Brown-Gitler Spectra – Brown-Gitler spectra were introduced by E.H. Brown, Jr. and Samuel Gitler [1] to study higher order obstructions to immersions of manifolds, but immediately found wide applicability in a variety of areas of homotopy theory, most notably in the stable homotopy groups of spheres ([9] and [4]), in studying homotopy classes of maps out of various classifying spaces ([3], [10], and [8]), and, as might be expected, in studying the Immersion Conjecture for manifolds ([2] and [5]).

The mod $p$ homology $H_*(X) = H_*(X, \mathbb{Z}/p\mathbb{Z})$ comes equipped with a natural right action of the Steenrod algebra $A$ which is unstable; at the prime $2$, for example, this means

$$0 = 
\operatorname{Sq}^i : H_{n} X \to H_{n-i} X, \quad 2i > n.$$

Write $\mathcal{U}_*$ for the category of all unstable right modules over $A$. This category has enough projectives; indeed, there is an object $G(n)$, $n \geq 0$, of $\mathcal{U}_*$ and a natural isomorphism

$$\Hom_{\mathcal{U}_*}(G(n), M) \cong M_n$$

where $M_n$ is the vector spaces of elements of degree $n$ in $M$. The module $G(n)$ can be explicitly calculated. For example, if $p = 2$ and $x_n \in G(n)$ is the universal class, then the evaluation map $A \to G(n)$ sending $\theta$ to $x_n \theta$ defines an isomorphism

$$\Sigma^n A / [\operatorname{Sq}^i : 2i > n] A \cong G(n).$$

These are the dual Brown-Gitler modules.

This pleasant bit of algebra can be only partly reproduced in algebraic topology. For example, for general $n$ there is no space whose (reduced) homology is $G(n)$; specifically, if $p = 2$, the module $G(8)$ cannot support the structure of an unstable coalgebra over the Steenrod algebra. However, after stabilizing, this objection does not apply and we have the following result from [1],[4],[6]: there is a unique $p$-complete spectrum $T(n)$ so that $H_* T(n) \cong G(n)$ and for all pointed CW complexes $Z$, the map

$$[T(n), \Sigma^n Z] \to \tilde{H}_n Z$$

sending $f$ to $f_*(x_n)$ is surjective. Here $\Sigma^n Z$ is the suspension spectrum of $Z$, the symbol $[\cdot, \cdot]$ denotes stable homotopy classes of maps, and $\tilde{H}$ is reduced homology. The spectra $T(n)$ are the dual Brown-Gitler spectra. The Brown-Gitler spectra themselves can be obtained by the formula

$$B(n) = \Sigma^n DT(n)$$

where $D$ denotes the Spanier-Whitehead duality functor. The suspension factor is a normalization introduced to put the bottom cohomology class of $B(n)$ in degree $0$. An easy calculation shows that $B(2n) \simeq B(2n+1)$ for all primes and all $n \geq 0$.

For a general spectrum $X$ and $n \neq \pm 1$ modulo $2p$, the group $[T(n), X]$ is naturally isomorphic to the group $D_n H_* \Omega^n X$ of homogeneous elements of degree $n$ in the Cartier-Dieudonné module $D_\ast H_* \Omega^n X$ of the abelian Hopf algebra $H_* \Omega^n X$. In fact, one way to construct the Brown-Gitler spectra is to note that the functor

$$X \mapsto D_{2n} H_* \Omega^n X$$

is the degree $2n$ group of an extraordinary homology theory; then $B(2n)$ is the $p$-completion of the representing spectrum. See [6]. This can be greatly, but not completely, destabilized. See [7].
References


