Homotopy Theory of Simplicial Abelian Hopf Algebras

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Abstract

We examine the homotopy theory of simplicial graded abelian Hopf algebras over a prime field \(\mathbb{F}_p\), \(p > 0\), proving that two very different notions of weak equivalence yield the same homotopy category. We then prove a splitting result for the Postnikov tower of such simplicial Hopf algebras. As an application, we show how to recover the homotopy groups of a simplicial Hopf algebra from its André-Quillen homology, which, in turn, can be easily computed from the homotopy groups of the associated simplicial Dieudonné module.

This paper is divided into two parts. The first and larger part, is also the more abstract; in it we undertake a thorough examination of the homotopy theory of simplicial graded abelian Hopf algebras over \(\mathbb{F}_p\), \(p > 0\). The second part is a calculational application, relying heavily on the first part and intended partly to demonstrate that the homotopy theory of simplicial Hopf algebras deserves consideration. In this second part we show that the homotopy groups of a simplicial abelian Hopf algebra support a very rich and rigid structure. This has implications for the cohomology spectral sequence of a variety of cosimplicial spaces. (See, for example, the work of the second author [20], or Dwyer’s spectral sequence as explained in [2, §4].)

To explain our results in more detail, fix a prime \(p\), and let \(\mathcal{H}A\) be the category of graded, bicommutative Hopf algebras \(A\) over \(\mathbb{F}_p\) which are connected in the sense that \(A_0 \cong \mathbb{F}_p\). The objects in \(\mathcal{H}A\) are the abelian objects in the category \(\mathcal{CA}\) of graded connected coalgebras over \(\mathbb{F}_p\); hence, we call an object in \(\mathcal{H}A\) an abelian Hopf algebra. Let \(s\mathcal{H}A\) be the category of simplicial objects in \(\mathcal{H}A\).

If \(f : A \to B\) is a morphism in \(s\mathcal{H}A\) there are two obvious ways to specify when \(f\) is a weak equivalence. On the one hand, if \(A \in s\mathcal{H}A\), it is, among other things, a graded abelian group and, as such, has graded homotopy groups \(\pi_* A \cong H_*(A, \partial)\), where

\[
\partial = \Sigma(-1)^i d_i : A_n \to A_{n-1}.
\]

We could demand that \(f\) be a weak equivalence if \(\pi_* f\) is an isomorphism. On the other hand, \(\mathcal{H}A\) is an abelian category with enough projectives, so \(s\mathcal{H}A\) acquires a notion of weak equivalence.

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equivalence from [14, §II.4]; essentially, \( f : A \to B \) is a weak equivalence if the morphism \( f \), regarded as a map of chain complexes over \( \mathcal{HA} \), becomes an isomorphism in the derived category of \( \mathcal{HA} \). Another way to say the same thing is this: the category \( \mathcal{HA} \) has a set of small projective generators and, hence, there is an equivalence of categories \( D_\ast : \mathcal{HA} \to D \) to a category of modules called Dieudonné modules. (See [16]. Dieudonné theory is summarized in section 3.) Then the second notion of weak equivalence is equivalent to specifying that

\[
\pi_\ast D_\ast f : \pi_\ast D_\ast A \to \pi_\ast D_\ast B
\]

be an isomorphism.

That these two notions of weak equivalence are very different is emphasized by the following example. Let \( K \in \mathcal{HA} \) and let \( B(K) \in s\mathcal{HA} \) be the bar construction on \( K \). Then

\[
\pi_\ast B(K) \cong \text{Tor}^K_\ast(\mathbb{F}_p, \mathbb{F}_p),
\]

but

\[
\pi_\ast D_\ast B(K) \cong D_\ast K
\]

concentrated in degree 1. Nonetheless, our first main result (Theorem 5.12) is that the two notions of weak equivalence are not so different after all.

**Theorem A.** Let \( f : A \to B \) be a morphism in \( s\mathcal{HA} \). Then \( \pi_\ast f : \pi_\ast A \to \pi_\ast B \) is an isomorphism if and only if \( \pi_\ast D_\ast f : \pi_\ast D_\ast A \to \pi_\ast D_\ast B \) is an isomorphism.

This would show, for example, that a map \( H \to K \) in \( \mathcal{HA} \) is an isomorphism if and only if

\[
\text{Tor}^H_\ast(\mathbb{F}_p, \mathbb{F}_p) \to \text{Tor}^K_\ast(\mathbb{F}_p, \mathbb{F}_p)
\]

is an isomorphism. Remember that we are working with graded, connected Hopf algebras.

We actually prove much more than Theorem A. We will show that \( s\mathcal{HA} \) has two closed model category structures with respectively, the two specified notions of weak equivalence and that, furthermore, the two resulting homotopy categories are equivalent. The reader will deduce that our methods are very model theoretic.

Our second main result is a decomposition theorem for Postnikov towers in \( s\mathcal{HA} \). If \( A \in s\mathcal{HA} \), then \( A \) has a Moore-Postnikov tower \( \{A(n)\}_{n \geq 0} \). The details are spelled out in section 5, but one can obtain this tower by taking the Moore-Postnikov tower of the associated Dieudonné module, regarded as a graded simplicial set, and then noticing that defines a tower of Hopf algebras. Let \( F(n) \) be the fiber at stage \( n \), defined by the pull-back
diagram of Hopf algebras

\[
\begin{array}{c c c c}
F(n) & \longrightarrow & A(n) \\
\downarrow & & \downarrow \\
\mathbb{F}_p & \longrightarrow & A(n-1)
\end{array}
\]

It is well-known (and an observation due to Moore) that if \( A \) is a simplicial abelian group, then \( A(n) \) is non-canonically the product of \( A(n-1) \) and \( F(n) \). This is essentially because the category of abelian groups has projective dimension 1. The category \( \mathcal{HA} \) has projective dimension 2, so no such splitting occurs in \( \mathcal{HA} \). However, we have the following. (See Theorem 7.1.) Note that tensor product is the product in \( \mathcal{HA} \).

**Theorem B.** Let \( A \in s\mathcal{HA} \) and suppose \( \pi_0 A \cong \mathbb{F}_p \). Then there is a weak equivalence of simplicial algebras between \( A(n) \) and \( F(n) \otimes A(n-1) \).

In other words, the Postnikov tower for such \( A \) splits as algebras, even though it may not split as Hopf algebras. The splitting is not canonical and depends on a choice of null-homotopy of a \( k \)-invariant. These invariants are explained in section 6, and the existence of the null-homotopy depends on Theorem A and some remarks on the homological algebra of Dieudonné modules from section 4.

We claim that Theorem B has strong implications for the homotopy groups \( \pi_* A \) with \( A \in s\mathcal{HA} \). We now fix \( p = 2 \). Because \( A \in s\mathcal{HA} \) is a simplicial algebra, \( \pi_* A \) is a \( D \)-algebra in the sense of [7] and [19]; in particular there are operations \( \delta_i : \pi_n A \to \pi_{n+i} A, \ 2 \leq i \leq n, \) doubling internal degrees and satisfying Cartan- and Adem-style relations. (See section 8.) Also \( A \) is a simplicial coalgebra, so \( \pi_* A \) is an unstable coalgebra over the Steenrod algebra. Here the Steenrod algebra is expanded in the sense that \( Sq^0 \neq 1 \); indeed, the action of \( Sq^0 \) on \( \pi_* A \) is induced by the Verschiebung on \( A \). These two structures are not unrelated and both interact with the Hopf algebra structure. In particular, \( \pi_* A \) is an unstable Hopf algebra over the Steenrod algebra. The details are spelled out in [19] and recapitulated in section 8. The resulting object is a Hopf \( D \)-algebra. For the investigation of \( \pi_* A \), we note that we may assume that \( \pi_0 A \cong \mathbb{F}_p \). For if \( A_+ \) is the kernel, in \( s\mathcal{HA} \), of the natural map \( A \to \pi_0 A \), then \( \pi_0 A_+ \cong \mathbb{F}_p \) and there is a natural isomorphism \( \pi_* A \cong \pi_* A_+ \otimes \pi_0 A \).

The first result concerns the \( D \)-algebra structure of \( \pi_* A \). If \( \Lambda \) is any \( D \)-algebra, the indecomposables \( QA \) are a module over the operations \( \delta_i \) above and we write \( \mathbb{F}_2 \otimes_{\Delta} QA \) for the quotient of \( QA \) by these operations. The augmentation ideal functor on \( D \)-algebras to bigraded vector spaces has a left adjoint \( S_D \); we say \( B \) is free as a \( D \)-algebra if \( B \cong S_D(V) \) for some \( V \). Note that in this case \( \mathbb{F}_2 \otimes_{\Delta} QB \cong V \). For the following result, see Remark 8.11.3.
Theorem C. Let $A \in s\mathcal{H}A$ and suppose $\pi_0 A \cong \mathbb{F}_p$. Then $\pi_* A$ is a free $D$-algebra and there is a natural isomorphism

$$\mathbb{F}_2 \otimes_{\Delta} Q\pi_* A \cong H_*^{Q} A$$

where $H_*^{Q} A$ is the André-Quillen homology of $A$.

We do not claim that there is a natural isomorphism $\pi_* A \cong S_D(H_*^{Q} A)$. The best naturality statement we have is in Proposition 8.6. The André-Quillen homology of $A$ is that of [15, §4]. One interpretation of Theorem C is that Quillen’s fundamental spectral sequence [15, §7] collapses. As an auxiliary to this result we claim that $H_*^{Q} A$ is easy to compute, especially if one knows $\pi_* D_* A$. See Section 4. To be fair, the statement about André-Quillen homology given in Theorem C is automatic, once one knows that $\pi_* A$ is a free $D$-algebra. We included this statement to emphasize computability.

Theorem C is closely related to one of the main technical results of [12]. (See Lemma 3.1 of that paper.) There the proof is algebraic; here we have understood the homotopy theoretic basis for this result. Also, from Theorem C, it is possible to recover the main calculational result of [12].

Let $\mathcal{H}D$ be the category of Hopf $D$-algebras, If $H \in \mathcal{H}D$ one can form the vector space $\mathbb{F}_2 \otimes_{\Delta} QH$. Although the Steenrod operations act on $QH$ such is the nature of the interaction with between these operations and the $\delta_i$ that $\mathbb{F}_2 \otimes_{\Delta} QH$ only inherits an $\mathcal{A}(0) = \mathbb{F}_2[S^q_0, S^q_1]/(S^q_1)^2$ action. Let $\mathcal{L}$ be the category of bigraded unstable $\mathcal{A}(0)$ modules. The functor

$$\mathbb{F}_2 \otimes_{\Delta} Q(-) : \mathcal{H}D \rightarrow \mathcal{L}$$

has a right adjoint $\Lambda$. Let $A \in s\mathcal{H}A$ be so that $\pi_* A \cong \mathbb{F}_2$. Then Theorem C and the arguments of [12] §3 imply that the natural map

$$\pi_* A \rightarrow \Lambda(\mathbb{F}_2 \otimes_{\Delta} Q\pi_* A)$$

is an isomorphism. The functor $\Lambda$ can be easily understood in terms of functors arising in the theory of unstable algebras over the Steenrod algebra. In fact, the forgetful functor from unstable coalgebras to modules over $S^q_0$ and $S^q_1$ has a right adjoint $\Gamma$. If $M$ is a module over $S^q_0$ and $S^q_1$, $\Gamma(M)$ is an unstable Hopf algebra, with diagonal obtained by applying $\Gamma$ to the diagonal $M \rightarrow M \times M$, and there is a natural isomorphism of unstable Hopf algebras

$$\Gamma(M) \cong \Lambda(M).$$
See [12], §1 for details. Finally, the isomorphism $F_2 \otimes_{\Delta} Q\pi_* A \cong H^Q A$ of Theorem C endows the André-Quillen homology of $A$ with the structure of an $A(0)$ module. At the end of section 8 we give an intrinsic definition of this action.

The reader sensitive to generalization will wonder to what extent our hypotheses are necessary. First the analogs of Theorems C and D hold at odd primes; however, for homotopy theoretic applications, we would want to consider topologically commutative algebras—that is $xy = (-1)^{|x||y|}yx$. In the absence of an immediately compelling application we chose not to venture into the resulting notational swamp. Secondly, one can work with any perfect field, not just prime fields. However, some care must be taken with Dieudonné modules. See [16] or [5]. Finally, one could drop the internal grading, at the price of adding hypotheses. Our methods apply explicitly to “connected” Hopf algebras ([18] or [5]); other types of Hopf algebras – group rings, for example – would have to be treated differently. And Theorem A would not be true as stated here: one would only have that $f : A \to B$ is a homotopy isomorphism if and only $D_* f : D_* A \to D_* B$ is a homotopy isomorphism after a suitable completion.

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1. Preliminaries on closed model category structures.

Our main structure result for simplicial abelian Hopf algebras uses homotopy theoretic techniques in an integral way. The purpose of this section is to spell out the necessary closed model category structures on simplicial algebras and simplicial abelian Hopf algebras. We assume familiarity with simplicial model categories as put forth in [14, Part II], [8], and many other sources.

Fix a prime field $F_p$ and let $sA$ be the category of $F_p$-algebras. Here and throughout this paper, the algebras (and coalgebras and Hopf algebras) we consider will be graded and connected – that is, isomorphic to $F_p$ in degree zero.

The category $sA$ of simplicial algebras over $F_p$ is a simplicial category in the sense of Quillen [14, §II.2]; in particular, there is a mapping space functor to simplicial sets

$$\text{map}_{sA}(\cdot, \cdot) : sA^{op} \times sA \to \mathcal{S}.$$ 

For fixed $A \in sA$, the functor $B \mapsto \text{map}_{sA}(A, B)$ has left adjoint $K \mapsto A \otimes K$, and for fixed $B \in sA$, the functor $A \mapsto \text{map}_{sA}(A, B)$ from $sA$ to $\mathcal{S}^{op}$ has right adjoint $K \mapsto K^B$.

If $A$ is a simplicial abelian group, let $NA$ be the normalized chain complex on $A$; thus

$$NA_n = A_n/\text{Im}(s_0) + \cdots + \text{Im}(s_{n-1})$$
and \( \partial = \sum_{i=1}^{n} (-1)^i d_i : NA_n \to NA_{n-1} \). Then \( \pi_* A = H_* NA \cong H_(A, \sum_{i=1}^{n} (-1)^i d_i) \).

**Proposition 1.1** [14, §II.4]. The category \( sA \) acquires the structure of a simplicial model category structure where a morphism \( f : A \to B \) is

1) a weak equivalence if \( \pi_* f : \pi_* A \to \pi_* B \) is an isomorphism;
2) a fibration if \( NA_n \to NB_n \) is onto for \( n \geq 1 \); and
3) a cofibration if \( f \) has the left lifting property with respect to all trivial fibrations.

A morphism is *trivial fibration* if it is at once a weak equivalence and a fibration. There is also a notion of trivial cofibration. A morphism \( f : A \to B \) has the left lifting property with respect to \( q : X \to Y \) if any diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow f & & \downarrow q \\
B & \longrightarrow & Y
\end{array}
\]

can be completed so that both triangles commute. Of course, the map \( q \) will have the right lifting property with respect to the map \( f \). The “simplicial” in simplicial model category means Quillen’s Axiom SM7 is satisfied: if \( j : A \to B \) is a cofibration and \( q : X \to Y \) is a fibration, then the induced map in \( S \)

\begin{equation}
\text{map}_{sA}(B, X) \to \text{map}_{sA}(B, Y) \times \text{map}_{sA}(A, Y) \text{map}_{sA}(A, X)
\end{equation}

is a fibration. It is a trivial fibration if \( j \) or \( q \) is a weak equivalence.

To clarify cofibrations somewhat, we introduce the notion of a *saturated* class of morphisms. This is a class of morphisms in a category \( C \) that is closed under isomorphisms, coproducts, retracts, cobase change, and sequential colimits. The last two conditions mean this: To be closed under cobase change means that if \( j : A \to B \) is the class and

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow j & & \downarrow i \\
B & \longrightarrow & Y
\end{array}
\]

is a push-out diagram, then \( i \) is in the class. To be closed under sequential colimits means that if \( K \) is an ordinal number regarded as a category with one morphism \( s \to t \) with \( s < t < K \), and \( X : K \to C \) is a functor with \( \colim_{s < t} X(s) \to X(t) \) an isomorphism for each limit ordinal \( t < K \), and if \( X(s) \to X(s + 1) \) is in the class for \( s < K \), then \( X(0) \to \colim_{s < K} X(s) \) is in the class.
In a closed model category, the class of cofibrations and the class of trivial cofibrations are both saturated.

The saturation of a set of morphisms is the smallest saturated class containing that set. A closed model category is cofibrantly generated [8] if both the cofibrations and the trivial cofibrations are the saturation of a set of morphisms.

Let $S(\cdot)$ by the symmetric algebra functor. It is left adjoint to the forgetful functor from $\mathcal{A}$ to the category $\mathcal{M}_{F_p}$ of vector spaces.

**Proposition 1.4** [14, §II.4]. The category $s\mathcal{A}$ is cofibrantly generated. The cofibrations are the saturation of the set $Sf : S(V) \to S(W)$ where $f : V \to W$ is a level wise inclusion of simplicial vector spaces and $NW$ is finite dimensional. The trivial cofibrations are the saturation of the set $Sg : S(V) \to S(W)$ where $g : V \to W$ is a level-wise inclusion of simplicial vector spaces, $NW$ is finite dimensional, and $\pi_*g$ is an isomorphism.

The reader will have noticed that the sets of morphisms listed in this result are not really sets. This problem can be rectified by taking one representative for each isomorphism class of the morphisms $f$ and $g$. We will do this elsewhere without comment.

Before placing a closed model category structure on simplicial abelian Hopf algebras, we recall a method of generating new closed model categories from old ones. This is spelled out in [8] although the rudiments of the method appear in many places and ultimately go back to Quillen’s small object argument. [14, §II.3].

The data is this. One has a cofibrantly generated model category $C$ and a functor $G : D \to C$ with left adjoint $F$. One assumes that $G$ commutes with filtered colimits and that if $C \to D$ is one of the generating cofibrations (or trivial cofibrations) of $C$, then the object $C$ is small, in the sense that Hom$_C(C, \cdot)$ commutes with colimits over large enough ordinals. Suppose $D$ has all limits and colimits. Then $D$ inherits a closed model category structure from $C$ in the sense that a morphism $f : A \to B$ in $D$ is a weak equivalence (or fibration) if $Gf$ is a weak equivalence (or fibration) in $C$, provided the following condition holds: $F$ preserves trivial cofibrations ($F$ automatically preserves cofibrations) and the class of trivial cofibrations is closed under coproducts push-outs and sequential colimits. Furthermore, one can identify the cofibrations in $D$. If $I = \{i_\alpha : C_\alpha \to D_\alpha\}$ is the generating set of cofibrations in $C$, then the set $\{Fi_\alpha : FC_\alpha \to FD_\alpha\}$ generates the cofibrations in $D$.

We next turn to the study of the homotopy theory of abelian Hopf algebras. Let $\mathcal{HA}$ be the category of abelian Hopf algebras over $\mathbb{F}_p$, and let $\mathcal{CA}$ be the category of cocommutative coalgebras over $\mathbb{F}_p$. The forgetful functor $\mathcal{HA} \to \mathcal{CA}$ has a left adjoint $S$; if $C \in \mathcal{CA}$, $S(C)$ is the symmetric algebra on $JC = \text{Ker}\{e : C \to k\}$ endowed with the
unique diagonal so that $C \to S(C)$ is a morphism of coalgebras. There is a diagram of functors

$$
\begin{align*}
\mathcal{HA} & \xrightarrow{S} \mathcal{CA} \\
\downarrow & \downarrow \downarrow \\
\mathcal{A} & \xrightarrow{S} \mathcal{M}_p
\end{align*}
$$

where the unlabeled arrows indicate forgetful functors. In this sense $S : \mathcal{CA} \to \mathcal{HA}$ “covers” the symmetric algebra functor $S$ to algebras, which is why we use the same notation for both functors.

We now endow $s\mathcal{HA}$ with its usual simplicial structure [14, §II.2].

**Proposition 1.5.** The category $s\mathcal{HA}$ becomes a simplicial model category where

1) a morphism $f : A \to B$ is a weak equivalence if $\pi_*f$ is an isomorphism;

2) the class of cofibrations is generated by the morphisms $S(C) \to S(D)$ where $C \to D$ is an inclusion of level-wise finite dimensional coalgebras;

3) a morphism $f : A \to B$ is a fibration if it has the right lifting property with respect to all trivial cofibrations.

**Proof:** In [10], the category $s\mathcal{CA}$ of simplicial cocommutative coalgebras was supplied with a simplicial model category structure where $f : A \to B$ is a weak equivalence if $\pi_*f$ is an isomorphism, a cofibration if it is an inclusion, and a fibration if it has the right lifting property with respect to all trivial cofibrations. Furthermore, $s\mathcal{CA}$ is cofibrantly generated. The generating set of cofibrations is the set of all inclusions $C \to D$ where $D_n$ is finite-dimensional for all $n$. The generating set of trivial cofibrations is all trivial cofibrations $C \to D$ so that $D_n$ has a basis of cardinality less than $\beta$, where $\beta$ is any fixed infinite regular cardinal.

The forgetful functor $G : s\mathcal{HA} \to s\mathcal{CA}$ has left adjoint $F = S : s\mathcal{CA} \to s\mathcal{HA}$, so the method above applies. The only serious point to check is that the class of trivial cofibrations has the requisite closure properties. But $S : s\mathcal{M}_{fp} \to s\mathcal{A}$ preserves all weak equivalences, so $S : s\mathcal{CA} \to s\mathcal{HA}$ preserves trivial cofibrations. Finally, the class of trivial cofibrations in $s\mathcal{HA}$ is closed under push-outs, etc., because the forgetful functor $s\mathcal{HA} \to s\mathcal{A}$ preserves weak equivalences and cofibrations, by Proposition 1.4, and it also preserves colimits. Since $s\mathcal{A}$ is already a closed model category, we are done.

To verify SM7, use SM7b [14, III.2], which holds in $s\mathcal{HA}$ because it holds in $s\mathcal{A}$. ■

**Remark 1.6:** The preceding proof uses, either implicitly or explicitly, the following facts.

1) If $C \in \mathcal{CA}$ is a coalgebra, then $C$ is the filtered colimit of its finite dimensional coalgebras. See [5] or [18]. This was used to get the generating set of cofibrations
in $s\mathcal{A}$.

2) The forgetful functor $s\mathcal{HA} \to s\mathcal{A}$ preserves cofibrations and weak equivalences. This functor has a right adjoint $S^*$ (by the special adjoint functor theorem, if nothing else) and $S^*$ preserves fibrations and weak equivalences among fibrant objects, by formal arguments. Since every object of $s\mathcal{A}$ is fibrant, $S^*$ preserves all weak equivalences.

We close this section with a short discussion of push-outs, homotopy push-outs and the like.

First consider the category $s\mathcal{A}$, and a push-out square

$$
\begin{array}{ccc}
A & \xrightarrow{j_2} & X_2 \\
\downarrow{j_1} & & \downarrow \\
X_1 & \rightarrow & Y.
\end{array}
$$

(1.7)

Thus $Y \cong X_1 \otimes_A X_2$. If $j_2$ is a cofibration in $s\mathcal{A}$, there is a spectral sequence

$$
\text{Tor}^*_p(\pi_* X_1, \pi_* X_2)_q \Rightarrow \pi_{p+q} Y.
$$

(1.8)

See [14, §II.6] or the proof of Proposition 1.9 below. This is a first quadrant homology spectral sequence and the grading labeled $q$ arises from the grading on homotopy. Note that one can transpose the roles of $j_1$ and $j_2$.

If the morphism $j_2$ is not a cofibration in 1.7, then we define the homotopy push-out of

$$
\begin{array}{ccc}
A_1 & \xrightarrow{j_1} & A \\
\downarrow{j_2} & & \downarrow \\
A_2 & \rightarrow & X_2
\end{array}
$$

as follows: factor $j_2$ as $A \to Z \to X_2$, where the first map is a cofibration and the second a weak equivalence and set the homotopy push-out to be $X_1 \otimes_A Z$. There is a spectral sequence

$$
\text{Tor}^*_p(\pi_* X_1, \pi_* X_2) \Rightarrow \pi_{p+q} (X_1 \otimes_A Z)
$$

and the homotopy push-out is independent of the choices up to weak equivalence. We will say that (1.7) is a homotopy push-out diagram if the induced map $X_1 \otimes_A Z \to X_1 \otimes_A X_2 \cong Y$ is a weak equivalence.

Now suppose (1.7) is a push-out diagram in $s\mathcal{HA}$. One could also form the homotopy push-out of $X_1 \leftarrow A \to X_2$ in $s\mathcal{HA}$ by appropriately factoring $j_2$ in $s\mathcal{HA}$; however, since, by Propositions 1.4 and 1.5, every cofibration and weak equivalence in $s\mathcal{HA}$ is also such in $s\mathcal{A}$, the result will be weakly equivalent to the homotopy push-out in $s\mathcal{A}$. However, the point of the next result is that one need not resort to cofibrations in $s\mathcal{HA}$.  

9
Proposition 1.9. Suppose one has a push-out diagram in \( s\mathcal{HA} \)
\[
\begin{array}{ccc}
A & \xrightarrow{j_2} & X_2 \\
\downarrow{j_1} & & \downarrow \\
X_1 & \rightarrow & Y
\end{array}
\]
and \( j_2 \) is a level-wise inclusion. Then there is a spectral sequence
\[
\text{Tor}^\pi_*(\pi_*X_1, \pi_*X_2)_q \Rightarrow \pi_{p+q}Y
\]
and the push-out diagram is a homotopy push-out diagram.

Proof: Given any “two-source” of algebras \( C_1 \leftarrow D \rightarrow C_2 \) one can form the bar construction \( B_\bullet(C_1, D, C_2) \); this is a simplicial algebra with
\[
B_q(C_1, D, C_2) \cong C_1 \otimes D^\otimes_q \otimes C_2
\]
and the usual face and degeneracy maps. Note
\[
\pi_*B\bullet(C_1, D, C_2) = \text{Tor}_*^D(C_1, C_2).
\]
Form the bisimplicial algebra
\[
B\bullet(X_1, A, X_2) = \{B_p((X_1)_q, A_q, (X_2)_q)\}.
\]
Filtering by degree in \( p \) gives a spectral sequence with
\[
E_2^{p,q} \cong \text{Tor}_p^\pi_*(\pi_*X_1, \pi_*X_2)_q.
\]
Filtering by degree in \( q \) gives a spectral sequence with
\[
E_1^{p,q} \cong \text{Tor}_p^{A_q}((X_1)_q, (X_2)_q).
\]
Since \( A_q \rightarrow (X_2)_q \) is an inclusion of Hopf algebras, \( (X_2)_q \) is a projective \( A_q \) module, so \( E_1^{p,q} = 0 \) if \( p > 0 \) and \( E_2^{0,*} = \pi_*(X_1 \otimes_A X_2) \). Thus we have the spectral sequence. It is exactly this method which yields spectral sequence of 1.8, so a spectral sequence comparison argument implies \( Y \) is weakly equivalent to a homotopy push-out.

A specialization of these ideas is the notion of a cofibration sequence. This happens when \( X_1 \cong \mathbb{F}_p \). Let us write \( X \) for \( X_2 \). Then \( A \rightarrow X \rightarrow Y \) is a cofibration sequence if
$A \rightarrow X$ is a cofibration and $Y \cong \mathbb{F}_p \otimes_A X$. The homotopy cofiber is the homotopy push-out of $\mathbb{F}_2 \leftarrow A \rightarrow X$. There is a spectral sequence 

$$\text{Tor}^{\pi_* A}(\mathbb{F}_p, \pi_* X) \Rightarrow \pi_* Z$$

if $Z$ is the homotopy cofiber. Again, if $A \rightarrow X$ is a morphism of Hopf algebras, it is, irrelevant, up to weak equivalence, whether we form the homotopy cofiber in $sA$ or $s\mathcal{H}A$. In any case, if $A \rightarrow X$ is a level-wise inclusion of simplicial Hopf algebras, then $\mathbb{F}_p \otimes_A X$ is, up to weak equivalence, the homotopy cofiber. This follows from Proposition 1.9. A sequence $A \rightarrow X \rightarrow Y$ will be called a *homotopy cofiber sequence* if there is a diagram in the homotopy category

$$\begin{array}{ccc}
X & \longrightarrow & Z \\
\downarrow & & \downarrow \cong \\
X & \longrightarrow & Y
\end{array}$$

where $Z$ is the homotopy cofiber. A homotopy cofiber sequence $A \rightarrow X \rightarrow Y$ in $sA$ will be called *split* if $X \rightarrow Y$ has a section in $\text{Ho}(sA)$ and the resulting map $A \otimes Y \rightarrow X$ is a weak equivalence.

Finally, there is suspension. If $A \in sA$, the *suspension* $\Sigma A$ of $A$ is the homotopy push-out of $\mathbb{F}_p \leftarrow A \rightarrow \mathbb{F}_p$. There is a spectral sequence

$$\text{Tor}^{\pi_* A}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_* \Sigma A.$$ 

A convenient model for $\Sigma A$, at least if $A$ is cofibrant, is given by $\bar{W}A$, where $\bar{W}A = \mathbb{F}_p \otimes_A WA$ and $WA$ is the contractible simplicial algebra with

$$(WA)_q = A_q \otimes A_{q-1} \otimes \cdots \otimes A_1 \otimes A_0$$

and face and degeneracy maps given as in [9]. If $A$ is a simplicial abelian Hopf algebra, then $\bar{W}A$ is a simplicial abelian Hopf algebra and there is a weak equivalence $\Sigma A \cong \bar{W}A$ regardless of whether $A$ is cofibrant. Note that because $\mathcal{H}A$ is an abelian category, the push-out diagram

$$\begin{array}{ccc}
A & \longrightarrow & WA \\
\downarrow & & \downarrow \\
\mathbb{F}_p & \longrightarrow & \bar{W}A
\end{array}$$

is also a pull-back diagram, so $A$ is a model for $\Omega \bar{W}A \cong \Omega \Sigma A$, at least if $\bar{W}A$ was fibrant and $WA \rightarrow \bar{W}A$ a fibration. We will show how to remove this “fibrancy-fibration” clause in Section 5.
2. Dieudonné Theory

The category $\mathcal{HA}$ is an abelian category with a set of small projective generators and, as such, is equivalent to a category of modules. Schoeller’s Theorem describes that category. We recapitulate that result here and use it to elaborate the homotopy theory of $s\mathcal{HA}$.

Let $F_p$ be our fixed field of characteristic $p$ and let $Z_p$ be the $p$-adic integers. We define the Dieudonné ring on $F_p$ to be the ring $R$ by quotient of the polynomial ring

$$R = Z_p[V,F]/(VF - p).$$

Note that $R$ may be graded by requiring $\deg(F) = 1$, $\deg(V) = -1$ and $Z_p = R_0$.

A graded Dieudonné module $M$ is a positively graded abelian group with an action of $V$ and $F$

$$V : M_n \to M_{n/p}$$
$$F : M_n \to M_{pn}$$

so that $VF(x) = px$, $FV(y) = py$. Note that $M$ becomes a graded $R$-module if we ask that for $x \in M_n$ and $b \in R_m$ that

$$\deg(bx) = p^m n$$

with the understanding that $bx = 0$ if $\deg(bx)$ is a fraction. This immediately implies $p^{s+1}M_n = 0$ if $n = p^s n_0$ and $(n_0, p) = 1$.

Let $D$ be the category of graded Dieudonné modules. The following is Schoeller’s result. See [16].

**Proposition 2.2.** There is an equivalence of categories

$$D_* : \mathcal{HA} \to D.$$

The functor $D_*$ is easily described. Write $n = p^s n_0$ with $(n_0, p) = 1$. Then the graded ring $Z[x_0, x_1, \ldots, x_s]$, $\deg(x_i) = p^i n_0$, has a unique Hopf algebra structure so that the Witt polynomials

$$w_i = x_0^i + px_1^{p^{i-1}} + \cdots + p^i x_i$$

are primitive. Let $H(n) = F_p \otimes_Z Z[x_0, x_1, \ldots, x_s] = F_p[x_0, x_1, \ldots, x_s]$ with this Hopf algebra structure. Then for $H \in \mathcal{HA}$, $D_* H = \{D_n H\}$ with

$$D_n H = \text{Hom}_{\mathcal{HA}}(H(n), H).$$

12
The operators $V$ and $F$ are induced, respectively, by the inclusion $H(n) \to H(pn)$ and the morphism $H(pn) \to H(n)$ sending $x_i$ to $x_{i-1}^p$. The Hopf algebras $H(n)$, $n \geq 1$, form a system of small, projective generators for $\mathcal{H}A$. Thus Proposition 2.2 follows once Schoeller calculates $D_mH(n)$ for all $m$ and $n$. Again, the following is in [16]. (Compare [12], Lemma 6.7.)

**Lemma 2.3.** Let $\mathbb{Z}_p[n]$, $n \geq 1$, be the graded $\mathbb{Z}_p$ module isomorphic to $\mathbb{Z}_p$ concentrated in degree $n$. Then there is an isomorphism of Dieudonné modules

$$D_sH(n) \cong R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[n].$$

The grading must be interpreted as in (2.1). Note that there are natural isomorphisms

$$\text{Hom}_{\mathbb{Z}_p}(D_sH(n), M) \cong \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[n], M) \cong M_n$$

Thus $D_sH(n)$ is “free” projective in $\mathcal{D}$, by which we mean this module represents the exact functor $M \mapsto M_n$.

There are also “free” injectives. The functor on $\mathcal{D}$ given by

$$M \mapsto \text{Hom}_{\mathbb{Z}_p}(M_n, \mathbb{Z}/p^\infty)$$

is exact and representable. To see this, fix $n$, and let $\text{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)$ denote the $\mathbb{Z}\left[\frac{1}{p}\right]$-graded $R$-module with

$$\text{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)_p^n = \text{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)$$

and if $\phi \in \text{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)$, then

$$(V\phi)(x) = \phi(Fx) \quad (F\phi)(x) = \phi(Vx).$$

Let $J(n) \subseteq \text{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)$ be the largest sub-$R$ module which is a Dieudonné module. Thus $J(n)$ is the sub-module of homomorphisms $\phi : R_i \to \mathbb{Z}/p^\infty$ so that

$$V^k\phi = 0$$

if $\deg(V^k\phi) = p^{i-k}n$ is a fraction. Standard adjoint functor arguments imply

$$\text{Hom}_{\mathcal{D}}(M, J(n)) \cong \text{Hom}_{\mathbb{Z}_p}(M_n, \mathbb{Z}/p^\infty),$$

so $J(n)$ is injective. If $\Lambda(n)$ is the (unique up to isomorphism) Hopf algebra so that $D_s\Lambda(n) \cong J(n)$, then $\Lambda(n)$ is injective in $\mathcal{H}A$.

Note that (2.4) and (2.5) supply explicit recipes for showing $\mathcal{D}$ (and hence $\mathcal{H}A$) has enough projectives and enough injectives. Schoeller adds:
Proposition 2.6. The categories $\mathcal{D}$ and $\mathcal{HA}$ have projective and injective dimension 2.

We actually give a proof of this fact in Corollary 3.6.

We now turn to the study of the homotopy theory of $s\mathcal{D}$, which, by Schoeller’s Theorem, is the equivalent to studying $s\mathcal{HA}$. There are several possible ways to put a closed model category structure on $s\mathcal{D}$—one supplied, for example, by combining Proposition 1.5 with Schoeller’s result. We shall use another also. Recall that the normalization functor

$$N : s\mathcal{D} \to ch_* \mathcal{D}$$

is an equivalence of categories between $s\mathcal{D}$ and non-negatively graded chain complexes over $\mathcal{D}$.

Proposition 2.7. The category $s\mathcal{D}$, with its standard simplicial structure, becomes a simplicial model category where $f : A \to B$ is

1) a weak equivalence if $\pi_* f = H_* N f$ is an isomorphism;
2) a fibration if $NA_n \to NB_n$ is surjective for $n \geq 1$; and
3) a cofibration if $A_n \to B_n$ is injective for all $n$ and the cokernel of $A_n \to B_n$ is projective in $\mathcal{D}$ for all $n$.

Proof: See [14, II.4]

Note that 3) could easily be rephrased to say $NA_n \to NB_n$ is injective for all $n$, and $\text{cok}(NA_n \to NB_n)$ is projective in $\mathcal{D}$ for all $n$.

We note that a morphism $f : A \to B$ in $s\mathcal{D}$ is a fibration if and only if the maps induced in internal (i.e. not simplicial) degree $n$ is a fibration of simplicial sets; that is, $f : A \to B$ is a fibration if and only if

$$f_* : \text{map}_s \mathcal{D}(D_* H(n), A) \to \text{map}_s \mathcal{D}(D_* H(n), B)$$

is a fibration of simplicial sets. See [14, II.4].

The category $s\mathcal{D}$ is also cofibrantly generated. The forgetful functor from $\mathcal{D}$ to graded abelian groups has a left adjoint

$$M \mapsto R \otimes_{\mathbb{Z}_p} M$$

with grading interpreted as in 2.1. The cofibrations in $s\mathcal{D}$ are generated by morphisms $1 \otimes f : R \otimes_{\mathbb{Z}_p} A \to R \otimes_{\mathbb{Z}_p} B$ where $f : A \to B$ is a level-wise inclusion of simplicial abelian groups with $NB$ free and finitely generated over $\mathbb{Z}$. Similarly the trivial cofibrations are generated by morphisms $1 \otimes g : R \otimes A \to R \otimes B$ with $g : A \to B$ a level-wise inclusion of simplicial abelian groups, $NB$ free and finitely generated over $\mathbb{Z}$, and $\pi_* g$ an isomorphism.

We close this section with a result intended to help characterize cofibrations in $s\mathcal{HA}$. 14
Lemma 2.8. Let $f : A \to B$ be a cofibration in $s\mathcal{HA}$. Then $D_*f : D_*A \to D_*B$ is injective in $s\mathcal{D}$ and $F$ operates injectively on the cokernel. In particular if $B \in s\mathcal{HA}$ is cofibrant, $F$ operates injectively on $D_*B$.

Proof: Notice that the class of injective morphisms $M \to N$ in $s\mathcal{D}$ so that $F$ operates injectively on the cokernel is saturated. Also notice that if $g : S(C) \to S(D)$ is any of the generators of the class of cofibrations in $s\mathcal{HA}$, then $D_*g$ is injective and $F$ acts injectively on the cokernel. This last is because the cokernel is isomorphic to $S(C/D)$. Thus since the class of cofibrations is saturated, the result follows.

We conjecture that the converse of Lemma 2.8 is true, but we won’t need that result here.

3. Homological algebra of Dieudonné modules

The other piece of information necessary for our splitting result is a vanishing criterion for certain Ext groups of Hopf algebras. This will be accomplished through the medium of Dieudonné modules.

We begin with indecomposables and primitives for Dieudonné modules.

Definition 3.1. Let $M \in \mathcal{D}$ be a Dieudonné module. Then the module of indecomposables is defined by

$$QM = \text{coker}\{F : M \to M\}$$

and the module of primitives is defined by

$$PM = \text{ker}\{V : M \to M\}.$$

Note that $QM$ inherits an action of $V$ and $PM$ inherits an action of $F$. These modules are so named because if $H \in \mathcal{HA}$, then there is a natural isomorphism of $\mathbb{F}_p[V]$ modules

(3.2.1) \quad QD_*H \cong QH,

where $QH$ inherits an action by $V$ from the Verschiebung $\xi : H \to H$, and there is a natural isomorphism of $\mathbb{F}_p[F]$ modules

(3.2.2) \quad PD_*H \cong PH

where $PH$ inherits an $F$ action from the $p^{th}$ power map. To see the isomorphism (3.2.1) for example, let $\Gamma(x)$ be the divided power algebra (dual to a polynomial algebra) on a primitive generator of degree $n$. Then

$$(QH)_n^* \cong \text{Hom}_{\mathcal{HA}}(H, \Gamma(x)) \cong \text{Hom}_{\mathcal{P}}(D_*H, D_*\Gamma(x)) \cong (QD_*H)_n^*,$$
where \((\cdot)^*\) denotes the \(F_p\)-dual. Equation (3.2.2) is proved similarly.

The functor \(Q\) on \(D\) is right exact and has left derived functors; the functor \(P\) is left exact and has right derived functors. Let

\[ \Phi: D \to D \]

be the “doubling” functor; thus

\[(\Phi M)_m = \begin{cases} M_n & m = pn \\ 0 & m \neq 0 \mod p \end{cases}\]

with inherited \(F\) and \(V\) action.

**Proposition 3.3.** 1) \(L_sQ = 0\) for \(s > 1\) and there is an exact sequence in \(D\)

\[ 0 \to L_1QM \to \Phi M \xrightarrow{F} M \to QM \to 0. \]

2) \(R^sP = 0\) for \(s > 1\) and there is a short exact sequence in \(D\)

\[ 0 \to PM \to M \xrightarrow{V} \Phi M \to R^1PM \to 0. \]

**Proof:** The argument is standard, once one observes \(F: \Phi M \to M\) is one-to-one for \(M\) projective, and \(V: M \to \Phi M\) is onto for \(M\) injective.

We can use the functors \(Q\) and \(P\) to characterize projective and injective objects in \(D\) and \(\mathcal{H}A\). Call a Dieudonné module \(M\) \(F\)-projective if \(F: M \to M\) is one-to-one; there is an obvious dual notion of \(V\)-injective: we require that \(V: M \to M\) be onto. The following is inspired by [11]; the argument is originally due to F. Morel.

**Proposition 3.4.** 1) A module \(M \in D\) is projective if and only if \(M\) is \(F\)-projective and \(QM\) is a projective graded \(F_p[V]\) module.

2) A module \(M \in D\) is injective if and only if \(M\) is \(V\)-injective and \(PM\) is an injective as an \(F_p[F]\) module.

**Proof:** We prove part 2. Part 1 is dual. If \(M\) is injective it is a retract of a product of the injectives \(J(n)\) of section 2, hence has the stipulated property. This leaves the converse.

Suppose \(j: M \to J\) is an one-to-one map in \(D\) to an injective \(J\). To show \(M\) is injective we need only show it splits. Since \(Pj: PM \to PJ\) is an injection and \(PM\) is an injective \(F_p[F]\) module, \(Pj\) splits. Thus if \(K\) is the cokernel of \(Pj\), we may choose a map of \(F_p[F]\) modules \(K \to PJ\) so that the composite \(K \to PJ \to K\) is the identity. Regard \(K \to PJ \to J\) as a morphism of Dieudonné modules, and let \(J_1\) be the cokernel. Then
if $PM \to PJ$ was a bijection in degrees $k \leq m$, the induced map $PM \to PJ_1$ is still an injection and a bijection in degrees $k \leq m + 1$. Repeat the process and recursively define Dieudonné modules $J_n$ and quotients $J_{n-1} \to J_n$ so that $PM \to PJ_n$ is an injection and a bijection in degrees $k \leq m + n$. Finally, let $J_\infty = \text{colim} J_n$. Then $PM \to PJ_\infty$ is an isomorphism. Since $M$ and $J_\infty$ are both $V$-injective, $M \to J_\infty$ is an isomorphism.

From this result one can get detailed information about projective and injective resolutions.

**Lemma 3.5.** 1) Let $M \in D$ be an $F$-projective Dieudonné module. Then $M$ has a projective resolution of length 1.

2) Let $M \in D$ be a $V$-injective Dieudonné module. Then $M$ has an injective resolution of length 1.

**Proof:** For part 1, let $0 \to K \to P \to M \to 0$ be the beginning of a projective resolution. Then $K$ is $F$-projective since $P$ is. Also

$$0 \to QK \to QP \to QM \to 0$$

is exact, since $L_1Q \to = 0$, by Proposition 3.3.1. Since $QP$ is a projective $F_p[V]$ module, so is $QK$. Hence $K$ is projective.

Part 2 is similar.

**Corollary 3.6.** Every object $M$ of $D$ has a projective resolution of length 2 and an injective resolution of length 2.

**Proof:** Let $0 \to K \to F_0 \to M \to 0$ be the beginning of a projective resolution. Then $K$ is $F$-projective, so has a projective resolution

$$0 \to F_2 \to F_1 \to K \to 0.$$

Splicing the resolutions together yields the result. The claim about injective resolutions is similar.

**Corollary 3.7.** Let $M, N \in D$. Then $\text{Ext}_D^s(M, N) = 0$ for $s > 2$ and if either $M$ is $F$-projective or $N$ is $V$-injective, then

$$\text{Ext}_D^2(M, N) = 0.$$  

**Remark 3.8:** We make several assertions here without proof. First, if $H \in \mathcal{HA}$, is an abelian Hopf algebra, then the following statements are equivalent:

1) $D_s H$ is $F$-injective;
2) \( L_1 QH = 0; \)
3) the Frobenius \((\cdot)^p : H \to H\) is injective; and
4) there is a vector space \( V \subseteq H\) and an isomorphism of algebras \( S(V) \cong H.\)

These facts, and Proposition 3.4.1 can be used to characterize projectives in \( \mathcal{H}A.\) For injectives, one has that the following statements are equivalent.

1') \( D_* H\) is \( V\)-injective;
2') \( R^1 PH = 0;\)
3') the Verschiebung \( \xi : H \to H\) is injective; and
4') there is a quotient vector space \( H \to V\) and automorphism of coalgebras \( H \cong S^* V.\)

Thus one can also characterize injectives in \( \mathcal{H}A.\) Statements of this type are addressed in [1] and [11, §1]. One line of proof might go, 1 \( \iff\) 2, by Proposition 4.3, 2 \( \iff\) 4 as in [1§3], and 3 \( \iff\) 4.

4. The total derived functors of indecomposables and primitives

This section extends the homological algebra of the previous section to the category \( sD.\) The total derived functors of indecomposables can be identified with \( \text{André-Quillen}\) homology. This will be done in the next section.

Define the total left derived functor \( LQ\) of the functor \( Q\) on \( sD\) be the formula

\[
LQM = QN
\]

where \( N \to M\) is a weak equivalence in \( sD\) with \( N\) cofibrant. The simplicial vector space \( LQM\) is well-defined up to weak equivalence. Since \( N\) is cofibrant, it is level-wise projective, so, by Proposition 4.4, there is a short exact sequence

\[
0 \to \Phi N \to N \to QN \to 0,
\]

hence a long exact sequence

\[
\cdots \to \Phi \pi_q M \to \pi_q LQM \to \Phi \pi_q M \to \Phi \pi_{q-1} M \to \cdots \pi_0 LQ \to 0
\]

Here we used that \( \Phi\) is exact and \( \pi_* N \cong \pi_* M.\)

**Lemma 4.3.** 1) If \( M \in sD\) is level-wise \( F\)-projective, then

\[
\pi_* LQM \cong \pi_* QM.
\]

2) For all \( M \in sD,\) there is a short exact sequence of Dieudonné modules

\[
0 \to Q \pi_q M \to \pi_q LQM \to L_1 Q \pi_{q-1} M \to 0.
\]
Proof: For part 1, choose $N \to M$ with $N$ cofibrant. Then there is a diagram of simplicial abelian groups

$$
0 \to \Phi N \xrightarrow{F} N \to QN \to 0
$$

$$
0 \to \Phi M \xrightarrow{F} M \to QM \to 0.
$$

The result follows from the induced map on long exact sequences and the five lemma. Part 2 follows from 4.2 and the definition of $Q$ and $L_iQ$ and Proposition 3.3.1.

There is a corresponding notion of total right derived functors of primitives, muddled by the fact that the functor $P$ on $sD$ does not preserve weak equivalences among fibrant objects—or, put another way, the model category structure we have chosen on $sD$ is built from projectives and thus is not well adapted to studying right derived functors. There are several ways out of this difficulty, including a closed model category structure on $sD$ built from injectives, but the following is quick.

Lemma 4.4. Let $M \in sD$. Then there is a natural morphism $\eta : M \to J(M)$ in $sD$ so that $\eta$ is a level-wise injection and a weak equivalence and the normalized Dieudonné modules $NJ(M)_q$ are $V$-injective for $q > 0$.

Proof: The forgetful functor from $D$ to graded pointed sets has a right adjoint $I$: if $X$ is a graded pointed set

$$
I(X) = \prod_n \prod_{X_n - *} J(n)
$$

where $*$ is the basepoint. If $K \in D$, we get a short exact sequence

$$
0 \to K \to I(K) \to I^1(K) \to 0
$$

(4.4.1)

with $I(K)$ and $I^1(K)$ both $V$-injective. If $K \in ch_\ast D$ is a chain complex of Dieudonné modules, one gets an augmented bicomplex

$$
K \to I^!(K)
$$

from 4.4.1. The total complex of $I^!(K)$ is given by

$$
tI^!(K)_q = \begin{cases} 
I(K_q) \times I^1(K_{q+1}) & q \geq 1 \\
\text{Ker} \{ I(K_0) \times I^1(K_1) \to I^1(K_0) \} & q = 0
\end{cases}
$$

The augmentation $K \to I^!(K)$ induces a level-wise injection $K \to tI^!(K)$ that is a homology isomorphism. Note that $tI^!(K)_q$ is $V$-injective if $q > 0$. If $M \in sD$, define $J(M)$ by the formula

$$
NJ(M) = tI^!(NM).
$$
We now simply define the total derived functor as primitives by the equation, for $M \in sD$,

$$RP(M) = PJ(M).$$

An immediate consequence is the existence of a long exact sequence

$$\cdots \to \pi_q RP(M) \to \pi_q M \xrightarrow{\Phi} \pi_{q-1} RP(M) \to \pi_{q-1} M \to \Phi \pi_{q-1} M \to \cdots.$$  

And we have the analog of Lemma 4.3.

**Lemma 4.7.** 1) If $M \in sD$ and $NM_q$ is $V$-injective for $q > 0$, then

$$\pi_q RP(M) \cong \pi_q PM.$$  

2) For all $M \in sD$ there is a short exact sequence of Dieudonné modules

$$0 \to R^1 P\pi_{q+1} M \to \pi_q RP(M) \to P\pi_q M \to 0.$$  

We can now prove the following technical lemma, which is crucial for our splitting result. Let $S^* : \mathcal{A} \to \mathcal{HA}$ be right adjoint to the forgetful functor.

**Lemma 4.8.** Let $A \in sA$. Then for all $q > 0$ the Dieudonné module $\pi_q D_n S^* A$ is $V$-injective.

**Proof:** Fix an integer $n$ and write $n = p^n n_0$ where $(n_0, p) = 1$. Then we have the projective Hopf algebra $H(n) = \mathbb{F}_p[x_0, \ldots, x_s]$ with Witt vector diagonal and if $H \in \mathcal{HA}$,

$$D_n H \cong \text{Hom}_{\mathcal{HA}}(H(n), H).$$

The homomorphism $V : D_n H \to D_{n/p} H$ is induced by the inclusion $H(n/p) \to H(n)$. The claim is that this map is a cofibration of constant Hopf algebras in $s\mathcal{HA}$. We prove this after completing our proof.

Assuming this fact, one has a cofibration sequence in $s\mathcal{HA}$

$$\mathbb{F}_p \to H(n/p) \to H(n) \to \mathbb{F}_p[x_s] \to \mathbb{F}_p$$

where $x_s \in \mathbb{F}_p[x_s]$ is a primitive generator of degree $n$. Thus, for $H \in s\mathcal{HA}$ fibrant, one has a fiber sequence of simplicial sets

$$\text{map}_{s\mathcal{HA}}(\mathbb{F}[x_s], H) \to \text{map}_{s\mathcal{HA}}(H(n), H) \to \text{map}_{s\mathcal{HA}}(H(n/p), H).$$
Since $\text{Hom}_{\mathcal{HA}}(\mathbb{F}_p[x_s], H) = (PH)_n$ (internal degree, not simplicial degree), this fiber sequence is
\[(PH)_n \to D_n H \to D_{n/p} H.\]
In particular, $ND_* H_q$ is $V$-injective for $q > 0$, since this is what it means for a morphism of simplicial abelian groups to be a fibration. [14, §II.4]. So, applying homotopy groups to 4.1 yields the long exact sequence 4.6.

Now let $H = S^* A$. Then $H$ is fibrant in $s\mathcal{CA}$, hence in $s\mathcal{HA}$. (See Proposition 1.5). Therefore, the sequence of 4.9 is a fibration sequence. By adjointness, this fibration sequence is isomorphic to
\[
\text{map}_{s\mathcal{HA}}(\mathbb{F}_p[x_s], A) \to \text{map}_{s\mathcal{HA}}(\mathbb{F}_p[x_0, \ldots, x_s], A) \to \text{map}_{s\mathcal{HA}}(\mathbb{F}_p[x_0, \ldots, x_{s-1}], A).
\]
Since $\mathbb{F}_p[x_0, \ldots, x_{s-1}] \to \mathbb{F}_p[x_0, \ldots, x_s]$ is split as algebras, this fibration sequence is split as simplicial sets (but not as simplicial groups). Hence
\[
\pi_q\text{map}_{s\mathcal{HA}}(H(n), S^* A) \to \pi_q\text{map}_{s\mathcal{HA}}(H(n/p), S^* A)
\]
is surjective—in fact, split surjective as groups for $q \geq 1$. Thus 4.6 breaks up into a sequence of short exact sequences.
\[
0 \to \pi_q A \to \pi_q D_* S^* A \to \Phi \pi_q D_* S^* A \to 0
\]
and the result follows.

To see $H(n/p) \to H(n)$ is a cofibration in $s\mathcal{HA}$, consider the diagram
\[
\begin{array}{ccc}
S(H(n/p)) & \to & S(H(n)) \\
\downarrow & & \downarrow \\
H(n/p) & \to & H(n).
\end{array}
\]
The vertical maps are onto, and the top is a cofibration. We show the bottom map is a retract of the top. Since $H(n)$ is a projective, we choose a splitting $\sigma : H(n) \to S(H(n))$ of $S(H(n)) \to H(n)$. Since
\[
S(H(n/p))_k \cong S(H(n))_k
\]
in degrees $k \leq n/p$, and all the generators of $H(n/p)$ are in degrees $k \leq n/p$, the splitting $\sigma$ restricts to a splitting of $S(H(n/p)) \to H(n/p)$.

**Example 4.10**: One cannot make the dual assertion that $\pi_q D_* S(C)$ is $F$-projective for $C \in s\mathcal{CA}$. For example, let $p = 2$ and $E(x)$ an exterior algebra on a generator $x$ of degree $k > 0$. Let $C = \overline{W E(x)}$ regarded as a coalgebra. Then a simple calculation shows
\[
\pi_1 D_* S(C) \cong \mathbb{F}_2
\]
concentrated in degree $k$.
5. Postnikov Towers and the equivalence of homotopy theories

Before stating and proving the main result on the decomposition of Postnikov towers of simplicial Hopf algebras, we give a description of what the Postnikov tower is for a general abelian category with enough projectives. We also prove one of our main results, namely, $\text{Ho}(s\mathcal{H}A) \cong \text{Ho}(s\mathcal{D})$.

Let $\mathcal{C}$ be an abelian category with enough projectives and let $X \in s\mathcal{C}$. Define the $n^{th}$ Postnikov section of $X$ as follows: for fixed $k$, let $I_{n,k} \to X_k$ be the kernel of the map

$$d : X_k \to \prod_{\phi : [n] \to [k]} X_n$$

where $\phi$ runs over all injections in the ordinal number category with $m \leq n$ and $d$ is induced by the maps $\phi^* : X_k \to X_n$. Define

$$(5.1) \quad X(n)_k = X_k/I_{n,k}.$$ 

Notice that there is a quotient map in $s\mathcal{C}$ from $X \to X(n)$ and that if $k \leq n$, $X(n)_k = X_k$. There are also quotient maps

$$(5.2) \quad q_n : X(n) \to X(n - 1)$$

and $X \cong \lim X(n)$. Let $F(n)$ be the fiber of $q_n$, defined by the pull-back diagram

$$(5.3) \quad \begin{array}{ccc} F(n) & \longrightarrow & X(n) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X(n - 1). \end{array}$$

Note that in an abelian category, this diagram is also a push-out diagram since $X(n) \to X(n - 1)$ is surjective. The reader familiar with the Moore-Postnikov tower of a simplicial set [13] will see that this new Postnikov tower agrees with the Moore-Postnikov tower if $\mathcal{C} = Ab$, the category of abelian groups. For if $x, y \in X_k$ are $k$-simplices, then $x = y$ in $X(n)_k$ if and only if $\phi^*x = \phi^*y$ for $\phi : [m] \to [k]$ as above, which occurs if and only if the $n^{th}$ faces of $x$ and $y$ agree.

We now describe the normalization $NX(n)$. Let $NX$ be the normalization of $X$, $Z_nX = \text{Ker}\{\partial : NX_n \to NX_{n-1}\}$

$$B_nX = \text{Im}\{\partial : NX_{n+1} \to NX_n\} \cong NX_{n+1}/Z_{n+1}X.$$ 

**Lemma 5.4.** The normalization $NX(n)$ of $X(n)$ is the quotient chain complex of $NX$

$$\cdots \to 0 \to B_nX \to NX_n \to \cdots \to NX_1 \to NX_0.$$
Proof: Use the description of $NX$ as the kernel of
\[ X_k \to \prod_{\phi:[m] \to [k]} X_m \]
where $\phi$ runs over all injections in $\Delta$ so that $\phi(0) = 0$ and $m < k$. If $k > n + 1$, then any injection $\phi : [m] \to [k]$ with $m < n$ can be written as a composition of injection $\phi = \phi' \psi$ where $\phi' : [k - 1] \to [k]$ and $\phi(0) = 0$. Thus $NX_k \subseteq I_{n,k}$. If $k = n + 1$, then $NX_{n+1} \cap I_{n,n+1} = Z_{n+1}X$, and the result follows.

As an immediate consequence we have

**Lemma 5.5.** The normalization $NF(n)$ of the fiber of $X(n) \to X(n-1)$ is
\[ \cdots \to 0 \to B_n X \to Z_n X \to 0 \to \cdots \to 0 \]
concentrated in degrees $n+1$ and $n$. Furthermore
\[ \pi_k F(n) \cong \begin{cases} \pi_n X & k = n \\ 0 & k \neq n \end{cases} \]

Note that the projection map $Z_n X \to \pi_n X$ defines a canonical weak equivalence

\[
F(n) \cong K(\pi_n X, n) \cong \otimes^n \pi_n X.
\]

A word on notation is in order here. If $C$ is any abelian category, then there is a functor $\otimes : sC \to sC$ characterized by the equation
\[
(N\otimes X)_n = (NX)_{n-1}
\]
where $NX_{-1} = 0$. It is this definition of $\otimes$ which is used in equation (5.6). The functor $\otimes$ is a model for suspension in $sC$. This functor $\otimes$ is different than the functor $\otimes$ defined on simplicial algebras at the end of section 1. But it is not that different: they are both models for suspension, and if $A$ is a simplicial abelian Hopf algebra, the two notions of $\otimes$ agree. Since both meanings are ingrained in the literature we will persist.

We now prove that $D_* : s\mathcal{H}A \to s\mathcal{D}$ induces an equivalence of homotopy category. We will, in fact prove more—we will show $A \to B$ in $s\mathcal{H}A$ is a weak equivalence in $s\mathcal{H}A$ if and only if $D_* A \to D_* B$ is a weak equivalence in $s\mathcal{D}$. We will also relate André-Quillen homology of $A \in s\mathcal{H}A$ to the total derived object $LQD_* A$. The proofs go by a sequence of lemmas.

**Lemma 5.7.** The functor $D_* : s\mathcal{H}A \to s\mathcal{D}$ preserves fibrations and trivial fibrations.
Proof: Let $H(n) \in \mathcal{H}A$ be the projective so that $D_nK = \text{Hom}_{\mathcal{H}A}(H(n), K)$. Then the constant simplicial object $H(n) \in s\mathcal{H}A$ is cofibrant, by Lemma 2.8, so if $q : X \to Y$ is a fibration in $s\mathcal{H}A$

$$q_* : \text{map}_{s\mathcal{H}A}(H(n), X) \to \text{map}_{s\mathcal{H}A}(H(n), Y)$$

is a fibration of simplicial sets. This map is isomorphic to the map of simplicial abelian groups $D_nX \to D_nY$. Since this morphism of simplicial abelian groups is a fibration of simplicial sets if and only if $(ND_nX)_q \to (ND_nY)_q$ is onto for $q > 0$,

$$D_*q : D_*X \to D_*Y$$

is a fibration. If $q$ is a trivial fibration, $q_*$ is a trivial fibration, hence $\pi_qD_nX \to \pi_qD_nY$ is an isomorphism for $q \geq 0$. Hence $D_*q$ is a trivial fibration. □

If $A \in sA$ is a simplicial algebra, the André-Quillen homology $H_*^QA$ is defined as follows. Choose a trivial fibration $X \to A$ in $sA$ with $X$ cofibrant. Then $H_*^QA = \pi_*QX$. Of course, $H_*^Q$ is a functor on the homotopy category of simplicial algebras. If $A$ is a simplicial Hopf algebra, we may take $X \to A$ to be a trivial fibration in $s\mathcal{H}A$ with $X$ cofibrant in $s\mathcal{H}A$. Then $X$ is cofibrant in $sA$, so $H_*^QA \cong \pi_*QX$.

Lemma 5.8. Let $A \in s\mathcal{H}A$. Then there is a natural isomorphism

$$H_*^QA \cong \pi_*LQD_*A.$$ 

Proof: Choose a trivial fibration $X \to A$ in $s\mathcal{H}A$ with $X$ cofibrant. By Lemma 2.8 and Lemma 5.7, $D_*X \to D_*A$ is a weak equivalence and $D_*X$ is $F$-projective. Thus Lemma 4.3.1 implies

$$\pi_*LQD_*A \cong \pi_*QD_*X.$$ 

But we proved in 3.2.1 that $QD_*X \cong QX$. □

The following result, which is crucial to our enterprise, uses the fact that we are working with graded connected objects in a crucial way – the last sentence of the proof is false without this assumption.

Lemma 5.9. Let $X \in s\mathcal{H}A$ be a simplicial abelian Hopf algebra. Then $\pi_*D_*X = 0$ if and only if $\pi_*X \cong \mathbb{F}_p$.

Proof: First assume $\pi_*D_*X = 0$. Then, for all $n \geq 0$, the inclusion

$$B_nD_*X \to Z_nD_*X$$

24
is an isomorphism. Thus $D_n F(n)$ has a simplicial contraction and, then, $F(n)$ has a simplicial contraction. Hence $\pi_\ast F(n) \cong \mathbb{F}_p$. Since (5.2) is a push-out diagram, Proposition 1.9 shows $X(n) \to X(n - 1)$ is a weak equivalence. Since $X(0) = F(0)$ we have $\pi_\ast X(n) \cong \mathbb{F}_p$. Now, for any $k$, $\pi_k X \cong \pi_k X(n)$ for $n$ sufficiently large, so $\pi_\ast X \cong \mathbb{F}_p$.

Conversely, suppose $\pi_\ast X \cong \mathbb{F}_p$. Thus the unit map $X \to F_p$ is a weak equivalence, and $H^Q_\ast X \cong H^Q_\ast F_p = 0$. Then Lemma 5.8 and the long exact sequence of 4.2 implies $F : \pi_q D_\ast X \to \pi_q D_\ast X$ is an isomorphism. This can only happen if $\pi_q D_\ast X = 0$.

**Lemma 5.10.** The functor $D_\ast : s\mathcal{H}A \to sD$ sends trivial cofibrations to weak equivalences and, therefore, preserves all weak equivalences.

**Proof:** Let $A \to B$ be a trivial cofibration. Then it is an inclusion. Let $C = F_p \otimes_A B$. Then

$$0 \to D_\ast A \to D_\ast B \to D_\ast C \to 0$$

is an exact sequence of Dieudonné modules and $\pi_\ast C \cong \mathbb{F}_p$, by Proposition 1.9. Hence $\pi_\ast D_\ast C = 0$ by Lemma 5.9 and $D_\ast A \to D_\ast B$ is a weak equivalence.

Since every weak equivalence in $s\mathcal{H}A$ can be factored as a trivial cofibration followed by a trivial fibration, the result follows from Lemma 5.7.

**Lemma 5.11.** The adjoint $D^{-1}_\ast : sD \to s\mathcal{H}A$ of $D_\ast$ preserves cofibrations and all weak equivalences.

**Proof:** By Lemma 5.7 and adjointness, $D^{-1}_\ast$ preserves cofibrations and trivial cofibrations. To show $D^{-1}_\ast$ preserves weak equivalences, it is equivalent to show that if $f : A \to B$ is a morphism in $s\mathcal{H}A$ so that $\pi_\ast D_\ast f$ is an isomorphism, then so was $\pi_\ast f$. Factor $f$ as $A \xrightarrow{j} X \xrightarrow{q} B$ with $j$ is a trivial cofibration and $q$ is a fibration. Then Lemma 5.10 shows $\pi_\ast D_\ast j$ is an isomorphism, so we have that $\pi_\ast D_\ast q$ is an isomorphism. We use this data to show $\pi_\ast q$ is an isomorphism. Since $D_\ast q$ is a fibration, by Lemma 5.7, and $\pi_\ast D_\ast q$ is an isomorphism, $D_\ast q$ is surjective. Hence $q$ is surjective. Consider the pull-back diagram

$$
\begin{array}{ccc}
F & \longrightarrow & X \\
\downarrow & & \downarrow q \\
\mathbb{F}_p & \longrightarrow & B.
\end{array}
$$

Since $q$ is surjective, this is also a push-out diagram. Since

$$0 \to D_\ast F \to D_\ast X \to D_\ast B \to 0$$

is exact, $\pi_\ast D_\ast F = 0$. Lemma 5.9 implies $\pi_\ast F = 0$, so Proposition 1.9 implies $\pi_\ast q$ is an isomorphism.

25
Theorem 5.12. The functor $D_* : sHA \to sD$ and its adjoint $D^{-1}_*$ both preserve all weak equivalences and define an adjoint equivalence of categories

$$D_* : \text{Ho}(sHA) \to \text{Ho}(sD)$$

**Proof:** Combine Lemmas 5.10 and 5.11 and Quillen’s result [14, §I-3].

We can use these results to make a simple application to homotopy pull-backs in $sHA$. If $X_1 \to A \leftarrow X_2$ is a diagram in $sHA$, the homotopy pull-back is defined in the same manner as the homotopy push-out: factor $X_2 \to A$ as $X_2 \xrightarrow{j} Z \xrightarrow{q} A$ where $j$ is a weak equivalence and $q$ is a fibration. Then the homotopy pull-back is the pull-back $X_1 \square_A Z$. It is well-defined up to weak equivalence. A pull-back diagram

$$\begin{array}{ccc}
Y & \xrightarrow{u} & X_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{w} & A
\end{array}$$

will be called a homotopy pull-back diagram if $Y$ is weakly equivalent to the homotopy pull-back. By analogy with Proposition 1.9 we have:

**Proposition 5.13.** Consider a pull-back diagram in $sHA$

$$\begin{array}{ccc}
Y & \xrightarrow{u} & X_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{w} & A.
\end{array}$$

If the map $q$ is surjective, it is a homotopy pull-back diagram.

**Proof:** There is a diagram of Meyer-Vietoris sequences

$$\begin{array}{ccccccccc}
0 & \xrightarrow{} & D_*Y & \xrightarrow{} & D_*X_1 \oplus D_*X_2 & \xrightarrow{} & D_*A & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & D_*(X_1 \square_A Z) & \xrightarrow{} & D_*X_1 \oplus D_*Z & \xrightarrow{} & D_*A & \xrightarrow{} & 0.
\end{array}$$

Since $\pi_*X_2 \cong \pi_*Z$, we have $\pi_*D_*X_2 \cong \pi_*D_*Z$, whence $\pi_*D_*Y \cong \pi_*D_*(X_1 \square_A Z)$. ■

6. On $k$-invariants and looped $k$-invariants

In the last section we introduced the Postnikov tower of a simplicial Hopf algebra. In this section, we give a short description of how $k$-invariants behave. Although we will use the looped $k$-invariant for our splitting result, we begin with the $k$-invariant, as it is more familiar and because we can use it to emphasize some of the subtleties.
Let $C$ be an abelian category, $X \in sC$, and $\{X(n)\}_{n \geq 0}$ the Postnikov tower of $X$. Let $F(n)$ be the fiber of $X(n) \to X(n-1)$.

For all $Z \in sC$, choose a natural inclusion $Z \to CZ$ so that the projection map $CZ \to 0$ induces a chain equivalence $NCZ \to 0$. Define the object $BF(n)$ and the $k$-invariant by the push-out diagram

$$
\begin{array}{ccc}
X(n) & \longrightarrow & CX(n) \\
\downarrow q & & \downarrow q \\
X(n-1) & \longrightarrow & BF(n).
\end{array}
$$

(6.1)

This is also a pull-back diagram, since $q$ is surjective and a homotopy pull-back diagram. The connection with what is normally known as a $k$-invariant is supplied by the following.

**Lemma 6.2.** There are natural weak equivalences in $sC$

$$BF(n) \leftarrow \bar{W}F(n) \rightarrow K(\pi_n X, n+1).$$

**Proof:** There is a push-out diagram

$$
\begin{array}{ccc}
F(n) & \longrightarrow & CF(n) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma F(n)
\end{array}
$$

and a map of Meyer-Vietoris sequences induced by the diagram of short exact sequences in $sC$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & F(n) & \longrightarrow & 0 \oplus CF(n) & \longrightarrow & \Sigma F(n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X(n) & \longrightarrow & X(n-1) \oplus CX(n) & \longrightarrow & BF(n) & \longrightarrow & 0.
\end{array}
$$

An examination of these Meyer-Vietoris sequences shows $\Sigma F(n) \to BF(n)$ is a weak equivalence. But $\bar{W}F(n)$ is a model for $\Sigma F(n)$ and there is a weak equivalence $\bar{W}F(n) \to \bar{W}K(\pi_n X, n) = K(\pi_n X, n+1)$, by 5.6. \(\blacksquare\)

**Remark 6.3:** Because (6.1) is a homotopy pull-back diagram, the weak homotopy type of $X(n)$ depends only on the class of $k_n$ in $[X(n-1), BF(n)]_{sC}$. This follows from a standard argument, recapitulated in some detail (in a dual situation) in the proof of one of our results. See 7.3. Now there is a spectral sequence (cf. 7.4 below)

$$\text{Ext}_{C}^{p}(\pi_q X(n-1), \pi_n X) \Rightarrow [\Sigma^{(n+1)-(p+q)} X(n-1), BF(n)]_{sC}.$$
So if \( \mathcal{C} \) has projective dimension 1, \([X(n-1), BF(n)]_{\mathcal{C}} = 0\) and there is a weak homotopy equivalence (non-canonical)

\[
X(n) \simeq X(n-1) \times F(n).
\]

This happens when \( \mathcal{C} \) is the category of abelian groups. If \( \mathcal{C} = \mathcal{HA} \), then

\[
[X(n-1), BF(n)] \cong \text{Ext}^2_{\mathcal{HA}}(\pi_{n-1}X, \pi_nX)
\]

and this group need not be zero. So \( X(n-1) \to X(n) \) need not split as Hopf algebras. ■

We now define the looped \( k \)-invariant. For all \( Z \in \mathcal{sC\mathcal{A}} \), choose an object \( \theta Z \) and a natural map \( \theta Z \to Z \) so that \( 0 \to \theta Z \) is a weak equivalence. Thus \( \theta Z \) is a model for the path space on \( Z \). If \( Z \) is simplicially connected in the sense that \( Z_0 = 0 \) or, more generally, if \( \pi_0 Z = 0 \), we may take \( \theta Z \to Z \) to be a surjection. Define \( Y(n-1) \) and the looped \( k \)-invariant \( \Omega k_n \) by the pull-back diagram

\[
\begin{array}{ccc}
Y(n-1) & \to & \theta X(n) \\
\downarrow \Omega k_n & & \downarrow \\
F(n) & \to & X(n).
\end{array}
\]

(6.4)

Note that \( Y(n-1) \to \theta X(n) \) is an injection, since \( F(n) \to X(n) \) is an injection. If \( X(n) \) is simplicially connected, \( \theta X(n) \to X(n) \) is a surjection, so this is a push-out diagram and a homotopy push-out diagram.

The following result says that \( Y(n-1) \) is a model for the loops on \( X(n-1) \).

**Lemma 6.5.** Suppose \( X \) is simplicially connected. Then each \( X(n) \) is connected and there is a natural weak equivalence \( \tilde{W}Y(n-1) \simeq X(n-1) \).

**Proof:** That each \( X(n) \) is simplicially connected is clear. Also, 6.4 is now also a push-out diagram, so the cokernel of \( Y(n-1) \to \theta X(n) \) is isomorphic to the cokernel of \( F(n) \to X(n) \), which is \( X(n-1) \). But \( \theta X(n) \simeq 0 \) and \( Y(n-1) \to \theta X(n) \) is an inclusion, so the cokernel is a model for \( \Sigma Y(n-1) \simeq \tilde{W}Y(n-1) \). ■

**Remark 6.6:** Under the isomorphism

\[
[Y(n-1), F(n)]_{\mathcal{sC\mathcal{A}}} \cong [X(n-1), BF(n)]_{\mathcal{sC\mathcal{A}}}
\]

supplied by Lemmas 6.2 and 6.5, the class of \( \Omega k_n \) is identified with the class of \( k_n \). We won’t need this fact so we won’t supply more details. But it explains our notation.
7. The splitting results

We will say a simplicial Hopf algebra $X$ is *simplicially connected* if $X_0 \cong \mathbb{F}_p$. Let $X$ be a simplicial abelian Hopf algebra and let \( \{X(n)\} \) be the Postnikov tower for $X$. Then each of the sequences, $n \geq 1$,

$$F(n) \to X(n) \xrightarrow{q_n} X(n - 1)$$

is a homotopy cofibration sequence. The point of this section is to prove

**Theorem 7.1.** Suppose $X$ is simplicially connected. Then this homotopy cofibration sequence is split in the category of simplicial algebras; that is, for all $n \geq 1$, there is an isomorphism in the homotopy category of simplicial algebras $X(n-1) \otimes F(n) \to X(n)$ that fits into a homotopy commutative diagram of cofibration sequences of simplicial algebras.

There is a further decomposition of the fiber $F(n)$ which we will present at the end of the section. We also discuss how to weaken the simplicially connected hypothesis in Lemma 8.3.

Theorem 7.1 is proved by calculating with the looped $k$-invariant $\Omega k_n : Y(n-1) \to F(n)$ as defined in section 3.

**Proposition 7.2.** Let $X$ be any simplicial abelian Hopf algebra. In the homotopy category $\text{Ho}(sA)$, the looped $k$-invariant $\Omega k_n : Y(n-1) \to F(n)$ is trivial for all $n \geq 2$.

**7.3. Proof of Theorem 7.1:** We assume Proposition 7.2. Recall that $Y(n-1)$ is defined by a pull-back diagram in $sHA$

$$
\begin{array}{ccc}
Y(n-1) & \longrightarrow & \theta X(n) \\
\Omega k_n \downarrow & & \downarrow \\
F(n) & \longrightarrow & X(n)
\end{array}
$$

(7.4.1)

where $\theta X(n)$ is a model for a contractible cover for $X(n)$. Since $X$ is simplicially connected, this is also a push-out diagram. Note that, for the same reason, we have that

$$\mathbb{F}_p \otimes Y_{(n-1)} \theta X(n) \cong \mathbb{F}_p \otimes F(n) X(n) \cong X(n-1).$$

Now if $\Omega k_n$ was actually the constant map, the universal property of push-outs would demonstrate that there was one isomorphism

$$X(n) \cong F(n) \otimes X(n-1).$$
The idea, then, is to deform $\Omega k_n$ to the constant map and claim that the resulting push-out weakly equivalent to $X(n)$. Since $Y(n - 1)$ may not be cofibrant, this essentially straightforward argument requires a little care.

Choose a cofibrant object $Y \in s \mathcal{HA}$ equipped with a weak equivalence $Y \to Y(n - 1)$ and factor the composite $Y \to Y(n-1) \to \theta X(n)$ and $Y \xrightarrow{j} C \to \theta X(n)$ where the first map is a cofibration and the second a weak equivalence. Define $X_1$ by the push-out diagram

$$
\begin{array}{ccc}
Y & \rightarrow & C \\
\Omega k_n \circ j & \downarrow & \downarrow \\
F(n) & \rightarrow & X_1.
\end{array}
$$

There is a map from (7.4.2) to (7.4.1) and a comparison of the homotopy spectral sequences implies $X_1 \to X(n)$ is a weak equivalence. For the same reason $F_p \otimes_Y C \to F_p \otimes_Y (n-1) \theta X(n) \cong X(n - 1)$ is a weak equivalence.

Since $Y$ is cofibrant, Proposition 7.2 implies there is a homotopy in $h : Y \otimes \Delta^1 \to F(n)$ in the category $s \mathcal{A}$ from $\Omega k_n \circ j$ to the constant map. Define $X_2$ by the push-out diagram

$$
\begin{array}{ccc}
Y \otimes \Delta^1 & \rightarrow & C \otimes \Delta^1 \\
\downarrow & & \downarrow \\
F(n) & \rightarrow & X_2
\end{array}
$$

The inclusion of the 0-simplex $\{0\} \to \Delta^1$ defines a map from 7.4.2 to 7.4.3. Since $Y \otimes \Delta^1 \to C \otimes \Delta^1$ is a morphism in $s \mathcal{HA}$, a comparison of the homotopy spectral sequences implies $X_1 \to X_2$ is a weak equivalence. Finally, the inclusion of the 0-simplex $\{1\} \to \Delta^1$ defines, in a similar manner, a weak equivalence $F(n) \otimes F_p \otimes_Y C \to X_2$. Finally one has a diagram

$$
\begin{array}{ccc}
F(n) & \rightarrow & F(n) \otimes F_p \otimes_Y C \rightarrow F_p \otimes_Y C \\
\downarrow & & \downarrow \\
F(n) & \rightarrow & X_2 \rightarrow F_p \otimes_Y C \\
\downarrow & & \downarrow \\
F(n) & \rightarrow & X_1 \rightarrow F_p \otimes_Y C \\
\downarrow & & \downarrow \\
F(n) & \rightarrow & X(n) \rightarrow X(n-1).
\end{array}
$$

All the vertical maps are weak equivalences and all the horizontal sequences are homotopy cofibration sequences.

We now start to work on Proposition 7.2. We begin with a standard lemma.
Lemma 7.4. Let \( X \in s\mathcal{H}A \) be any simplicial abelian Hopf algebra so that \( \pi_*D_*X \cong M \) concentrated in degree \( n \). Then for any \( A \in s\mathcal{H}A \) there is a first quadrant cohomology spectral sequence

\[
\text{Ext}^p_\mathcal{D}(\pi_qD_*A, M) \Rightarrow [\Sigma^{n-(p+q)}A, X]_{s\mathcal{H}A}.
\]

This spectral sequence is equipped with an edge homomorphism

\[
\text{Ext}^2_\mathcal{D}(\pi_qD_*A, M) \rightarrow [\Sigma^{n-(q+2)}A, X]_{s\mathcal{H}A}.
\]

Proof: See [4, Chap. XVII]. The edge homomorphism arises because \( \text{Ext}^p_\mathcal{D}(N, M) = 0 \) for \( p > 2 \).

Lemma 7.5. Let \( X \in s\mathcal{H}A \) have the property that \( \pi_qD_*X = 0 \) for \( q \geq n-1 \) and suppose \( Y \in s\mathcal{H}A \) has the property that \( Y(q) \cong F_p \) for \( q < n \). Then

\[
[X, Y]_{s\mathcal{H}A} \cong \text{Ext}^2_\mathcal{D}(\pi_{n-2}D_*X, \pi_nD_*Y).
\]

Proof: Let \( \{Y(q)\} \) be the Postnikov tower for \( Y \) and let \( F(q) \) be the fiber of \( Y(q) \rightarrow Y(q-1) \). We can assume \( D_*X \) is fibrant in \( s\mathcal{D} \). Then there is a tower of fibrations of simplicial sets

\[
\{\text{map}_\mathcal{D}(D_*X, D_*Y(q))\}.
\]

The claim is that

\[
\pi_0\text{map}_\mathcal{D}(D_*X, D_*Y(q)) = \text{Ext}^2_\mathcal{D}(\pi_{n-2}D_*X, \pi_nD_*Y)
\]

if \( q \geq n \) and that

\[
\lim^1 \pi_1\text{map}_\mathcal{D}(D_*X, D_*Y(q)) = 0.
\]

The result will follow since

\[
[X, Y]_{s\mathcal{H}A} \cong [D_*X, D_*Y]_{s\mathcal{D}} \cong \pi_0\text{map}_\mathcal{D}(D_*X, D_*Y(q)).
\]

For (7.6.1), we argue inductively. If \( q = n \), then \( Y(q) = F(n) \) and (7.6.1) follows from Lemma 7.4. For larger \( q \), the homotopy pull-back square

\[
\begin{array}{ccc}
Y(q+1) & \rightarrow & CY(q+1) \\
\downarrow & & \downarrow \\
Y(q) & \rightarrow & BF(q)
\end{array}
\]

31
of 6.1 yields a homotopy fibration sequence

\[ \text{map}_s \mathcal{D}(D_*X, D_*Y(q+1)) \to \text{map}_s \mathcal{D}(D_*X, D_*Y(q)) \to \text{map}_s \mathcal{D}(D_*X, D_*BF(q)). \]

Thus, Lemma 7.4 implies

\[ \pi_t \text{map}_s \mathcal{D}(D_*X, D_*Y(q+1)) \to \pi_t \text{map}_s \mathcal{D}(D_*X, D_*Y(q)) \]

is an isomorphism for \( t \leq q - n \). Thus at once supplies the inductive step for 7.6.1 and proves 7.6.2.

**Proof of Proposition 7.2:** We need to show that the class of \( \Omega k_n \) in \( [Y(n-1), F(n)]_{s \mathcal{H}A} \) goes to zero in \( [Y(n-1), F(n)]_{s \mathcal{A}} \). We show, in fact, that

\[ [Y(n-1), F(n)]_{s \mathcal{A}} = 0. \]

The forgetful functor \( s \mathcal{H}A \to s \mathcal{A} \) and its right adjoint \( S^* \) satisfy the hypotheses of Quillen’s adjoint functor theorem [14, §1.3]; hence there is an isomorphism

\[ [Y(n-1), F(n)]_{s \mathcal{A}} \cong [Y(n-1), S^*F(n)]_{s \mathcal{H}A}. \]

Since \( F(n)_q \cong \mathbb{F}_p \) for \( q < n \), \( S^*F(n)_q \cong \mathbb{F}_p \) for \( q < n \); therefore, Lemma 5.5 implies

\[ [Y(n-1), S^*F(n)]_{s \mathcal{H}A} \cong \text{EXT}^2_{s \mathcal{D}}(\pi_{n-1}D_*X(n-1), \pi_n D_*S^*F(n)) \]

However, the \text{EXT} groups are zero by Lemma 4.8 and Corollary 3.7.

**Remark 7.8:** A consequence of Theorem 7.1 is that for all \( n \geq 0 \), there is an isomorphism in \( \text{Ho}(s \mathcal{A}) \), for \( X \in s \mathcal{H}A \) with the property that \( \pi_0 X \cong \mathbb{F}_p \), there is a splitting

\[ \phi_n : X(n) \xrightarrow{\cong} \bigotimes_{q=1}^n F(q) \]

and diagrams in \( \text{Ho}(s \mathcal{A}) \)

\[ \begin{array}{ccc}
X(n) & \xrightarrow{\phi_n} & \bigotimes_{q=1}^n F(q) \\
\downarrow & & \downarrow \\
X(n-1) & \xrightarrow{\phi_{n-1}} & \bigotimes_{q=1}^{n-1} F(q).
\end{array} \]
Since, in any given simplicial degree $k$, the towers $\{X(n)_k\}$ and $\{\bigotimes_{q=1}^n F(q)_k\}$ stabilize in the sense that $X(n)_k \cong X(n-1)_k$ for $n$ sufficiently large, we may write an isomorphism in $\text{Ho}(sA)$

$$\phi: X \xrightarrow{\cong} \bigotimes_{q=1}^\infty F(q).$$

Note that the tensor product makes sense since $F(q)_k \cong \mathbb{F}_p$ if $k < q$. Also, by (6.5), there is a weak equivalence in $sA$

$$F(q) \cong \tilde{W}^q H(q)$$

where $D_s H(q) \cong \pi_q D_s H$.

The next proposition further decomposes $F(q)$. This splitting result is a minor variation on a result noticed by the second author [19].

**Proposition 7.9.** Let $H \in \mathcal{H}A$. Then there is a homotopy cofibration sequence of simplicial algebras

$$\tilde{W}S(QH) \to \tilde{W}H \to \tilde{W}^2 S(L_1QH)$$

and this sequence is split. Furthermore, the sequence is natural up to homotopy in maps $H \to K$ in $\mathcal{H}A$, but not the splitting.

**Proof:** We use the fact that $[\tilde{W}^k S(V), A]_{sA} \cong \text{Hom}_{\mathbb{F}_p}(V, \pi_k A)$. Choose a weak equivalence $A \to \tilde{W}H$ in $sA$ with $A$ almost free in the sense of [14, §II.4]. Then $A$ is cofibrant in $sA$. Filter $A$ by powers of the augmentation ideal and get a spectral sequence

$$E_1^{s, t} \cong \pi_s S_t(QA) \Rightarrow \pi_s A.$$ 

Here $S_t(W) \cong W^\otimes t / \Sigma_t$ is the $t$\textsuperscript{th} homogeneous piece of the symmetric algebra functor. This is Quillen’s fundamental spectral sequence. Since $E_1^{s, t} \cong \pi_s Q A \cong HQ_s(\tilde{W}H) \cong L_{s-1} QH$ we have

$$E_1 \cong E(QH) \otimes \Gamma(L_1QH)$$

where $E(\cdot)$ denotes the exterior algebra and $\Gamma(\cdot)$ denotes the divided power algebra. Since Quillen’s spectral sequence is a spectral sequence of algebras, it must collapse. Hence $\pi_1 A \cong \pi_1 \tilde{W}H \cong QH$. Choose a map in $sA$

$$j: \tilde{W}S(QH) \to \tilde{W}H$$

inducing an isomorphism on $\pi_1$. This map is unique and natural up to homotopy. Factor $j$ as

$$\tilde{W}S(QH) \xrightarrow{i} B \xrightarrow{q} \tilde{W}H$$
where $i$ is a cofibration in $sA$ and $q$ is a weak equivalence, and let $C = F_{p} \otimes \tilde{W}_{S(QH)} B$. This is the homotopy cofiber of the morphism $j$. The spectral sequence

$$\text{Tor}_{p}(F_{2}, \pi_{*} \tilde{W} H) \Rightarrow \pi_{*} C$$

shows that $\pi_{*} C \cong \Gamma(L_{1}QH)$. Thus there is a weak equivalence $\tilde{W}^{2}S(L_{1}QH) \to C$, unique and natural up to homotopy. Thus it only remains to show that the sequence is split. However $\pi_{2}B \cong \pi_{2} \tilde{W} H \to \pi_{2} C \cong L_{1}QH$ is onto, so there is a lift

$$\begin{array}{c}
\tilde{W}^{2}S(L_{1}QH) \\
\downarrow \cong
\end{array}$$

and the map

$$\tilde{W} S(QH) \otimes \tilde{W}^{2}S(QH) \to B$$

is a weak equivalence by direct calculation.

**Corollary 7.10.** Let $H \in \mathcal{HA}$. Then for all $n \geq 1$ there is a homotopy cofibration sequence of simplicial algebras

$$\tilde{W}^{n}S(QH) \to \tilde{W}^{n}H \to \tilde{W}^{n+1}S(L_{1}QH)$$

and this sequence is split. Furthermore, the sequence is natural up to homotopy in morphisms $H \to K$ in $\mathcal{HA}$.

**Proof:** Apply $\tilde{W}^{n-1}$ to the sequence of Proposition 7.9. 

8. The homotopy of simplicial abelian Hopf algebras: the $D$-algebra structure

In this section and the next, we give our major application: the calculation of $\pi_{*}A$ as a functor of $H_{*}^{Q}A$, for $A \in s\mathcal{HA}$. We work only at the prime 2, as this simplifies the presentation considerably.

Let $A$ be a simplicial abelian Hopf algebra over $F_{2}$. Then $\pi_{*}A$ comes equipped with a great deal of structure. The totality of that structure makes $\pi_{*}A$ into what is known as a Hopf $D$-algebra$^{1}$ [19]. We now give a brief review.

$^{1}$The authors apologize for using the letter $D$ both here and for Dieudonné modules. These usages are both in the literature. We hope this causes no difficulty.
To begin with, $\pi_* A$ is a bigraded abelian Hopf algebra. Also, since $A$ is a simplicial cocommutative coalgebra, $\pi_* A$ is an unstable coalgebra over the Steenrod algebra. Thus there are operations on the right

$$(\cdot) Sq^i : \pi_n A \to \pi_{n-i} A$$

which halve the internal degree and are subject to the usual instability, Cartan, and Adem formulas. In particular, $Sq^i = 0$ if $2i > n$, $Sq^{n/2}$ is the Verschiebung of the Hopf algebra $\pi_* A$ and $Sq^0$ is induced from the Verschiebung of the Hopf algebra $A_n$. Note that $Sq^0 \neq 1$. Also, since the simplicial algebra multiplication $A \otimes A \to A$ is a morphism of simplicial cocommutative coalgebras, the algebra multiplication $\pi_* A \otimes \pi_* A \to \pi_* A$ is a morphism of unstable coalgebras over the Steenrod operations; that is, the multiplicative Cartan formula holds. We say $\pi_* A$ is an unstable Hopf algebra over the Steenrod algebra.

Next, since $A$ is a simplicial commutative algebra, $\pi_* A$ supports higher divided power operations — operations that go back to work of Cartan and are studied in detail in [3] and [7]:

$$\delta_i (\cdot) : \pi_n A \to \pi_{n+i} A, \quad 2 \leq i \leq n$$

doubling internal degree and defined only for $2 \leq i \leq n$. The following formulas hold:

8.1.1): $\delta_i$ is a homomorphism for $2 \leq i < n$ and $\delta_n = \gamma_2$, the divided square, so $\delta_n (x+y) = \delta_n x + \delta_n y + xy$;

8.1.2): $\delta_i (xy) = 0$ unless degree of $x$ or the degree of $y$ is zero, when $\delta_i (xy) = x^2 \delta y$ (if the degree of $x$ is zero) or $\delta_i (xy) = \delta_i (x) y^2$ (if the degree of $y$ is zero);

8.1.3): If $i < 2j$, $\delta_i \delta_j (x) = \sum_{\frac{i+j}{2} \leq s \leq \frac{i+j}{2}} \left( \frac{j-i+s-1}{j-s} \right) \delta_{i+j-s} \delta_s (x)$. Note that 8.1.1 forces $x^2 = 0$ if $x \in \pi_n A, n \geq 2$. In fact:

8.1.4): if $x \in \pi_n A, n \geq 1$, then $x^2 = 0$.

A bigraded algebra equipped with operations $\delta_i$ satisfying 8.1.1)—4) is known as a $D$-algebra. Since the diagonal $A \to A \otimes A$ is a morphism of simplicial commutative algebras, $\pi_* A \to \pi_* A \otimes \pi_* A$ commutes with the action of the $\delta_i$, where $\delta_i (x \otimes y)$ can be computed using 8.1.2. One might call such an object a $D$-Hopf algebra except that this would cause some confusion with Hopf $D$-algebras, so we avoid this terminology.

Hopf $D$-algebras arise from the observation the $\delta_i$ and the $Sq^i$ are not independent, but satisfy some Nishida-style relations. These relations are the subject of [19], and we refer the reader to that reference for details. We record here only the following relation, because it makes clear that if $\pi_* A$ is free as a $D$-algebra, then many Steenrod operations...
must be non-zero. Let $\xi$ be the Verschiebung on $\pi_* A$ and $\gamma_2$ the second divided power operation; then if $x \in \pi_n A$ and $2 \leq j \leq n$

\[(\delta_j x) Sq^j = \xi \gamma_2(x) + \sum_{2s > j} \delta_s(x Sq^s).\]  

Since $\xi \gamma_2(x)$ is often non-zero, the implications of the formula are very strong. Note, however, that $j \geq 2$, so this formula says nothing about $Sq^0$ or $Sq^1$.

The rest of this section is devoted to studying the $D$-algebra structure of $\pi_* A$, with $A \in s\mathcal{H}A$.

We begin our calculations by reducing to the case where $A$ is simplicially connected. For any $A \in s\mathcal{H}A$, regard $\pi_0 A$ as a constant object in $s\mathcal{H}A$ and define $A_+ \in s\mathcal{H}A$ by the short exact sequence in $s\mathcal{H}A$

$$\mathbb{F}_2 \rightarrow A_+ \rightarrow A \rightarrow \pi_0 A \rightarrow \mathbb{F}_2.$$  

**Lemma 8.3.** The object $A_+ \in s\mathcal{H}A$ is weakly equivalent to a simplicially connected simplicial Hopf algebra and there is a natural isomorphism of Hopf $D$-algebras

$$\pi_* A \cong \pi_0 A \otimes \pi_* A_+.$$  

**Proof:** To see $A_+$ is weakly equivalent to a simplicially connected object, note that $D_* \pi_0 A \cong \pi_0 D_* A$ concentrated in degree 0. Hence $\pi_0 D_* A_+ = 0$. There is a simplicial Dieudonné module $M$ and a weak equivalence (in $sD$) $M \rightarrow D_* A_+$ so that $M_0 = 0$. Let $B \in s\mathcal{H}A$ be so that $D_* B \cong M$ and $B \rightarrow A_+$ the induced map. Then $B$ is simplicially connected and weakly equivalent to $A_+$. The statement about $\pi_* A$ follows from the short exact sequence of Hopf $D$-algebras

$$\mathbb{F}_p \rightarrow \pi_* A_+ \rightarrow \pi_* A \rightarrow \pi_0 A \rightarrow \mathbb{F}_p.$$  

**Remark 8.4:** Let $A \in s\mathcal{H}A$. Then $D_* \pi_0 A \cong \pi_0 D_* A$, so to recover $\pi_* A$ from $\pi_* D_* A$, we need only worry about the case where $A$ is simplicially connected. This will be in the hypothesis of all our results in the rest of the section.

We now calculate $\pi_* A$ with its action by the operations $\delta_i$. If $B$ is any simplicial commutative algebra, then the bigraded algebra $\pi_* B$ equipped with the action by the $\delta_i$, the associated Cartan and Adem formulas and the fact that $x^2 = 0$ if $x \in \pi_n B$, $n \geq 1$, is an example of a $D$-algebra [19, §2]. There is a category of such and the augmentation
ideal functor $I$ from $D$-algebras to bigraded vector spaces has a left adjoint $S_D$; in fact, if $V$ is a simplicial graded vector space

\begin{equation}
S_D(\pi_\ast V) \cong \pi_\ast S(V).
\end{equation}

By [6], this formula determines $S_D$.

Now let $A \in \mathcal{HA}$. Then $\pi_q D_\ast A$ is a Dieudonné module and we can form $Q\pi_q D_\ast A$ and $L_1 Q\pi_q D_\ast A$ as in 3.0. Let $Q\pi_\ast D_\ast A$ be the resulting bigraded vector space and let $\Sigma L_1 Q\pi_\ast D_\ast A$ be the bigraded vector space with

$$(\Sigma L_1 Q\pi_\ast D_\ast A)_{p,q} = (L_1 Q\pi_{q-1} D_\ast A)_p.$$ 

Thus we have shifted degrees in homotopy.

**Proposition 8.6.** Let $A \in \mathcal{HA}$ be a simplicially connected abelian Hopf algebra. Then there is a natural exact sequence of $D$-algebras

$$\mathbb{F}_p \rightarrow S_D(Q\pi_\ast D_\ast A) \overset{i_\ast}{\rightarrow} \pi_\ast A \rightarrow S_D(\Sigma L_1 Q\pi_\ast D A) \rightarrow \mathbb{F}_p.$$ 

The sequence is split, but not naturally.

Before proving this, we will construct the map $S_D(Q\pi_\ast D_\ast A) \rightarrow \pi_\ast A$. To begin with we produce a map $\pi_\ast D_\ast A \rightarrow \pi_\ast A$. Since

$$\pi_q D_n A \cong \pi_q \text{map}_{\mathcal{HA}}(H(n), A),$$ 

an element $x \in \pi_q D_n A$ determines a unique homotopy class of maps

$$f_x : \tilde{W}^q H(n) \rightarrow A$$

and, hence, a unique map, $\pi_\ast f_x : \pi_\ast \tilde{W}^q H(n) \rightarrow \pi_\ast A$. Since $H(n)$ is a polynomial algebra,

\begin{equation}
\pi_\ast \tilde{W}^q H(n) \cong S_D(\Sigma^q Q H(n))
\end{equation}

by (8.5). Here $\Sigma^q Q H(n)$ means $Q H(n) \cong \pi_q \tilde{W}^q H(n)$. Let $i_n \in Q H(n)_n$ be the top class. Then the assignment $x \mapsto \pi_\ast f_x(i_n)$ defines a homomorphism $\pi_q D_n A \rightarrow (\pi_q A)_n$.

The operator $F : \pi_q D_n A \rightarrow \pi_q D_{p,n} A$ is defined by the Hopf algebra map $H(pn) \rightarrow H(n)$ sending $x_i \in H(pn)$ to $x_{i-1}^p \in H(n)$. Hence if $x = F y \in \pi_q D_{p,n} A$, then 8.7 implies

$$\pi_\ast f_x(i_{pn}) = 0 \in (\pi_q A)_{pn}$$

37
and we have defined a natural
\[ Q\pi_q D_* A \to \pi_q A. \]

This extends to a map of \( D \)-algebra
\[
i_A : S_D(Q\pi_* D_* A) \to \pi_* A. \tag{8.8} \]

This is the map of Proposition 8.6.

8.9: Proof of Proposition 8.6: By combining Theorem 7.1 and Corollary 7.10, we immediately have that \( i_A \) is injective. Let
\[
C(A) \cong \mathbb{F}_p \otimes_{S_D(Q\pi_* D_* A)} \pi_* A.
\]

We need only show there is a natural isomorphism
\[
C(A) \cong S_D(\Sigma L_1 Q\pi_* D_* A). \tag{8.9.1} \]

Then the sequence of 8.6 is split, because \( C(A) \) is a free \( D \)-algebra.

Suppose, for a moment, that we can construct such a natural isomorphism for \( A \) simplicially connected and \( \pi_* D_* A \) concentrated in a single degree. Then we induct over the Postnikov tower to get a natural isomorphism
\[
C(A(n)) \cong S_D(\Sigma L_1 Q\pi_* D_* A(n)).
\]

Indeed, if one has such an isomorphism for \( n - 1 \), one need only complete the diagram
\[
\begin{array}{c}
\mathbb{F}_2 \\
\cong \\
\mathbb{F}_2
\end{array}
\begin{array}{c}
\rightarrow C(F(n)) \\
\rightarrow C(A(n)) \\
\rightarrow C(A(n - 1)) \\
\rightarrow \mathbb{F}_2
\end{array}
\begin{array}{r}
\cong \\
\cong
\end{array}
\begin{array}{c}
\mathbb{F}_2 \\
\mathbb{F}_2
\end{array}
\begin{array}{c}
\rightarrow S_D(\Sigma L_1 Q\pi_* D_* F(n)) \\
\rightarrow S_D(\Sigma L_1 Q\pi_* D_* A(n)) \\
\rightarrow S_D(\Sigma L_1 Q\pi_* D_* A(n - 1)) \\
\rightarrow \mathbb{F}_2
\end{array}.
\]

But both rows are uniquely split as \( D \)-algebras, as \( C(F(n)) \cong \mathbb{F}_2 \) in degrees less than \( n + 1 \) and the generators on \( C(A(n - 1)) \) as a \( D \)-algebra all lie in degrees less than \( n + 1 \). Thus the result follows from the next lemma.

Lemma 8.10. Let \( A \in s\mathcal{H}A \) be simplicially connected and suppose \( \pi_* D_* A \cong M \) concentrated in degree \( m \). Then
\[
C(A) \cong S_D(\Sigma L_1 Q M).
\]

Furthermore this isomorphism is natural with respect to maps in \( s\mathcal{H}A \) among simplicially connected objects with one non-vanishing group in \( \pi_* D_*(\cdot) \).
**Proof:** The existence of the isomorphism follows from Corollary 7.10. We prove naturality. Suppose $A$ is as stated and $B$ is simplicially connected with $\pi_* D_* B \cong N$ concentrated in degree $n$. By Lemma 7.4 $[A, B]_{s\mathcal{H}A} = 0$ unless $0 \leq n - m \leq 2$, whence

$$[A, B]_{s\mathcal{H}A} \cong \operatorname{Ext}^{n-m}_D(M, N).$$

If $m - n = 1$ or $2$, then $C(A) \to C(B)$ is the trivial map, since $C(B) \cong E_2$ in degree less than $n + 1$ and the generators $C(A)$ as a $D$-algebra lie in degree $m + 1$. If $m = n$, let $H, K \in \mathcal{H}A$ be so that $D_* H \cong M$ and $D_* K \cong N$. Then a model for $f : A \to B$, up to homotopy is given by

$$\tilde{W}^n g : \tilde{W}^n H \to \tilde{W}^n K$$

where $g$ is the image of $f$ under the isomorphism

$$[A, B]_{s\mathcal{H}A} \cong \operatorname{Hom}_D(M, N) \cong \operatorname{Hom}_{\mathcal{H}A}(H, K).$$

The result now follows from the naturality clause of Corollary 7.10.

**Remark 8.11:**

1) It is easy to construct examples where the splitting of Proposition 8.6 is not natural. For example, if $A$ is a cofibrant model for $\tilde{W}E(x)$ where $E(x)$ is the exterior algebra on a primitive generator of degree $n$ and $\mathbb{F}_p[y]$ is the polynomial algebra on a primitive generator of degree $2n$, then there is a map $A \to \tilde{W}^2 \mathbb{F}_2[y]$ which is surjective on homotopy. However, both the end terms of 8.6 must map trivially.

2) One consequence of Proposition 8.6 is that the Quillen spectral sequence for $\pi_* A$ collapses. Assuming $A$ is cofibrant, one filters $A$ by powers of the augmentation ideal (cf. the proof of 7.9) and gets a spectral sequence

$$\pi_* S(QA) \Rightarrow \pi_* A.$$

Assuming $A$ is simplicially connected, one uses 8.5 to get

$$\pi_* S(QA) \cong S_D(H_*^Q A)$$

and Proposition 8.6 to get that the spectral sequence collapses. Here one uses the exact sequence

$$0 \to Q\pi_q D_* A \to H_*^Q A \to L_1 Q\pi_{q-1} D_* A = 0.$$

Note that the non-naturality of the splitting in Proposition 8.6 is absorbed in the filtration of the spectral sequence.
3) The previous remark and Proposition 8.6 immediately imply Theorem C of the introduction.

Fix \( A \in s\mathcal{H}A \) and suppose that \( A \) is simplicially connected. Then formula 8.1.2) implies that the vector space of indecomposables \( Q\pi_*A \) supports an action by the operations \( \delta_i \) (now all homomorphisms by 8.1.1) subject to the relations of 8.1.3. Let \( F_2 \otimes_\Delta Q\pi_*A \) be the quotient of \( Q\pi_*A \) by this action. In light of Remark 8.11.2, there is a natural isomorphism:

\[
F_2 \otimes_\Delta Q\pi_*A \cong H_*^Q A.
\]

The module \( Q\pi_*A \) is also an unstable module over the Steenrod operations; however, because of formula 8.2, \( F_2 \otimes_\Delta Q\pi_*A \) does not get an induced unstable structure. Nonetheless, \( F_2 \otimes_\Delta Q\pi_*A \) is a module over the sub-algebra \( \mathcal{A}(0) = F_2[Sq^0, Sq^1]/(Sq^1)^2 \subseteq \mathcal{A} \) of the Steenrod algebra generated by \( Sq^0 \) and \( Sq^1 \) and the quotient map

\[
Q\pi_*A \to F_2 \otimes_\Delta Q\pi_*A \cong H_*^Q A
\]

is a module over \( \mathcal{A}(0) \). As mentioned in the introduction, the Hopf \( D \)-algebra \( \pi_*A \) is determined by this \( \mathcal{A}(0) \) module. The reader may prefer, on aesthetic grounds, to have a description of the \( \mathcal{A}(0) \) action internal to \( H_*^Q A \), rather than have it imposed from the outside form \( \pi_*A \). We will sketch such a description.

Choose a weak equivalence of simplicial Hopf algebras \( B \to A \) so that \( F \) acts injectively on \( D_*B \). Then by Lemmas 4.3 and 5.8, there is a natural isomorphism

\[
H_*^Q A \cong \pi_* QD_*B
\]

where \( QD_*B \) fits into the short exact sequence

\[
0 \to \Phi D_*B \overset{\epsilon}{\to} D_*B \to QD_*B \to 0.
\]

Define the action of \( Sq^0 \) on \( H_*^Q A \) to be the action induced by \( V \) on \( \pi_* QD_*B \). Define the action of \( Sq^1 \) to be the “Bockstein” of the short exact sequence of (8.13); thus \( Sq^1 \) is the composite

\[
\pi_n QD_*B \to \pi_{n-1} \Phi D_*B \cong \pi_{n-1} D_*B \to \pi_{n-1} QD_*B.
\]

Note that \( Sq^1Sq^1 = 0 \) and \( Sq^0Sq^1 = Sq^1Sq^0 \). Both \( Sq^1 \) and \( Sq^0 \) divide internal degree by 2.

The claim is that this \( \mathcal{A}(0) \) action on \( H_*^Q A \) is the same as that considered above in 8.12. The proof is by the method of universal examples. Let \( \Gamma(x_k) \) be the divided power algebra on a single primitive generator of degree \( k \).
Lemma 8.14. Let $A \in s\mathcal{H}A$. Then there is a natural isomorphism

$$[A, \tilde{W}^n\Gamma(x_k)]_{s\mathcal{H}A} \cong [H^n Q A]_k^*.$$  

Proof: In the category of Dieudonné modules

$$\text{Hom}_D(D_s A, D_s \Gamma(x_k)) = [Q D_s A]_k^*.$$  

Choose a weak equivalence $B \to A$ in $s\mathcal{H}A$ so that $F$ acts injectively on $D_s B$. Then

$$[A, \tilde{W}^n\Gamma(x_k)]_{s\mathcal{H}A} \cong [D_s B, \tilde{W}^n D_s \Gamma(x_k)]_{sD}$$

and the result follows.  

Now to show that the two definitions of the $\mathcal{A}(0)$ structure on $H^* Q A$ agree for all $A$, we need only show they agree on $H^* Q W^n \Gamma(x_k)$ for all $n$ and $k \geq 1$. This is a calculation, and we omit it. It is similar, but far easier, than the similar calculation given in [12]. §6.

References

13. J.P. May, 