Toward a Kozmic Galois group

[A talk on joint work with A. Blumberg & K. Hess, at the Midwest Topology Seminar, Northwestern University, 24 April 2010]

I Classical motives & cosmic Galois groups

II BGT's big category of motives [`arXiv:1001.2282`]

III Hess's homotopy-theoretic descent [`arXiv:1001.1556`]

IV Proposed applications [`arXiv:0908.3124`]

Some background reading:

Y. André, **Un introduction aux motifs**, Panoramas et Syntheses 17, Soc. Math. France (2004)

K. Igusa, **Higher Franz-Reidemeister torsion**, AMS/IP Studies in Advanced Mathematics 31, AMS (2002)

§I Arithmetic vs geometric motives

Voevodsky's work has made **geometric** motives (over fields) more accessible to homotopy theorists, but **arithmetic** motives (eg over Dedekind rings such as $\mathbb{Z}$), and their 'cosmic' Galois groups, are still under construction.

Here is some of the backstory:

1.1 When I was a grad student I heard Atiyah remark that zeta functions were the 19th century's way of doing homological algebra. In 1987 Deligne defined a symmetric monoidal $\mathbb{Q}$-linear abelian category of **systems of realizations**, consisting of

• filtered vector spaces over $\mathbb{Q}$, with a compatible mixed Hodge structures, together with

• a compatible $\widehat{\mathbb{Z}} \otimes \mathbb{Q}$-module, with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, plus

• for each finite prime p, a compatible $\mathbb{Q}_p$-vector space with a lift of Frobenius

as analogs of Euler products of the form

$$\xi(s) := \pi^{s/2}\Gamma(\tfrac{1}{2}s) \cdot \prod (1 - p^{-s})^{-1} = \xi(1-s) \; .$$

By construction this category admits a faithful forgetful functor ω to $\mathbb{Q}$-vector spaces, and Deligne showed (under some finiteness hypotheses, roughly equivalent to the existence of duals) that the functor

$$(\mathbb{Q} - \mathrm{alg}) \ni A \mapsto \mathrm{Aut}_\otimes(\omega \otimes A) := \mathrm{Aut}_\otimes(\omega)(A) \in (\mathrm{Groups})$$

is representable, defining a **fully** faithful lift

$$\boldsymbol{\omega} : (\mathrm{Systems}) \to (\mathrm{Aut}_\otimes(\omega) - \mathrm{Reps})$$

identifying his category with the representations of some **motivic** group-scheme $\mathrm{Aut}_\otimes(\omega)$.

[For comparison, consider mod p cohomology as a $\mathbb{Z}/2\mathbb{Z}$-graded monoidal functor

$$H^*(-,\mathbb{F}_p) : (\text{Finite CW}) \to (\mathbb{F}_p - \text{Vect}) \; .$$

It has the group

$$A \in (\mathbb{F}_p){-}\text{alg} \mapsto \text{Aut}_\otimes(A) := \{\sum_{k\geq 0} a_k X^{p^k} \mid a_k \in A, \; a_0 \in A^\times\} = \text{Aut}(\hat{\mathbb{G}}_a)(A)$$

(under composition) of automorphisms of the additive group as its even multiplicative automorphisms, represented by the Hopf algebra $\mathbb{F}_p[\xi_k | k \geq 1]$, with

$$\Delta \xi_k = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j$$

dual to the algebra of Steenrod powers. This is **not** faithful; instead of Deligne's identification we get the Adams spectral sequence.]

1.2 This Tannakian philosophy, which identifies a suitably nice monoidal category in terms of representations of some group object, is usually the second step in the construction of a version of a theory of motives. In fact Deligne proposed a (symmetric monoidal &c &c) category MTM of **mixed Tate motives** over $\mathbb{Z}$ as the smallest such category containing systems of representations 'of a geometric nature'; in particular, those coming from the hyperplane decomposition

$$\mathbb{P}^n = 1 \oplus T \oplus \cdots \oplus T^{\otimes n} \; .$$

of projective space. Later work with Goncharov provided a construction of this category, together with a vestigial spectral sequence

$$\text{Ext}^*_{\mathsf{MTM}}(\mathbf{1}, T^{\otimes n}) \Rightarrow K_{2n-*}(\mathbb{Z}) \otimes \mathbb{Q}$$

$[0 \leq * \leq 1]$ (suggested by conjectures of Beilinson) which leads to an identification of the motivic group of this category as an extension

$$1 \to (\text{prounipotent free}) \to \cdots \to \mathbb{G}_m \to 1 \; .$$

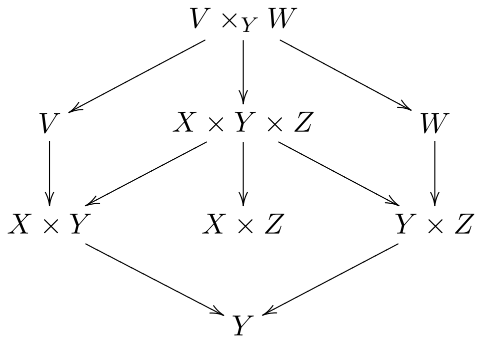
The action of the multiplicative group defines a grading on the Lie algebra of the pro-unipotent kernel: it is free, with generators $e_{2k+1}, \; k > 0$ corresponding (noncanonically) to the generators of $K^{\text{alg}}_{4k+1}(\mathbb{Z}) \otimes \mathbb{Q}$ (and hence (by Borel) to the (presumably transcendental) number $\zeta(2k+1)$). The Hopf algebra of functions on the kernel can be identified with the (rationalized) algebra of quasi-symmetric functions, which occurs (by work of Baker and Richter) in topology as $H^*(\Omega\Sigma\mathbb{C}P^\infty, \mathbb{Z})$.

[We can sometimes think of classical Galois groups as Zariski-dense subgroups of points in associated motivic groups. For example, Shafarevich conjectured the existence of an exact sequence

$$1 \to (\text{profinite free}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widetilde{\mathbb{Z}}^\times \to 1$$

analogous to Deligne's presentation above.]

To see how K-theory got into this, we need to look back at what is usually the **first** step in the construction of a category of motives.If X and Y are nice algebraic varieties, and $V \to X \times Y$ an algebraic cycle, its correspondence composition with $W \to Y \times Z$ is defined by the diagram



of fiber products. The graph of an algebraic map defines an algebraic cycle, and if $V = \text{graph } (f : X \to Y)$, $W = \text{graph } (g : Y \to Z)$ are two such, then the correspondence composition

$$V \times_Y W = \text{graph } (g \circ f : X \to Z)$$

is the graph of their composition as morphisms. Any geometric cycle theory, in the sense of Sullivan, thus defines an additive monoidal symmetric [?] category with nice varieties as objects, and abelian groups [eg $A^*(X \times Y)$ or $K_*^{\text{alg}}(X \times Y)$] as Hom-objects, together with a monoidal functor

$$(\text{Varieties}) \to (\text{proto} - \text{Motives}) \; .$$

The final category is then constructed by further refinements, such as adding kernels and cokernels, inverting $\mathbb{P}_1$, splitting idempotents ....

Possible variations involve replacing the Chow or K-theory groups with some nice **bivariant** functor, following eg Fulton-MacPherson, Kasparov, Quillen, **B. Williams**, Dundas-Østvær, Barwick-Rognes, ... In fact the theory of correspondences is basically a ruse for producing bivariant functors from more classical ones.

1.3 To a topologist, these conjectures of Beilinson immediately call up Adams' work on the image of the J-homomorphism: a real vector bundle over S^{4k} defines a stable cofiber sequence

$$S^{4k-1} \xrightarrow{\alpha} S^0 \longrightarrow \mathrm{cof}\ \alpha \longrightarrow S^{4k} \cdots$$

and hence an extension

$$[0 \to KO(S^{4k}) \to \cdots \to KO(S^0) \to 0]$$

in the group

$$\mathrm{Ext}_{\mathrm{Adams}}(KO(S^0), KO(S^{4k}) \cong H^1_c(\hat{\mathbb{Z}}^\times, \hat{K}O(S^{4k})) \cong (\zeta(1-4k)\cdot\mathbb{Z})/\mathbb{Z}\ ,$$

where the Adams operation $\psi^\alpha,\ \alpha \in \hat{\mathbb{Z}}^\times$ acts on $\hat{K}O(S^{2k})$ by $\psi^\alpha(b^k) = \alpha^k b^k$. This (essentially Galois) cohomology can be evaluated, via von Staudt's theorem, in terms of Bernoulli numbers.

The corresponding **odd** zeta-values appear in differential topology in the classification of smooth ('Euclidean') cell bundles over the $4k+2$-sphere. There, both even and odd zeta-values can be seen as having a common origin, summarized by a diagram (where, implicitly, $n \to \infty$)

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ B\mathrm{O}_n & \longrightarrow & B\mathrm{Diff}(E^n) & \longrightarrow & B\mathrm{Diff}_c(\mathbb{R}^n) \\ \downarrow & & & & \downarrow \\ BQ(S^0) & & & & \Omega\mathrm{Wh}(*)\ . \end{array}$$

The space $\mathrm{Wh}(*)$ on the bottom right is Waldhausen's smooth pseudoisotopy space, which appears in

$$K(\$) = A(*) = \$ \vee \mathrm{Wh}(*)\ .$$

The shift by a double suspension in the cell versus vector bundle story is explained by the factor B on the lower left, and Ω on the lower right. The odd zeta-values appear in both geometry and topology because the natural map

$$K(\$) \to K(\mathbb{Z})$$

is a rational equivalence. This suggests that some of the ideas of differential topology might be usefully reformulated in terms of a category of 'motives over \$' analogous to the arithmetic geometers' motives over $\mathbb{Z}$, with the algebraic K-spectrum of the integers replaced by Waldhausen's A-theory: these zeta-values might then provide a trail of breadcrumbs leading us to some deeper insights.

1.3 Integrals of algebraic forms over algebraic varieties define interesting fields ('of periods') larger than those of algebraic numbers: the polyzeta values

$$\int_{\Delta_k} (x^{-1}dx)^{k_n-1}(1-x)^{-1}dx \ldots (x^{-1}dx)^{k_1-1}(1-x)^{-1}dx =$$

$$= \sum_{0<m_1<\cdots<m_n} m_1^{-k_n} \ldots m_n^{-k_1} := \zeta(k_1, \ldots, k_n)$$

($k = \sum k_m$) defined by iterated Chen integrals are important examples. When the class of algebraic varieties defining such integrals has good monoidal properties, the associated motivic groups can be regarded (cf. Nori) as generalized Galois groups.

Work of Connes, Kreimer, and Marcolli identifies a very closely related motivic group with the symmetries of BPHZ renormalization in physics. Very roughly, Feynman's rules for calculating the probabilities of physical processes often lead to divergent integrals. Millions of man-hours by working physicists have provided an internally consistent scheme for regularizing these integrals, and in a remarkably large class of examples the coefficients of the divergencies can be expressed as polyzeta values. Whether the coupling constants of physics such as

$$\alpha = \frac{e^2}{\hbar c} \sim 1/137.0359997\ldots$$

might be explicitly calculable as periods of algebraic integrals is an old question, and Cartier has suggested the possible existence of a 'cosmic Galois theory' behind this. The work of CKM raises the possibility that reality might be defined over something like Spec $\mathbb{Z}[i]$.

§II Blumberg, Gepner, & Tabuada's big category of motives

Category theory is in the midst of a revolution, which you guys in the audience know much more about than I do; so I'll try only to sketch some basic facts. For example, Lurie [HTT 4.2.2.4] associates a K-theory spectrum to any small pointed ∞-category with finite colimits. Recall also that an ∞-category is **stable** if it has finite limits and colimits, and pullbacks and pushouts coincide [DAG I §2.9, 4.4]. Such a category has a suspension functor, which is a Quillen equivalence of the category with itself; and the Schwede-Shipley theorem asserts that any stable model category with a set of compact generators has an enrichment over spectra.

G. Tabuada [`arXiv:0801.4524`] says:

" The main idea is to replace the monoidal category of complexes of abelian groups by the monoidal category of symmetric spectra, which one should imagine as 'complexes of abelian groups up to homotopy' . . . spectral categories provide a non-additive framework for non-commutative algebraic geometry in the sene of Bondal, Drinfeld, Kapranov, Kontsevich, Toën, Van den Bergh . . . In this way several different subjects such as: equivariant homotopy theory, stable motivic theory of schemes, . . . and all the classical algebraic situations[1] fit in the context of spectral categories. . . "

BGT construct a monoidal symmetric stable ∞-category $\mathcal{M}_A$ with small stable ∞-categories as objects, such that

$$\mathrm{Hom}_{\mathcal{M}_A}(\mathbf{A},\mathbf{B}) = K(\mathrm{Fun}^{\mathrm{ex}}(\mathbf{A},\mathbf{B}))$$

when $\mathbf{A}$ is compact and idempotent-complete [roughly: in which Morita equivalences are invertible [HTT §4.4.5]; alternately, in which idempotents split. More precisely: if the image of the Yonenda embedding

$$\mathbf{C} \to \mathrm{Fun}(\mathbf{C}, \Delta^{\mathrm{op}}(\mathrm{Sets}))$$

[1]To which we can add recent work (eg by Cohen, Jones, Segal, Manolescu . . .) on Floer homology; cf `http://en.wikipedia.org/wiki/All_your_base_are_belong_to_us`.

in the category of presheaves of simplicial sets on C is closed under retracts.]

Examples of small stable idempotent-complete categories include

- perfect complexes over a scheme, and
- compact R-module spectra.

If $\mathbf{A}, \mathbf{B}$ are **both** idempotent-complete then $\mathrm{Fun}^{\mathrm{ex}}(\mathbf{A}, \mathbf{B})$ is equivalent to the category of right-compact $\mathbf{A}^{\mathrm{op}} \otimes \mathbf{B}$-modules [where $\mathbf{C} - \mathrm{Mod}$ denotes $\mathrm{Fun}^{\mathrm{ex}}(\mathbf{C}, \$)$].

Since $\mathcal{M}_A$ is monoidal,

$$\mathrm{Hom}_{\mathcal{M}_A}(\$, \$) \times \mathrm{Hom}_{\mathcal{M}_A}(\mathbf{A}, \mathbf{B}) \to \mathrm{Hom}_{\mathcal{M}_A}(\mathbf{A}, \mathbf{B})$$

defines an enrichment over the category of $\mathrm{Hom}_{\mathcal{M}_A}(\$, \$) = K(\$)$-modules.

Recall two theorems of Waldhausen:

- $\$ \to \mathbb{Z}$ induces a $\mathbb{Q}$-equivalence $K(\$) \to K(\mathbb{Z})$, and
- The Dennis trace $K \to THH$ makes

$$K(\$) \cong \$ \vee \mathrm{Wh}$$

a kind of equicharacteristic local ring over \$; more generally, $A(X) \sim Q(X_+) \wedge \mathrm{Wh}(X)$.

§III Hess's homotopy-Tannakian categories

If $E \to F$ is a Galois extension of fields, then

$$F \otimes_E F \cong \mathrm{Fns}(\mathrm{Gal}(F/E), F) \in (\text{Commutative Hopf algebras}) ,$$

and there is a fully faithful functor

$$(E - \mathrm{Mod}) \ni M \mapsto F \otimes_E M \in (\mathrm{Fns}(\mathrm{Gal}(F/E), F) - \mathrm{Comod}) .$$

This identifies the Galois group with the discrete algebraic groupscheme $\mathrm{Spec}\ \mathrm{Fns}(\mathrm{Gal}(F/E), F)$.

More generally, if $A \to B$ is a homomorphism of (commutative) ring-spectra, a factorization

$$A \rightarrowtail Q(B) \cong B$$

as a cofibration followed by a weak equivalence (in classical terms, a projective resolution) defines the **Hessian** Hopf object

$$\mathcal{H}_{B/A} \ := \ Q(B) \wedge_A Q(B) \in {}_{Q(B)}\mathrm{Mod}_{Q(B)} \ .$$

underlying a monad on the category of $Q(B)$-bimodules, whose algebras are $Q(B)$-linear representations of a groupoid Spec $\mathcal{H}_{B/A} = \mathcal{G}(B/A)$.

Example: if

$$A = \mathbb{Z}_p \to \mathbb{F}_p = B$$

then we can take $Q(B)$ to be

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\ p\ } \mathbb{Z}_p \longrightarrow 0 \ ,$$

which has the primitively generated exterior algebra

$$\mathrm{Tor}^*_{\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p[\beta]/(\beta^2)$$

as its Hessian.

Under reasonable assumptions [§5.4, 5.15, 6.9], the derived pushforward

$$\mathfrak{h} : Q(B) \otimes_A - : (A - \mathrm{Mod}) \to (\mathcal{H}_{B/A} - \mathrm{Comod}) = (\mathcal{G}(B/A) - \mathrm{Reps})$$

satisfies a homotopy-theoretic version of descent: the mapping space between a pair of (cofibrant, complete ...) objects in the domain category is weakly equivalent to the space of mappings between their image objects in the target category. In particular, there is an Adams spectral sequence

$$\mathrm{Ext}^*_{\mathcal{G}(B/A)-\mathrm{Reps}}(\mathfrak{h}(M), \mathfrak{h}(M')) \Rightarrow R^*\mathrm{Hom}_{A-\mathrm{Mod}}(M, M')$$

converging to (the associated graded to) the maps to the completion of M'. This (in principle) reconstructs a certain category of complete A-modules (ie ∞-algebras, in some sense [4.15]) from its image under $\mathfrak{h}$.

Example, cont'd: The Bockstein spectral sequence

$$\mathrm{Ext}^*_{E(\beta)}(H_*(C, \mathbb{F}_p), H_*(C', \mathbb{F}_p)) \Rightarrow \mathrm{Hom}^*_{D(\mathbb{Z}_p)}(C, C') \ .$$

[This goes back to Tate's study, in the late 50's, of the functor

$$A \mapsto \mathrm{Tor}^k_*(A, A) = \mathcal{H}_{k/A}$$

of local rings $A \to k$; but the resulting **Koszul duality**

$$\mathfrak{h} : D(A) \to D(\mathcal{G}(k/A) - \mathrm{Reps})$$

seems never to have really caught on ...]

Example, pour amuser les topologues: the Thom spectrum $M\mathrm{U}$ represents cobordism of complex-oriented manifolds, $M\mathrm{SO}$ represents cobordism of oriented manifolds, and $M\mathrm{Sp}$, unfortunately, represents cobordism of (stably almost-...) quaternionic manifolds. There are forgetful homomorphisms

$$M\mathrm{Sp} \to M\mathrm{U} \to M\mathrm{SO}$$

of ring-spectra. Away from the prime two, the map

$$\begin{array}{ccc} & & \mathrm{Sp}/U \\ & \nearrow & \downarrow \\ M & \longrightarrow & BU \\ & \searrow & \downarrow \\ & & B\mathrm{Sp} \end{array}$$

classifying a complex orientation splits, defining an odd-primary equivalence

$$M\mathrm{U} \ \sim [\tfrac{1}{2}] \ M\mathrm{Sp} \wedge \mathrm{Sp}/\mathrm{U}_+ \ ;$$

thus

$$\mathcal{H}_{M\mathrm{U}/M\mathrm{Sp}} = M\mathrm{U} \wedge^L_{M\mathrm{Sp}} M\mathrm{U} \ \sim [\tfrac{1}{2}] \ M\mathrm{U} \wedge \mathrm{Sp}/U_+ \ ,$$

ie $M\mathrm{Sp} \to M\mathrm{U}$ has the 'group ring' $\$[\mathrm{Sp}/U_+]$ as Rognes Hopf-Galois object. A similar splitting of $B\mathrm{U}$ as the product (away from two) of $B\mathrm{SO}$ with SO/U implies that

$$\mathcal{H}_{M\mathrm{SO}/M\mathrm{U}} = M\mathrm{SO} \wedge^L_{M\mathrm{U}} M\mathrm{SO} \ \sim [\tfrac{1}{2}] \ M\mathrm{SO} \wedge^L_{M\mathrm{SO} \wedge \mathrm{SO}/\mathrm{U}_+} M\mathrm{SO} \ ;$$

this is equivalent to

$$\Omega(\mathrm{SO}/U_+) \wedge M\mathrm{SO} \sim M\mathrm{SO} \wedge \mathrm{U}/\mathrm{Sp}_+ \ .$$

Consequently the composition

$$M\mathrm{Sp} \ \xrightarrow{(\mathrm{Sp}/\mathrm{U})^*} \ M\mathrm{U} \ \xrightarrow{(\mathrm{U}/\mathrm{Sp})^*} \ M\mathrm{SO}$$

is (away from two) a homotopy-equivalence.

§IV Applications

Proposition: The trace $K(\$) \to THH(\$) \sim \$$ has

$$\mathcal{H}_{\$/K(\$)} \otimes \mathbb{Q} \cong \mathrm{Tor}_*^{K_*(\mathbb{Z})\otimes\mathbb{Q}}(\mathbb{Q}, \mathbb{Q})$$

dual to the enveloping algebra of a free Lie algebra on odd-degree generators. This suggests an interpretation of the cosmic Galois group as a Hess-Rognes-Galois-Hopf object

$$\mathrm{Spec}\ (\$ \wedge^L_{A(*)} \$) := \mathcal{G}(\$/K(\$))$$

defined by derived automorphisms of the sphere spectrum, as an algebra over $K(\$)$ (regarded as a ring of deformations of $\$$).

Here is a program to make this more precise:

? $\vdash$: $\exists$ a Quillen-equivalent version $\widetilde{\mathcal{M}}_A$ of BGT's category, whose morphism-objects are cofibrant $K(\$)$-module spectra.

? $\vdash$: $\exists$ a monoidal 'underlying space' functor from $\widetilde{\mathcal{M}}_A$ to a category (Betti) with the same objects, but with

$$\mathrm{Hom}_{\mathrm{Betti}}(\mathbf{A}, \mathbf{B}) := \$ \wedge_{K(\$)} \mathrm{Hom}_{\widetilde{\mathcal{M}}}(\mathbf{A}, \mathbf{B}) \ .$$

? $\vdash$: $\exists$ On the subcategory of complete objects,

$$\widetilde{\mathcal{M}}_A \to (\mathcal{G}(\$/K(\$) - \mathrm{Reps})$$

satisfying homotopic descent, and

? $\vdash$: if X is a finite Poincaré complex, with Spanier-Whitehead dual $X_D = \mathrm{Fns}(X, \$)$, then

$$\$ \wedge_{K(\$)} K(X_D^{\mathrm{op}} \wedge Y_D - \mathrm{Mod}) \sim X_D^{\mathrm{op}} \wedge Y_D$$

(by a variant of Waldhausen's splitting, cf [`arXiv:0912.1670`]).

[This would imply that the 'big motive' defined (for a finite Poincaré complex) by the category of compact X_D-modules has, as its 'underlying' Betti object, the stable homotopy type of the complex.]

Further remarks and loose ends:

- There is a version of these constructions built from TC rather than K, which may be of interest to analysts. One expects

$$\mathcal{G}(TC(\$)/K(\$)) \sim \mathbb{Z}/2\mathbb{Z}$$

(modulo noise), based on the identification $TC(\$) \sim \Sigma \mathbb{C}P^{\infty}_{-1}$ (after p-adic completion) and Rognes' identification $K(\$) \sim \Sigma \mathbb{H}P^{\infty}$ at regular primes.

- I don't know if Voevodsky's derived category of motives maps naturally to BGT's category, nor do I know how his motivic cohomology might be related to the hypothetical Betti functor suggested here.

Note that this program only identifies (some version of) $\mathcal{M}_A$ as Tannakian over some underlying spectral category (conjecturally containing some small category of spectra); but it doesn't further identify that underlying category.