Realizing Families of Landweber Exact Theories

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Summary

The purpose of this talk is to give a precise statement of

1. The Hopkins-Miller Theorem on Topological Modular Forms
2. Lurie's Theorem on realizing structure sheaves on algebraic stacks

Since I have to talk about stacks, let's begin with:

The moduli stack of formal groups

The moduli stack $\mathcal{M}_{\mathbf{fg}}$ classifies all smooth 1-dimensional formal groups and their isomorphisms:

If R is a ring, then morphisms

$$G : \mathrm{Spec}(R) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

are in one-to-one correspondence with formal groups G over R. And 2-commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}(S) & \overset{H}{\searrow} & \\ {\scriptstyle f}\downarrow & & \mathcal{M}_{\mathbf{fg}} \\ \mathrm{Spec}(R) & \underset{G}{\nearrow} & \end{array}$$

correspond to pairs $(f : R \to S,\ \phi : H \xrightarrow{\cong} f^*G\ )$.

Formal group laws

Example

If $F(x,y) = x +_F y \in R[[x,y]]$ is a formal group law over R, then the assignment

$$\begin{aligned} R[[x]] &\longrightarrow R[[x,y]] \\ x &\mapsto F(x,y) \end{aligned}$$

gives $G = \mathrm{Spf}(R[[x]])$ the structure of a formal group over R.

The choice of x is a *coordinate*. Every formal group can be given a coordinate locally.

Isomorphisms can be expressed locally as invertible power series $\phi(x)$ is x; that is, $\phi'(0)$ is a unit.

Complex cobordism

If $L \cong \mathbb{Z}[a_1, a_2, \ldots]$ is the Lazard ring, then the universal formal group law over L gives a faithfully flat morphism

$$\mathrm{Spec}(L) \longrightarrow \mathcal{M}_{\mathbf{fg}}.$$

This cover is *presentation*. We note $L = MUP_0$, where MUP is a 2-periodic version of complex cobordism.

Landweber exact homology theories

Example

Let E be a 2-periodic complex orientable cohomology theory. Thus $E^*(X)$ has commutative cup products, E^* is in even degrees and there is a unit $u \in E_2 = E^{-2}$. Then

$$G_E \stackrel{\mathrm{def}}{=} \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty)$$

is a formal group over over E_0.

Conversely if $G : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is *flat*, there is a 2-periodic complex orientable cohomology theory $E(R, G)$ with $E_0 = R$ and

$$\mathrm{Spf}(E(R, G)^0\mathbb{C}\mathrm{P}^\infty) \cong G.$$

The Landweber exact functor theorem gives conditions for flatness; hence we call these *Landweber exact* theories.

Invariant differentials

If G is a formal group over R, then we get an R-module ω_G of invariant differentials. Locally these are the differentials $f(x)dx$ so that

$$f(x +_F y)d(x +_F y) = f(x)dx + f(y)dy.$$

The module ω_G is locally free of rank 1 with local generator given by

$$\frac{dx}{F_x(0, x)}$$

where $F_x = \partial F/\partial x$.

Example

If $x +_F y = x + y + xy$ is the *multiplicative formal group law* then this generator is

$$\frac{dx}{1 + x} = \frac{d}{dx} \log(1 + x).$$

A 2-commuting diagram with $\phi : f^*G \xrightarrow{\cong} H$

$$\begin{array}{ccc} \mathrm{Spec}(S) & \xrightarrow{H} & \\ {\scriptstyle f}\downarrow & & \mathcal{M}_{\mathbf{fg}} \\ \mathrm{Spec}(R) & \xrightarrow{G} & \end{array}$$

yields an isomorphism

$$d\phi : S \otimes_R \omega_G = f^*\omega_G \longrightarrow \omega_H$$

given locally by multiplication by $\phi'(0)$.
Thus the assignment

$$\omega(G : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}) = \omega_G$$

defines a *quasi-coherent* sheaf on $\mathcal{M}_{\mathbf{fg}}$.

Complex orientable theories, redux

The sheaf ω on $\mathcal{M}_{\mathbf{fg}}$ is locally free of rank 1 and we have all tensor powers $\omega^{\otimes t}$, $t \in \mathbb{Z}$.

Example

Let E be a 2-periodic complex orientable cohomology theory with associated formal group over E_0,

$$G_E = \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty).$$

Then

$$\omega_G^{\otimes t} \cong E_{2t}.$$

A choice of unit $u \in E_2$ is a choice of generating invariant differential in ω_G.

If $f : S$ is a scheme, or more generally, an algebraic stack, then a morphism $S \to \mathcal{M}_{\mathbf{fg}}$ parametrizes a *family* of formal groups over S. If the morphism f is flat, then we get a family of cohomology theories, one for each flat map

$$\mathrm{Spec}(R) \to S.$$

By Hovey-Strickland we get an associated diagram in the stable homotopy category.

Example

Even simple examples can give bad families: take the formal group law

$$x +_F y = x + y + bxy$$

over $\mathrm{Spec}(\mathbb{Z}[b]) = \mathbb{A}^1$. This is multiplicative over $\mathbb{A}^1 - \{0\}$ (where b is a unit) and additive at 0. The map $\mathbb{A}^1 \to \mathcal{M}_{\mathbf{fg}}$ is not flat.

If S is an algebraic stack, we would want the morphism $S \to \mathcal{M}_{\mathbf{fg}}$ to be "scheme-like" or *representable*.

The Realization Problem

Let $S \to \mathcal{M}_{\mathbf{fg}}$ be a representable and flat morphism from an algebraic stack. Is there a pre-sheaf of E_∞-ring spectra $\mathcal{O}^S_{\mathbf{top}}$ with an isomorphism of associated sheaves

$$(\mathcal{O}^S_{\mathbf{top}})_* \cong \mathcal{O}^S_* = \{\omega_S^{\otimes t}\}?$$

If so, how unique is this? What is the homotopy type of the space of all realizations?

Topologies

Remark

I have been using the flat or (*fpqc*)-topology. There are too many flat maps, it turns out, so we should really use the étale topology. In this case, we need S to be a Deligne-Mumford stack; that is, have an étale presentation.

Elliptic curves

Let $\mathcal{M}_{e\ell\ell}$ be the moduli stack of elliptic curves; thus, a morphism

$$\mathrm{Spec}(R) \longrightarrow \mathcal{M}_{e\ell\ell}$$

corresponds to an elliptic curve C over R.

There is a compactification $\bar{\mathcal{M}}_{e\ell\ell}$ of $\mathcal{M}_{e\ell\ell}$ where we allow nodal, but not cusp, singularities. Thus

$$y^2 = x^2(x-1)$$

is allowed, but $y^2 = x^3$ is not.

The Hopkins-Miller Theorem

All of these curves are 1-dimensional and have a group structure on the smooth locus; the assignment

$$\bar{\mathcal{M}}_{e\ell\ell} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$
$$C \mapsto \hat{C}_e = \text{formal neighborhood of } e$$

is flat . Here is the best theorem I know:

Theorem (Hopkins-Miller)

This realization problem has a solution: there is a presheaf $\mathcal{O}^{e\ell\ell}_{\mathbf{top}}$ of E_∞-ring spectra realizing the graded structure sheaf on the étale topology. The space of realizations is path connected.

If we define **TMF** to be the homotopy global sections:

$$\mathbf{TMF} \stackrel{\mathrm{def}}{=} \operatorname*{holim}_{\bar{\mathcal{M}}_{e\ell\ell}} \mathcal{O}^{e\ell\ell}_{\mathbf{top}}$$

then there is a descent spectral sequence

$$H^s(\bar{\mathcal{M}}_{e\ell\ell}, \omega^{\otimes t}) \Longrightarrow \pi_{2t-s}\mathbf{TMF}$$

Modular forms are, by definition,

$$H^0(\bar{\mathcal{M}}_{e\ell\ell}, \omega^{\otimes *}) \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = (12)^3\Delta).$$

Hence "topological modular forms". The question of which modular forms are homotopy classes is quite interesting. For example, c_6 and the discriminant Δ are not; however $2c_6$ and 24Δ are. Note that **tmf** is usually defined to be the zero-connected cover of **TMF**.

Shimura varieties

A shameless advertisement for an upcoming talk.

Example (Topological Automorphic Forms)

Mark Behrens and Tyler Lawson solve the realization problem for certain Shimura varieties, which are moduli stacks of highly structured abelian varieties.

The stable sphere

The identity map $\mathcal{M}_{\mathbf{fg}} \to \mathcal{M}_{\mathbf{fg}}$ is flat, and we could ask whether the realization can be solved for all of $\mathcal{M}_{\mathbf{fg}}$.

Ignore the following fundamental issue: $\mathcal{M}_{\mathbf{fg}}$ is not an algebraic stack, let alone a Deligne-Mumford stack. It has no presentation with good finiteness properties.

If the realization problem could be solved, we would have an equivalence

$$S^0 \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{M}_{\mathbf{fg}}} \mathcal{O}^{\mathbf{fg}}_{\mathbf{top}}$$

and the descent spectral sequence

$$H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t}) \Longrightarrow \pi_{2t-s} S^0.$$

The Adams-Novikov Spectral Sequence

By considering the Čech complex of the cover $\mathrm{Spec}(L) \to \mathcal{M}_{\mathbf{fg}}$ we have

$$\mathrm{Ext}^s_{MU_*MU}(\Sigma^{2t} MU_*, MU_*) \cong H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t})$$

and we would have an algebraic geometric version of the Adams-Novikov Spectral Sequence.

Alas, $\mathcal{O}^{\mathbf{fg}}_{\mathbf{top}}$ probably doesn't exist: we would have to use the *fqpc* topology and there are too many flat maps.

Why E_∞-ring spectra?

Remark

Why do I insist on highly structured ring spectra in the realization problem? There are two reasons:

1. **(Practical)** Asking for E_∞-ring spectra allows for algebraic geometry (i.e., ring theoretic) input into the constructions; and
2. **(Aesthetic)** The stack $\mathcal{M}_{e\ell\ell}$ with its E_∞ structure sheaf becomes a central exhibit in the world of derived algebraic geometry: we learn something inherently new about elliptic curves.

p-divisible groups

The formal group of an elliptic curve is part of a richer and more rigid structure. At this point we pick a prime p and work over $\mathrm{Spf}(\mathbb{Z}_p)$; that is, p is implicitly nilpotent in all our rings. This has the implication that we will we working in the p-complete stable category.

Definition

*Let R be a ring and G a sheaf of abelian groups on R-algebras. Then G is a p-**divisible group** of **height** n if*

1. $p^k : G \to G$ *is surjective for all* k;
2. $G(p^k) = \mathrm{Ker}(p^k : G \to G)$ *is a finite and flat group scheme over R of rank* p^{kn};
3. colim $G(p^k) \cong G$.

Examples of p-divisible groups

Formal Example: Any formal group over a field or complete local ring is p-divisible.

Étale Example: Let $\mathbb{Z}/p^n = \mathrm{Spec}(\mathsf{map}(\mathbb{Z}/p^n, R))$ and $\mathbb{Z}/p^\infty = \mathsf{colim}\, \mathbb{Z}/p^n$.

Remarks on p-divisible groups I

1.) If G is a p-divisible group, then completion at $e \in G$ gives an abelian formal group $G_{\mathrm{for}} \subseteq G$, not necessarily of dimension 1. The quotient G/G_{for} is étale over R; thus we get a natural short exact sequence

$$0 \to G_{\mathrm{for}} \to G \to G_{\mathrm{et}} \to 0.$$

This is split over fields, but not in general.

2.) If C is a smooth elliptic curve, then $C(p^\infty) = \operatorname{colim} C(p^n)$ is p-divisible of height 2 with formal part of dimension 1.

3.) (Rigidity) If G is a p-divisible group over a scheme S, the height remains constant at all points of S; this is not true of formal groups. If G is p-divisible of height n with G_{for} of dimension 1, then the height of G_{for} is between 1 and n.

4.) Formal groups need not be p-divisible groups as there is no reason to suppose

$$\operatorname{colim} G(p^k) \cong G.$$

Nor can one assume that a formal groups is a sub-group a p-divisible group. For example, it is not all obvious that the the formal group over the Johnson-Wilson theory $E(n)_*$, $n > 1$, has anything to do with a p-divisible group.

Moduli stack of p-divisible groups

Define $\mathcal{M}_p(n)$ to the moduli stack of p-divisible groups of

1. height n and
2. with $\dim G_{\mathrm{for}} = 1$.

There is a flat morphism

$$\begin{aligned} \mathcal{M}_p(n) &\longrightarrow \mathcal{M}_{\mathbf{fg}} \\ G &\longmapsto G_{\mathrm{for}} \end{aligned}$$

Remark (Good news and bad news)

- B The stack $\mathcal{M}_p(n)$ is not algebraic, just as $\mathcal{M}_{\mathbf{fg}}$ is not. Both are "pro-algebraic".
- G The stack $\mathcal{M}_p(n)$ has geometry very like, but more rigid than, the open substack $\mathcal{U}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$ of formal groups of height less than or equal to n.

Theorem (Lurie)

Let $\mathcal{M}$ be an algebraic stack equipped with a formally étale morphism

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n).$$

Then the realization problem for the composition

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

has a canonical solution; that is, the space of all solutions is connected and has a preferred basepoint.

The fine print: since we are working over $\mathbb{Z}_p$, one must take care with notion of an algebraic stack: it's formally algebraic over $\mathrm{Spf}(\mathbb{Z}_p)$.

Example (Serre-Tate theory)

Lurie points out the morphism $\epsilon : \mathcal{M} \to \mathcal{M}_p(n)$ is étale if it satisfies the Serre-Tate theorem; thus, for example, we recover the Hopkins-Miller Theorem, at least for smooth elliptic curves. This is the condition that Behrens-Lawson check as well.

Serre-Tate says the following: let $A_0/\mathbb{F}$ be an $\mathcal{M}$-object over a field $\mathbb{F}$, necessarily of characteristic p. Let $q : R \to \mathbb{F}$ be a morphism of rings with nilpotent kernel. A *deformation* of A_0 to R is an $\mathcal{M}$-object A over R and an ismorphism $A_0 \to q^*A$. These form a category $\mathbf{Def}_{\mathcal{M}}(\mathbb{F}, A_0)$. Serre-Tate then says

$$\mathbf{Def}_{\mathcal{M}}(\mathbb{F}, A_0) \longrightarrow \mathbf{Def}_{\mathcal{M}_p(n)}(\mathbb{F}, \epsilon A_0)$$

is an equivalence.

Formal groups are still essential

Warning!

Do not develop too many hopes for generalization. It is essential to Lurie's argument that the other Hopkins-Miller theorem holds:

There is an *a priori* construction of an E_∞-ring spectrum structure on the Morava E-theories E_n; the space of all such realizations is contractible.

The geometry of $\mathcal{M}_p(n)$

The moduli stack $\mathcal{M}_p(n)$ stack is something of an mysterious object, despite years of work by Messing, Fontaine, etc. As an example of what is known, it has one geometric point (i.e., isomorphism class of an algebraically closed field) for each integer k, $1 \leq k \leq n$, given by the p-divisible group

$$G_k = \Gamma_k \times (\mathbb{Z}/p^\infty)^{n-k}$$

where Γ_k is a formal group of height k over an algebraically closed field of characteristic p. The automorphism group of G_k is

$$\mathrm{Aut}(\Gamma_k) \times \mathrm{Gl}_{n-k}(\mathbb{Z}_p).$$