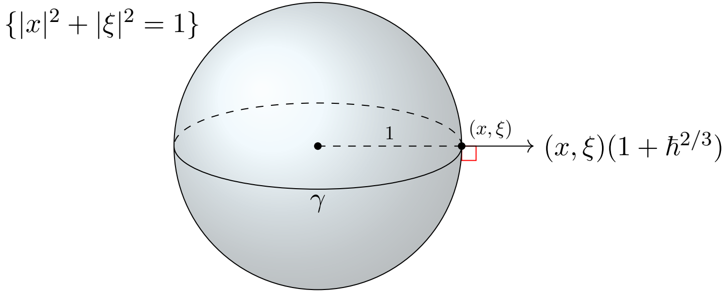
Theorem 2 (Scaling Asymptotics, L. ’23).

Fix $E = 1/2$ and let γ a H.O. If $I = v^2\hbar^2$ and $I^\perp = s^2\hbar$, then

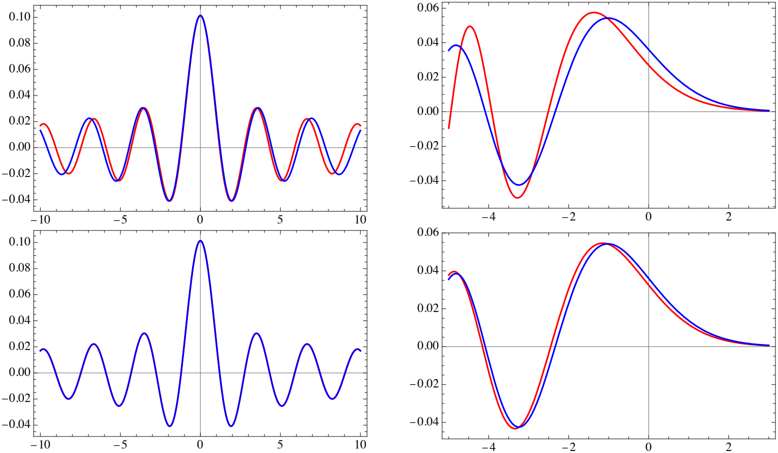
$$\hbar^d W^{\hbar}_{\varphi^{\gamma}_{\hbar,N}}(x,\xi) \sim c_N e^{-s^2} J_0(2v)$$

OTOH, if $I = 1 + u\hbar^{2/3}$ and $I^\perp = s^2\hbar$, then

$$\hbar^{d-1/3} W^{\hbar}_{\varphi^{\gamma}_{\hbar,N}}(x,\xi) \sim \frac{1}{\pi^d} e^{-s^2}\,\mathrm{Ai}(u)$$



First result addressing of a conjecture of M. V. Berry ’89



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0.04
0.02
0.00
−0.02
−0.04
−4
−2
0
2

Proof Idea

Use metaplectic covariance of Wigner distributions:

$$W_{\mu(\mathcal{A})[f]}(x,\xi) = W_f(\mathcal{A}^{-1}(x,\xi))$$

where $\mathcal{A} \in \mathrm{Sp}(6,\mathbb{R})$. If $U \in \mathrm{SU}(3) \subset \mathrm{Sp}(6,\mathbb{R})$, then

$$\mu(U)[\varphi^{\gamma}_{\hbar,N}] = \varphi^{U\cdot\gamma}_{\hbar,N}.$$

Method 1: Rotate γ to include $x_0 = e_1, \xi_0 = 0$ and use known asymptotics of Laguerre polynomials ([FW88] or Olver)
Method 2: Rotate γ to include $x_0 = e_1, \xi_0 = e_2$ to create a semiclassical parametrix and do stationary phase [CFU56]

$$\int_{\mathbb{R}} g(z)e^{Nf(z,\alpha)}dz \sim N^{-1/3}\,\mathrm{Ai}(N^{2/3}\zeta) + N^{-2/3}\,\mathrm{Ai}'(N^{2/3}\zeta)$$

Coulomb Schrödinger Operator

Kepler's Hamiltonian goes to Schrödinger's Hydrogen

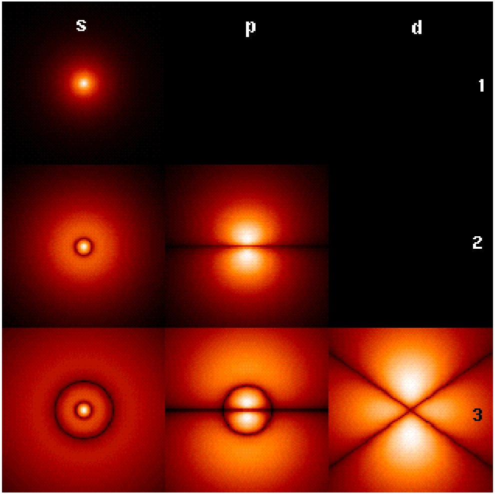
$$H^{\odot}(x,\xi) = \frac{|\xi|^2}{2} - \frac{1}{|x|} \rightsquigarrow \widehat{H}^{\odot}_{\hbar} := -\frac{\hbar^2}{2}\Delta - \frac{1}{|x|}, \qquad \hbar > 0.$$

$$\operatorname{spec} \widehat{H}^{\odot}_{\hbar} = \{-\tfrac{1}{2\hbar^2(N+1)^2} \mid N = 0,1,\ldots\} \cup \{0\} \cup \{\tfrac{1}{2\hbar^2\lambda^2} \mid \lambda > 0\},$$

- The dimension of the $-\frac{1}{2\hbar^2(N+1)^2}$-eigenspace is $(N+1)^2$.
- $$\psi_{N,\ell,m}(r,\theta,\varphi) = C_{N,\ell,\hbar} r^{\ell} e^{-r/2} L^{2\ell+1}_{N-\ell}(r) Y^m_{\ell}(\theta,\varphi)$$
  where $\ell = 0,\ldots,N$, $m = -\ell,\ldots,\ell$

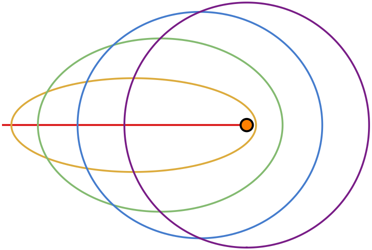
$$\psi_{N,\ell,m}(r,\theta,\varphi) = C_{N,\ell,\hbar} r^{\ell} e^{-r/2} L_{N-\ell}^{2\ell+1}(r) Y_{\ell}^{m}(\theta,\varphi)$$

where $\ell = 0, \ldots, N$, $m = -\ell, \ldots, \ell$

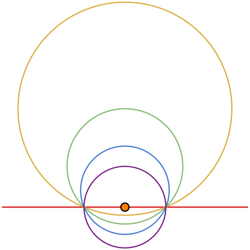


What are the semiclassical measures of $\widehat{H}_\hbar^\odot := -\frac{\hbar^2}{2}\Delta - \frac{1}{|x|}$?

The Hamiltonian orbits of $H^\odot(x,\xi) = \frac{|\xi|^2}{2} - \frac{1}{|x|}$ obey Kepler's laws.



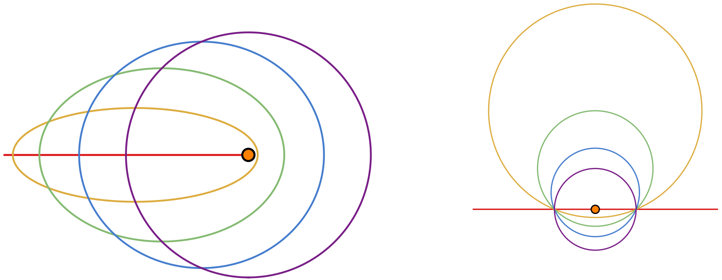
(a) The position graphs



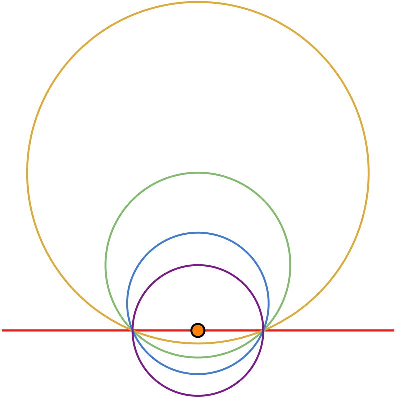
(b) The velocity/momentum graphs

Theorem 3 (L. 2024).

For any $E < 0$ and any Radon probability measure μ on $\{H^{\odot} = E\}$ invariant under $e^{t\mathcal{J}\nabla H^{\odot}}$, there exists $u_\hbar$ such that $\widehat{H}_\hbar^{\odot} u_\hbar = E u_\hbar$ where $u_\hbar \rightharpoonup \mu$ as $\hbar \to 0, N \to \infty, -\frac{1}{2\hbar^2(N+1)^2} = E$.



(a) The position graphs (b) The velocity/momentum graphs



Theorem M (Moser '70).

Let $E < 0$. The map $\mathcal{M}_E : T^*\mathbb{R}^3 \to T^*\mathbb{S}^3_{\neq \mathrm{NP}}$ given by

$$\mathcal{M}_E := \omega^* \circ S_{(-2E)^{1/2}} \circ \mathcal{D}_{(-2E)^{-1/2}} \circ R_{-\pi/2}$$

where $\omega : \mathbb{R}^3 \to \mathbb{S}^3_{\neq \mathrm{NP}}$ is stereographic projection,

$$\begin{aligned} R_{-\pi/2}(x,\xi) &:= (\xi, -x), \\ \mathcal{D}_k(x,\xi) &:= (kx, k^{-1}\xi), \\ S_k(x,\xi) &:= (x, k\xi), \end{aligned}$$

satisfies

- $\mathcal{M}_E$ *is a (conformal) symplectomorphism that pulls back the symplectic form on $T^*\mathbb{S}^3_{\neq \mathrm{NP}}$ to $\sqrt{-2E}dx \wedge d\xi$*
- $\mathcal{M}_E|_{\Sigma_E^{\odot}} = \mathbb{S}^*\mathbb{S}^3_{\neq \mathrm{NP}}$
- $\mathcal{M}_E$ *sends the Kepler Hamiltonian flow to the (co)geodesic flow on $\mathbb{S}^*\mathbb{S}^3_{\neq \mathrm{NP}}$ (up to time reparametrization).*

‘Quantization’ of Moser’s map

Theorem F (Fock ’35).

Define eigenspace

$$\mathcal{E}_N(\hbar) := \{\psi \in L^2(\mathbb{R}^3) : \widehat{H}_\hbar^{\odot}\psi = -\tfrac{1}{2\hbar^2(N+1)^2}\psi\}.$$

There exists a map $\mathcal{V}_{N,\hbar} : \mathcal{E}_N(\hbar) \to \mathcal{H}_3^{(N)}$ *given by*

$$\mathcal{V}_{N,\hbar} = \widehat{\omega^{-1}} \circ \mathsf{S} \circ \widehat{\mathcal{D}}_{[\hbar(N+1)]^{-1}} \circ \mathcal{F}_\hbar, \quad \mathsf{S}[f] := \Big(\frac{|\bullet|^2+1}{2}\Big)^{1/2} f$$

(where $\mathcal{H}_3^{(N)}$ *are the harmonic polynomials of degree* N *on* $\mathbb{S}^d$*) such that* $\mathcal{V}_{N,\hbar}$ *is unitary with*

$$\mathcal{V}_{N,\hbar}\widehat{H}_\hbar^{\odot} = -\frac{1}{2\hbar^2}\Big(-\Delta_{\mathbb{S}^3}+1\Big)^{-1}\mathcal{V}_{N,\hbar}$$

Proof Idea

Theorem 4 (L. 2024).

For any $E < 0$ and any Ham. orbit γ on $\Sigma_E^{\odot}$, there exists $\Psi_{\hbar,N}^{\gamma}$ such that $\widehat{H}_{\hbar}^{\odot}\Psi_{\hbar,N}^{\gamma} = E\Psi_{\hbar,N}^{\gamma}$ where $\Psi_{\hbar,N}^{\gamma} \rightharpoonup \delta_{\gamma}$ as $\hbar \to 0, N \to \infty, -\frac{1}{2\hbar^2(N+1)^2} = E$. Namely,

$$\langle \mathrm{Op}_{\hbar}(a)\Psi_{\hbar,N}^{\gamma}, \Psi_{\hbar,N}^{\gamma}\rangle \xrightarrow{\hbar\to 0} \int_{\gamma} a$$

for any $a \in C_c^{\infty}(\mathbb{R}^6)$.

Using this and the Krein-Milman theorem proves the original theorem.

$$\Psi_{\hbar,N}^{\gamma} := \mathcal{V}_{\hbar,N}^{-1}[\Phi_N^{\mathcal{M}_E(\gamma)}]$$

Φ_N were used in [JZ99]

Continuous Spectrum?

$$\widehat{H}_\hbar^{\odot} := -\frac{\hbar^2}{2}\Delta - \frac{1}{|x|}, \qquad \hbar > 0.$$

Want solutions ψ in the continuous spectrum:

$$\widehat{H}_\hbar^{\odot}\psi = E_\hbar^{\odot}(\lambda)\psi, \qquad E_\hbar^{\odot}(\lambda) := \frac{1}{2\hbar^2\lambda^2}.$$

How big is this solution set? HUGE
How do we parameterize the solution set? Scattering Theory
Example: $\mathcal{E}_{\mathbb{R}^2}(\lambda) := \{u \in C^\infty(\mathbb{R}^2) : -\Delta_{\mathbb{R}^2} u = \lambda^2 u\}$

- $\mathcal{S}'(\mathbb{R}^2) \cap \mathcal{E}_{\mathbb{R}^2}(\lambda) \leftrightarrow \mathcal{D}'(\mathbb{S}^1)$
- $\mathcal{E}_{\mathbb{R}^2}(\lambda) \leftrightarrow E'(\mathbb{S}^1)$ [Helgason '74, Morimoto '81]

Example Cont.

Example: $\mathcal{E}_{\mathbb{R}^2}(\lambda) := \{u \in C^\infty(\mathbb{R}^2) : -\Delta_{\mathbb{R}^2} u = \lambda^2 u\}$

- $\mathcal{S}'(\mathbb{R}^2) \cap \mathcal{E}_{\mathbb{R}^2}(\lambda) \leftrightarrow \mathcal{D}'(\mathbb{S}^1)$
- $\mathcal{E}_{\mathbb{R}^2}(\lambda) \leftrightarrow E'(\mathbb{S}^1)$ [Helgason '74, Morimoto '81]

Fortunately, the image of $P(\lambda) : L^2(\mathbb{S}^1) \to \mathcal{E}_{\mathbb{R}^2}(\lambda)$

$$P(\lambda)[f](x) := \int_{\mathbb{S}^1} f(\theta) e^{i\lambda x \cdot \theta} d\theta$$

is dense in $\mathcal{E}_{\mathbb{R}^2}(\lambda)$. For $f \in C^\infty(\mathbb{S}^1)$,

$$P(\lambda)[f](x) \sim c_{\lambda,d} e^{i\lambda|x|} |x|^{-\frac{d-1}{2}} f(\tfrac{x}{|x|}) + c_{-\lambda,d} e^{-i\lambda|x|} |x|^{-\frac{d-1}{2}} \widetilde{f}(\tfrac{x}{|x|}).$$

Call $S(\lambda)[f] := \widetilde{f}$ the *scattering matrix*.

$$S(\lambda) = P(-\lambda)^{-1} P(\lambda)$$

$$P(\lambda)[f](x) := \int_{\mathbb{S}^1} f(\theta) e^{i\lambda x\cdot\theta} d\theta, \qquad S(\lambda) = P(-\lambda)^{-1} P(\lambda)$$

What about $(-\frac{1}{2}\Delta + V)u = \lambda^2 u$? Construct solution

$$\psi_{\lambda,V}(x,\theta) \sim e^{i\lambda x\cdot\theta} + e^{i\lambda|x|}|x|^{-\frac{d-1}{2}} a_\lambda(\tfrac{x}{|x|},\theta).$$

Define

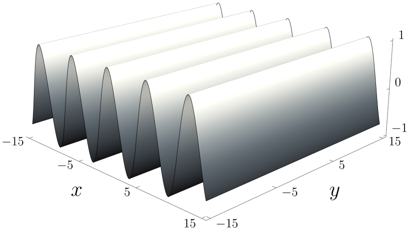
$$P_V(\lambda)[f](x) := \int_{\mathbb{S}^{d-1}} f(\theta)\psi_{V,\lambda}(x,\theta) d\theta$$
$$S_V(\lambda) := P_V(-\lambda)^{-1} P_V(\lambda).$$

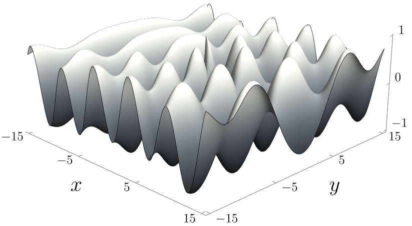
$\widehat{H}_\hbar^\odot \psi := (-\frac{\hbar^2}{2}\Delta - \frac{1}{|x|})\psi = E_\hbar^\odot(\lambda)\psi$ is more complicated...

$$\psi_{\hbar,\lambda}(x,\theta) := e^{i\lambda x\cdot\theta} g(|x| - x\cdot\theta) \quad \text{for some } g \in C^\infty(\mathbb{R}).$$

$$P_\hbar(E)[f](x) := \int_{\mathbb{S}^2} f(\theta)\psi_{\hbar,\lambda}(x,\theta)d\theta, \quad S_\hbar(E) = P_\hbar(-E)^{-1}P_\hbar(E)$$

$$\psi_{\hbar,\lambda}(x,\theta) := e^{i\lambda x\cdot\theta} g(|x| - x\cdot\theta) \quad \text{for some } g \in C^\infty(\mathbb{R}).$$





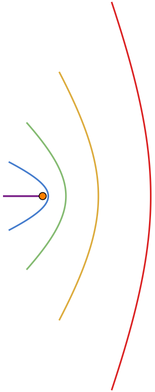
Theorem Y (Yafaev '97).

$$S_\hbar(E_\hbar^{\odot}(\lambda)) = \frac{c_\lambda}{|\theta - \theta'|^{2-2\lambda i}} = S_{\mathbb{H}^3}(\lambda)$$

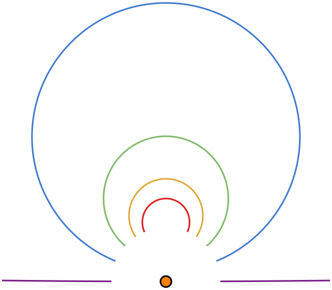
Classical Mechanics

$E > 0$: Hamiltonian flow of $H(x, \xi) = \frac{|\xi|^2}{2} - \frac{1}{|x|}$ in position space

1. is a planar hyperbola with the origin at one focus,
2. sweeps equal areas in equal times,



(a) Position graphs



(b) Momentum graphs

Theorem (Osipov '72,'77, Belbruno '77,'81).

Let $E > 0$*. The map* $\mathcal{M}_E : \mathbb{R}^3 \times \{|\xi| > \sqrt{2E}\} \to T^*\mathbb{H}^3_{\neq 0}$ *given by*

$$\mathcal{M}_E := \iota^* \circ S_{(-2E)^{1/2}} \circ \mathcal{D}_{(-2E)^{-1/2}} \circ R_{-\pi/2}$$

where $\iota : \mathbb{R}^3_{>1} \to \{0 < |\bullet| < 1\}$ *is inversion,*

$$\begin{aligned} R_{-\pi/2}(x,\xi) &:= (\xi, -x), \\ \mathcal{D}_k(x,\xi) &:= (kx, k^{-1}\xi), \\ S_k(x,\xi) &:= (x, k\xi), \end{aligned}$$

satisfies

- $\mathcal{M}_E$ *is a (conformal) symplectomorphism that pulls back the symplectic form on* $T^*\mathbb{H}^3_{\neq 0}$ *to* $\sqrt{2E}dx \wedge d\xi$
- $\mathcal{M}_E|_{\Sigma_E^{\odot}} = \mathbb{S}^*\mathbb{H}^3_{\neq 0}$
- $\mathcal{M}_E$ *sends the Kepler Hamiltonian flow to the (co)geodesic flow on* $\mathbb{S}^*\mathbb{H}^3_{\neq 0}$ *(up to time reparametrization).*

‘Quantization’?

“ The geometric interpretation of the Schrödinger equation is less intuitive in the case of the continuous spectrum than in the case of the point spectrum. For applications, it is therefore more advantageous to first derive the formulas for the discrete spectrum and only in the final result consider the principal quantum number n as purely imaginary. ” (Fock, 1935)

[BI66,PP66] wrote down one direction of this map (with many caveats...):

$$\mathcal{V}_{\lambda,\hbar} = \widehat{\iota^{-1}} \circ \mathsf{S} \circ \widehat{\mathcal{D}}_{(\hbar\lambda)^{-1}} \circ \mathcal{F}_{\hbar}, \quad \mathsf{S}[f] := \left| \frac{|\bullet|^2 - 1}{2} \right|^{1/2} f$$

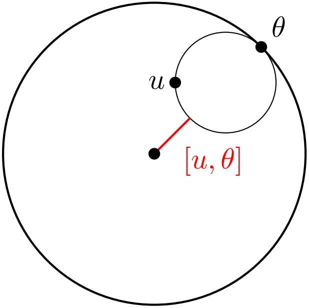
Spectral Analysis of $\mathbb{H}^3$

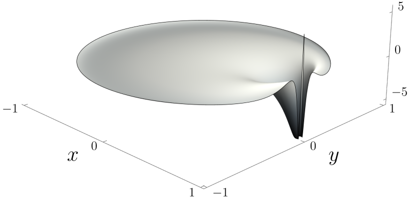
$$\mathbb{H}^3 = \{u \in \mathbb{R}^3 : |u| < 1\}, \quad du^2 = \tfrac{4}{(1-|x|^2)^2} dx^2$$

$\mathrm{spec}(-\Delta_{\mathbb{H}^3}) = [1, \infty)$. Plane waves:

$$-\Delta_{\mathbb{H}^3} e_\lambda(u,\theta) = (\lambda^2 + 1) e_\lambda(u,\theta),$$

$$e_\lambda(u,\theta) := e^{(1-i\lambda)[u,\theta]} = \left(\frac{1-|u|^2}{|u-\theta|^2}\right)^{1-\lambda i}$$





Theorem 5 (L. '24).

For any $\lambda \neq 0, \hbar > 0$, there is an explicit unitary operator $\mathcal{V}_{\hbar,\lambda}$ making the following diagram commute:

$$
\begin{array}{ccc}
 & L^2(\mathbb{S}^2) & \\
 {\scriptstyle P_\hbar(E_\hbar^{\odot}(\lambda))} \swarrow & & \searrow {\scriptstyle P_{\mathbb{H}^3}(\lambda)} \\
\mathscr{H}_\hbar(E_\hbar^{\odot}(\lambda)) & \xrightarrow[\mathcal{V}_{\hbar,\lambda}]{} & \mathscr{H}_{\mathbb{H}^3}(\lambda)
\end{array}
$$

Namely,

$$\mathcal{V}_{\hbar,\lambda} = \widehat{\iota^{-1}} \circ \mathsf{S} \circ R_{\{|\bullet|>1\}} \circ \widehat{\mathcal{D}}_{(\hbar\lambda)^{-1}} \circ \mathcal{F}_\hbar, \quad \mathsf{S}[f] := \left|\frac{|\bullet|^2 - 1}{2}\right|^{1/2} f$$

$$\mathcal{V}_{\hbar,\lambda}^* = \mathcal{V}_{\lambda,\hbar}^{-1} = \mathcal{F}_\hbar^{-1} \circ \widehat{\mathcal{D}}_{\hbar\lambda} \circ E \circ \mathsf{S}^{-1} \circ \widehat{\iota}.$$

HYPERBOLI.1- HELGSONTOWN
EVIL PHYSICIST LAND
HYEROLIC HELIGASONTOWN
EVIL PHYSICIST LAND

Proof Sketch

Idea: Show $\mathcal{V}_{\hbar,\lambda}[\psi_{\hbar,\lambda}(x,\theta_0)] = e_\lambda(u,\theta_0)$. Away from θ_0,

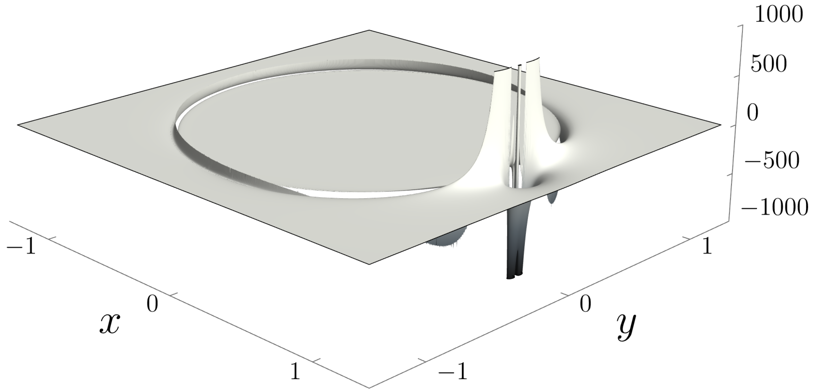
$$(\widehat{\mathcal{D}}_{(\hbar\lambda)^{-1}} \circ \mathcal{F}_\hbar)[\psi_{\hbar,\lambda}](\xi) = \frac{c}{(|\xi|^2 - 1 - 0i)^{1+\lambda i}|\xi - \theta_0|^{2-2\lambda i}}.$$

- The regularization at θ_0 is unique in $H^{-3/2-0}$ up to adding $c\delta_{\theta_0}$. Might be a paired Lagrangian!
- The proof of the above uses Feynman's identity [Feynman '49]

$$\frac{1}{z^\alpha w^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{(zt + w(1-t))^{\alpha+\beta}} dt,$$

whenever $\Re\alpha, \Re\beta > 0, 0 \notin \operatorname{conv}(z,w) \subset \mathbb{C}$.

$$
\begin{aligned}
&(\widehat{\mathcal{D}}_{(\hbar\lambda)^{-1}} \circ \mathcal{F}_\hbar)[\psi_{\hbar,\lambda}(\bullet;\theta_0)](\xi) \\
&= \lim_{\varepsilon\to 0^+} \frac{4(\varepsilon+\lambda i)}{\lambda} \frac{\varepsilon^2\left(1+\frac{(1-\varepsilon-\lambda i)(|\xi|^2-1+\varepsilon^4-2\varepsilon^2 i)}{(\varepsilon+\lambda i)(|\xi-\theta_0|^2+\varepsilon^4)}\right)-i}{(|\xi|^2-1+\varepsilon^4-2\varepsilon^2 i)^{1+\varepsilon+\lambda i}(|\xi-\theta_0|^2+\varepsilon^4)^{1-\varepsilon-\lambda i}}
\end{aligned}
$$



1000
500
0
−500
−1000
−1
0
1
x
−1
0
1
y

Corollaries

Corollary 1.

$\mathcal{V}_{\hbar,\lambda}\widehat{H}_{\hbar}^{\odot}\mathcal{V}_{\hbar,\lambda}^{-1} = \frac{1}{2\hbar^2}\Big(-\Delta_{\mathbb{H}^3}-1\Big)^{-1}$ on $\mathscr{H}_{\mathbb{H}^3}(\lambda)$ for $\hbar > 0, \lambda \in \mathbb{R}_{\neq 0}$.

Corollary 2.

$S_\hbar(E_\hbar^{\odot}(\lambda)) = S_{\mathbb{H}^3}(\lambda)$ for $\hbar > 0, \lambda \in \mathbb{R}_{\neq 0}$.

Corollary 3.

For $\psi \in \mathscr{H}_\hbar(E_\hbar^{\odot}(\lambda))$ and $0 < |\xi| < 1$,

$$(\widehat{\mathcal{D}}_{\frac{1}{\hbar\lambda}} \circ \mathcal{F}_\hbar)[\psi](\xi) = -\frac{e^{-\pi|\lambda|}}{|\xi|^4}(\widehat{\mathcal{D}}_{\frac{1}{\hbar\lambda}} \circ \mathcal{F}_\hbar)[\psi]\Big(\frac{\xi}{|\xi|^2}\Big).$$

Corollary 4.

For $f \in L^2(\mathbb{S}^2)$*,*

$$P_\hbar(E_\hbar^\odot(\lambda))[f] = \mathcal{F}_\hbar^{-1} \circ \widehat{\mathcal{D}}_{\hbar\lambda} \int_{\mathbb{S}^2} \frac{4}{|\bullet - \theta|^{2-2\lambda i}(|\bullet|^2 - 1)^{1+\lambda i}} f(\theta) d\theta$$

Corollary 5.

The infinite dimensional unitary representation $\rho_\hbar(E_\hbar^\odot(\lambda)) : \mathrm{SO}_e(1,3) \to \mathrm{U}(\mathscr{H}_\hbar(E_\hbar^\odot(\lambda))$ *of the restricted Lorentz group* $\mathrm{SO}_e(1,3)$ *given by*

$$\rho_\hbar(E_\hbar^\odot(\lambda)) := \mathcal{V}_{\hbar,\lambda}^{-1} \circ \rho_{\mathbb{H}^3}(\lambda) \circ \mathcal{V}_{\hbar,\lambda},$$

where $\rho_{\mathbb{H}^3}(\lambda)[A](f) := f(A^{-1}\bullet)$*, is irreducible.*

Future Projects

- Is the theorem true with $L^2(\mathbb{S}^2)$ replaced by $\mathscr{A}'(\mathbb{S}^2)$?
- What about $\lambda \in \mathbb{C}$ such that $1 + \lambda i \notin \mathbb{Z}_{\leq 0}$?
- Is there a Helgason-type theorem for $\{\psi \in C^\infty(\mathbb{R}^3 \setminus 0) : \widehat{H}_\hbar^\odot \psi = E_\hbar^\odot(\lambda)\psi\}$?
- Can we handle perturbations of $\widehat{H}_\hbar$?
- For a suitably nice function f on $\mathbb{R}^3$, define the *Coulomb-Helgason-Fourier transform*

$$\widetilde{\mathcal{F}}_\hbar[f](\lambda, \theta) = \int_{\mathbb{R}^3} f(x)\psi_{\hbar,\lambda}(x, \theta)dx.$$

  Can we prove theorems about this transform? Is it different than the usual Helgason-Fourier transform?

Thank you for your attention!