The Adams-Novikov Spectral Sequence and the Homotopy Groups of Spheres

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Abstract

These are notes for a five lecture series intended to uncover large-scale phenomena in the homotopy groups of spheres using the Adams-Novikov Spectral Sequence. The lectures were given in Strasbourg, May 7–11, 2007.

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Contents

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A note on sources: I have put some references at the end of these notes, but they are nowhere near exhaustive. They do not, for example, capture the role of Jack Morava in developing this vision for stable homotopy theory. Nor somehow, have I been able to find a good way to record the overarching influence of Mike Hopkins on this area since the 1980s. And, although, I've mentioned his name a number of times in this text, I also seem to have short-changed Mark Mahowald – who, more than anyone else, has a real and organic feel for the homotopy groups of spheres. I also haven't been very systematic about where to find certain topics. If I seem a bit short on references, you can be sure I learned it from the absolutely essential reference book by Doug Ravenel [27] – "The Green Book", which is not green in its current edition. This book contains many more references. Also essential is the classic paper of Miller, Ravenel, and Wilson [22] and the papers those authors wrote to make that paper go – many of which I haven't cited here. Another source is Steve Wilson's sampler [35], which also has some good jokes.

1 The Adams spectral sequence

Let X be a spectrum and let $H^*(X) = H^*(X, \mathbb{F}_2)$ be its mod 2 cohomology. Then H^*X is naturally a left module over the mod 2 Steenrod algebra A.

We will assume for the moment that all spectra are *bounded below* and of *finite type*; that is $\pi_i X = 0$ if i is sufficiently negative and $H^n X$ is a finite vector space for all n. These are simply pedagogical assumptions and will be removed later.

The *Hurewicz map* is the natural homomorphism

$$\begin{aligned}[X,Y] &\longrightarrow \mathrm{Hom}_A(H^*Y, H^*X)\\ f &\mapsto f^* = H^*(f).\end{aligned}$$

and we would like to regard the target as relatively computable. However, the map is hardly ever an isomorphism.

There is, however, a distinguished class of spectra for which it *is* an isomorphism. Suppose V is a graded $\mathbb{F}_2$-vector space, finite in each degree and bounded below; then by Brown representability, there is a spectrum KV – a generalized mod 2 *Eilenberg-MacLane spectrum* – and a natural isomorphism

$$[X, KV] \cong \mathrm{Hom}_{\mathbb{F}_2}(H_*X, V).$$

Such a spectrum has the following properties.

1. There is a natural isomorphism $\pi_* KV \cong V$ (set $X = S^n$ for various n);
2. H^*KV is a projective A-module – indeed, any splitting of the projection $H^*KV \to V^*$ defines an isomorphism $A \otimes V^* \to H^*KV$.

Then for all X (satisfying our assumptions) we have

$$[X, KV] \cong \mathrm{Hom}_{\mathbb{F}_2}(V^*, H^*X) \cong \mathrm{Hom}_A(H^*KV, X).$$

Let me isolate this as a paradigm for later constructions:

1.1 Lemma. *There is a natural homomorphism*

$$h : [X, Y] \longrightarrow \mathrm{Hom}_A(H^*X, H^*Y)$$

which is an isomorphism when Y is a generalized mod 2 Eilenberg-MacLane spectrum. Furthermore, for such Y, H^*Y is projective as an A-module.

The idea behind the Adams Spectral Sequence (hereinafter know as the clASS) and all its variants is that we resolve a general Y by Eilenberg-MacLane spectra to build a spectral sequence for computing $[X, Y]$ with edge homomorphism h.

1.2 Definition. *Let Y be a spectrum. Then an Adams resolution for Y is a sequence of spectra*

$$Y \xrightarrow{d} K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \to \cdots$$

where each K^s is an mod 2 Eilenberg-MacLane spectrum and for all mod 2 Eilenberg-MacLane spectra, the chain complex of vector spaces

$$\cdots \to [K^2, K] \to [K^1, K] \to [K^0, K] \to [Y, K] \to 0$$

is exact.

Such exist and are unique up to a notion of chain equivalence which I leave you to formulate. It also follows – by setting $K = \Sigma^n H\mathbb{Z}_2$ for various n – that

$$\cdots \to H^*K^1 \to H^*K^0 \to H^*Y \to 0$$

is a projective resolution of H^*Y as an A-module.

1.3 Remark (The Adams Tower). From any Adams resolution of Y we can build a tower of spectra under Y

$$\cdots \longrightarrow Y_2 \longrightarrow Y_2 \longrightarrow Y_0$$

with the properties that

1. $Y_0 = K^0$ and $Y \to Y_0$ is the given map d;
2. there are homotopy pull-back squares for $s > 0$

$$\begin{array}{ccc} Y_s & \longrightarrow & * \\ \downarrow & & \downarrow \\ Y_{s-1} & \longrightarrow & \Sigma^{1-s}K^s; \end{array}$$

3. the induced maps

$$\Sigma^{-s}K^s \to Y_{s+1} \to \Sigma^{-s}K^{s+1}$$

are the given maps d; and

4. the induced map $\Sigma^{-s}K^s \to Y_s$ induces a short exact sequence

$$0 \to H^*Y \to H^*Y_s \to \Sigma^{-s}M_s \to 0$$

where

$$M_s = \mathrm{Ker}\{d^* : H^*K^s \to H^*K^{s-1}\} = \mathrm{Im} d^* : H^*K^{s+1} \to H^*K^s.$$

The short exact sequence is split by the map $Y \to Y_s$.

From the tower (under Y remember) we get an induced map

$$\mathrm{colim}\, H^*Y_s \to H^* \lim Y_s \to H^*Y$$

and the composite is an isomorphism. Under my hypotheses of bounded below and finite-type, the first map is an isomorphism as well; from this and from the fact that homotopy inverse limits of H_*-local spectra are local, it follows that $Y \to \lim Y_s$ is the H_*-localization of Y, which I will write $Y_2^\wedge$, for if the homotopy groups of Y are finitely generated, the homotopy groups of the localization are gotten by completion at 2 from the homotopy groups of Y.

We now get a spectral sequence built from the exact couple

$$\begin{array}{ccccc}
[\Sigma^{t-s}X, Y_{s+1}] & \longrightarrow & [\Sigma^{t-s}X, Y_s] & \longrightarrow & [\Sigma^{t-s}, Y_{s-1}] \\
\uparrow & \swarrow & \uparrow & \swarrow & \\
[\Sigma^{t-s}X, \Sigma^{-s-1}K^{s+1}] & & [\Sigma^{t-s}X, \Sigma^{-s}K^s] & &
\end{array}$$

where the dotted arrows are of degree -1. This spectral sequence has E_1 term

$$\begin{aligned}
E_1^{s,t} &\cong [\Sigma^{t-s}X, \Sigma^{-s}K^s] \\
&\cong [\Sigma^t X, K^s] \\
&\cong \mathrm{Hom}_A(H^*K^s, \Sigma^t H^*X)
\end{aligned}$$

and from this and Remark 1.3.3, we have that

$$\begin{aligned}
E_2^{s,t} &\cong H^s\mathrm{Hom}_A(H^*K^\bullet, \Sigma^t H^*X) \\
&\cong \mathrm{Ext}_A^s(H^*Y, \Sigma^t H^*X).
\end{aligned}$$

It abuts to $[\Sigma^{t-s}X, \lim Y_s]$; thus we write the clASS as

$$\mathrm{Ext}_A^s(H^*Y, \Sigma^t X) \Longrightarrow [\Sigma^{t-s}X, Y_2^\wedge].$$

The differentials go

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

This is a left half-plane spectral sequence. For large r, we have

$$E_\infty^{s,t} \subseteq E_r^{s,t}$$

but it can happen that there is no finite r for which this is an equality. Thus we need to distinguish between *abuts* and *converges*.

Write $e_\infty^{s,t}$ as the quotient of

$$\mathrm{Ker}\{[\Sigma^{t-s}X, Y_2^\wedge] \to [\Sigma^{t-s}X, Y_{s-1}]\}$$

by

$$\mathrm{Ker}\{[\Sigma^{t-s}X, Y_2^\wedge] \to [\Sigma^{t-s}X, Y_s]\}.$$

Then, all the word "abuts" signifies is that there is a map

$$e_\infty^{s,t} \longrightarrow E_\infty^{s,t}$$

where the target is the usual E_∞ term of the spectral sequence.

1.4 Definition. *The spectral sequence* **converges** *if*

1. *this map is an isomorphism; and*
2. $\lim^1[\Sigma^n X, Y_s] = 0$ *for all* n.

We get convergence if, for example, we know that for all (s,t) there is an r so that $E_\infty^{s,t} = E_r^{s,t}$. In particular, this will happen (although this is not obvious) if H_*X is *finite* as a graded vector space. See Theorem 2.3 below. A thorough discussion of convergence and the connections with localization is contained in Bousfield's paper [4]. A source for the Adams spectral sequence and the Adams-Novikov Spectral Sequence is [1], although the point of view I'm adopting here is from the first section of [20].

2 Classical calculations

In any of the Adams spectral sequences, there are two computational steps: (1) the algebraic problem of calculating the E_2-term; and (2) the geometric problem of resolving the differentials. Both problems are significant and neither has been done completely. There has been, at various times, optimism that resolving the first would lead to a resolution of the second, but the following principle, first named by Doug Ravenel, seems to hold universally:

The Mahowald Uncertainty Principle: Any spectral sequence converging to the homotopy groups of spheres with an E_2-term that can be named using homological algebra will be infinitely far from the actual answer.

Nonetheless, we soldier on. For the clASS we have at least three tools for computing the E_2-term of

$$\mathrm{Ext}_A^s(H^*Y, \Sigma^t\mathbb{F}_2) \Longrightarrow \pi_{t-s}Y_2^\wedge.$$

They are

1. minimal resolutions, which can be fully automated, at least at the prime 2 (see [6]);
2. the May spectral sequence (still the best method for by-hand computations; see [32] and [27] §3.2); and
3. the Λ-algebra, which picks up the interplay with unstable phenomena (see [5]);

I won't say much about them here, but all of them can be a lot of fun.

We now concentrate on calculating the clASS

$$E_2^{s,t}(Y) = \mathrm{Ext}_A(H^*Y, \Sigma^t\mathbb{F}_2) \Longrightarrow \pi_{t-s}Y$$

where Y is a homotopy associative and homotopy commutative H_*-local *ring* spectrum. For such Y the the homotopy groups of Y are a graded commutative ring and the Adams spectral sequence becomes a spectral sequence of appropriate graded commutative rings. In particular, differentials are derivations. There are also Massey products and Toda brackets.

Let me say a little more about these last. Suppose $a, b, c \in \pi_*Y$ have degree m, n, and p respectively and suppose $ab = 0 = bc$. Then, by choosing null-homotopies we get a map

$$\langle a, b, c\rangle : S^{m+n+p+1} = C^{m+n+1} \wedge S^p \coprod_{S^m \wedge S^n \wedge S^p} S^m \wedge C^{n+p+1} \to Y$$

where C^{n+1} is the cone of S^n. This bracket is well-defined up to the choices of null-homotopy; that is, well-defined up to the indeterminacy

$$a[\pi_{n+p+1}Y] + [\pi_{m+n+1}Y]c.$$

There is a similar construction algebraically; that is, given

$$x, y, z \in \mathrm{Ext}_A^*(H^*Y, \Sigma^*\mathbb{F}_2)$$

with $xy = 0$ and $yz = 0$, there is a Massey product

$$\langle x, y, z\rangle \in \mathrm{Ext}_A^*(H^*Y, \Sigma^*\mathbb{F}_2)$$

well-defined up to indeterminacy. Furthermore, if the Massey product detects the Toda bracket via the spectral sequence.

2.1 Remark (The cobar complex). The Massey product is computed when the Ext groups are given as the cohomology of a differential graded algebra. We now describe the usual example of such a dga.

The forgetful functor from A-modules to graded vector spaces has a left adjoint $V \mapsto A \otimes V$. The resulting cotriple resolution of an A-module M is the *bar* complex $B_\bullet(M) \to M$. This is an augmented simplicial A-module with

$$B_n(A) = A^{\otimes(n+1)} \otimes M$$

and face operators given by

$$d_i(a_0 \otimes \dots a_n \otimes x) = \begin{cases} a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots a_n \otimes x, & i < n; \\ a_0 \otimes \dots \otimes a_{n-1} \otimes a_n x, & i = n. \end{cases}$$

The map

$$a_0 \otimes \dots a_n \otimes x \to 1 \otimes a_0 \otimes \dots a_n \otimes x$$

gives this complex a contraction as a simplicial vector space; hence we have a natural projective resolution. Applying $\mathrm{Hom}_A(-, \Sigma^* \mathbb{F}_2)$ to the bar complex yields the cobar complex $C^\bullet(M_*)$ where

$$C^n(M_*) = A_*^{\otimes n} \otimes M_*$$

where I have written A_* and M_* for the duals. The coface maps now read

$$d^i(\alpha_1 \otimes \dots \alpha_n \otimes y) = \begin{cases} 1 \otimes \alpha_1 \otimes \dots \alpha_n \otimes y, & i = 0; \\ \alpha_1 \otimes \dots \otimes \Delta(\alpha_i) \otimes \dots \otimes \alpha_n \otimes y, & 0 < i < n+1; \\ \alpha_1 \otimes \dots \otimes \alpha_n \otimes \psi(y). & i = n+1. \end{cases}$$

Here $\Delta : A_* \to A_* \otimes A_*$ and $\psi : M_* \to A_* \otimes M_*$ are the diagonal and comultiplication respectively. If $M_* = H_* Y$, where Y is a ring spectrum, then $C^\bullet(H_* Y)$ is a cosimplicial $\mathbb{F}_2$-algebra and

$$\pi^* C^\bullet(H_* Y) = \mathrm{Ext}_A^*(H^* Y, \Sigma^* \mathbb{F}_2).$$

For computational purposes, we often take the normalized bar construction $\bar{C}^\bullet(M_*)$ where

$$\bar{C}^n(M_*) \cong \bar{A}_*^{\otimes n} \otimes M_*$$

where $\bar{A}_* \subseteq A_*$ is the augmentation ideal. If $M = H_* Y$, then $\bar{C}^\bullet(H_* Y)$ is a differential graded algebra and a good place to compute Massey products.

Some beginning computational results are these.

2.2 Theorem. *1.)* $\mathrm{Hom}_A(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2$ *generated by the identity. If* $t \neq 0$*, then* $\mathrm{Hom}_A(\mathbb{F}_2, \Sigma^t \mathbb{F}_2) = 0$.

2.) $\mathrm{Ext}_A^1(\mathbb{F}_2, \Sigma^t \mathbb{F}_2) \cong \mathbb{F}_2$ *if* $t = 2^i$ *for some* i*; otherwise this group is zero.*

Proof. The short exact sequence of A-modules

$$0 \to \bar{A} \to A \to \mathbb{F}_2 \to 0$$

and the resulting long exact sequence in Ext shows that

$$\begin{aligned} \mathrm{Ext}_A^1(\mathbb{F}_2, \Sigma^* \mathbb{F}_2) &\cong \mathrm{Hom}_A(\bar{A}, \Sigma^* \mathbb{F}_2) \\ &\cong \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2 \otimes_A \bar{A}, \Sigma^* \mathbb{F}_2) \\ &\cong \mathrm{Hom}_{\mathbb{F}_2}(\bar{A}/\bar{A}^2, \Sigma^* \mathbb{F}_2) \end{aligned}$$

and the Adem relations show that $\bar{A}/\bar{A}^2$ is zero except in degrees 2^i where it is generated by the residue class of Sq^{2^i}. □

The non-zero class is $\mathrm{Ext}^1_A(\mathbb{F}_2, \Sigma^{2^i}\mathbb{F}_2)$ is called h_i.

The following result shows that the clASS for the sphere actually converges. See Definition 1.4 and the remark following. Part (3) is usually phrased as saying there is a vanishing line of slope 2. The following result was originally due to Adams [3].

2.3 Theorem. *We have the following vanishing result for the clASS:*

1. $Ext^s_A(\mathbb{F}_2, \Sigma^t\mathbb{F}_2) = 0$ *all* (s,t) *with* $t - s < 0$.
2. $Ext^s_A(\mathbb{F}_2, \Sigma^s\mathbb{F}_2) = \mathbb{F}_2$ *generated by* h_0^s*;*
3. $Ext^s_A(\mathbb{F}_2, \Sigma^t\mathbb{F}_2) = 0$ *for all* (s,t) *with*

$$0 < t - s < 2s + \epsilon$$

where

$$\epsilon = \begin{cases} 1, & s \equiv 0, 1 \bmod (4); \\ 2, & s \equiv 2 \bmod (4); \\ 3, & s \equiv 3 \bmod (4);. \end{cases}$$

We have the following low dimensional calculations. The first is from [2].

2.4 Theorem. *The* 2*-line* $Ext^2_A(\mathbb{F}_2, \Sigma^*\mathbb{F}_2)$ *is the graded vector space generated by* h_ih_j *subject to the relations* $h_ih_j = h_jh_i$ *and* $h_ih_{i+1} = 0$.

The relation in this last result immediately allows for Massey products; a fun exercise is to show

$$\langle h_0, h_1 h_0 \rangle = h_1^2 \qquad \text{and} \qquad \langle h_1, h_0, h_1 \rangle = h_0 h_2.$$

Deeper results include the following from [34]:

2.5 Theorem. *The* 3*-line* $Ext^3_A(\mathbb{F}_2, \Sigma^*\mathbb{F}_2)$ *is the graded vector space generated* $h_ih_jh_k$ *subject to the further relations*

$$h_i h_{i+2}^2 = 0 \qquad \text{and} \qquad h_i^2 h_{i+2} = h_{i+1}^2$$

and the classes

$$c_i = \langle h_{i+1}, h_i, h_{i+2}^2 \rangle$$

2.6 Remark. Surprisingly few of the low dimensional classes actually detect homotopy classes. The classes h_i, $0 \le i \le 3$ detect $\times 2 : S^0 \to S^0$ and the stable Hopf maps η, ν, and σ respectively. Otherwise we have, from [2], that

$$d_2 h_i = h_{i-1} h_0^2.$$

The first of these, when $i = 4$ is forced by the fact that $2\sigma^2 = 0$, as it is twice the square of an odd dimensional class in a graded commutative ring. The others can be deduced from Adams's original work on secondary operations or by a clever inductive argument, due to Wang [34], which relies on knowledge of the 4-line.

On the 2-line, the classes h_0h_2, h_0h_3, h_2h_4, h_i^2, $i \leq 5$, and h_jh_1, $j \geq 3$ survive and detect homotopy classes. The infinite family detected by the classes h_jh_1 is the family known as Mahowald's η_j-family [18]; they provided a counterexample to the "Doomsday conjecture" – which posited that there could only be finitely many non-bounding infinite cycles on any s-line of the clASS.

The only other classes on the 2-line which may survive are the Kervaire invariant classes h_i^2, $i \geq 6$. This remains one of the more difficult problems in stable homotopy theory. If you want to work on this question make sure you are in constant contact with Mark Mahowald, Fred Cohen, and/or Norihiko Minami. Preferably "and", not "or". There are many standard errors and seductive blind alleys; an experienced guide is essential. See [19] for many references. One world view suggests that not only is h_i^2 a permanent cycle, but it is also the stabilization of a class $x \in \pi_* S^{2^{i+1}-1}$ with the property that $2x$ is the Whitehead product. It is this last problem we would really, really like to understand.[1]

The first large scale family of interesting classes detecting homotopy elements is in fact near the vanishing line. We have that $h_0^4 h_3 = 0$, so if

$$x \in \mathrm{Ext}_A^s(\mathbb{F}_2, \Sigma^t \mathbb{F}_2)$$

with x so that $h_0^4 x$ is above the vanishing line, we can form the Massey product $\langle x, h_0^4, h_4 \rangle$. The following result is the first result of Adams periodicity [3].

2.7 Theorem. *The operator*

$$P(-) = \langle -, h_0^4, h_3 \rangle : \mathrm{Ext}_A^s(\mathbb{F}_2, \Sigma^t \mathbb{F}_2) \to \mathrm{Ext}_A^{s+4}(\mathbb{F}_2, \Sigma^{t+12} \mathbb{F}_2)$$

is an isomorphism whenever $(s+4, t+4)$ *is above the vanishing line.*

There is actually a stronger result. For all $n > 1$, we have $h_0^{2^n} h_{n+1} = 0$ and, again near the vanishing line, we can form the operator

$$\langle -, h_0^{2^n}, h_{n+1} \rangle : \mathrm{Ext}_A^s(\mathbb{F}_2, \Sigma^t \mathbb{F}_2) \to \mathrm{Ext}_A^{s+2^n}(\mathbb{F}_2, \Sigma^{t+3\cdot 2^n} \mathbb{F}_2)$$

and this tends to be an isomorphism where it is universally defined, which is approximately above a line of slope 1/5. Thus in this "wedge", between the lines of slope 1/5 and slope 1/2, we have considerable hold over the clASS.

The operator $P(-)$ of Theorem 2.7 takes permanent cycles to permanent cycles; however, there is no guarantee that they are not boundaries – indeed, being so high up there are lot of elements that can support differentials below. In order to show they are not boundaries, we need to use other homology theories.

[1] In my youth, I once suggested to Fred Cohen that this problem wasn't so important any more. He became visibly upset and suggested I didn't understand homotopy theory. He was right – even allowing for the self-importance of the new guy in town, this was a remarkably misguided statement. I apologize, Fred.

3 The Adams-Novikov Spectral Sequence

In working with generalized (co-)homology theories, it turns out to be important to work with homology rather than cohomology. This is even evident for ordinary homology – cohomology is the dual of homology (but not the other way around) and cohomology is actually, thus, a graded topological vector space. This is the reason for the finite type hypotheses of the previous sections. There is no simple hypothesis to eliminate this problem for generalized homology theories: E^*E will almost always be a topological E^*-module. The price we pay is that we must work with comodules rather than modules and, while the two notions are categorically dual, the notion of a comodule is much less intuitive (at least to me) than that of a module.

Let E_* be a homology theory. We make many assumptions about E as we go along. Here's the first two.

1. The associated cohomology theory E^* has graded commutative cup products. Thus the representing spectrum E is a homotopy associative and commutative ring spectrum.
2. There are two morphisms, the left and right units

$$\eta_L, \eta_R : E_* \longrightarrow E_*E$$

interchanged by the conjugation $\chi : E_*E \to E_*E$. We assume that for one (and hence both) of these maps, E_*E is a flat E_*-module.

The second of these conditions eliminates some of the standard homology theories: $H_*(-, \mathbb{Z})$ and connective K-theory, for example.

Under these assumptions, there is a natural isomorphism of left E_*-modules

$$E_*E \otimes_{E_*} E_*X \xrightarrow{\cong} E_*(E \wedge X)$$

sending $a \in E_mE$ and $x \in E_nX$ to the composition

$$S^n \wedge S^m \xrightarrow{a \wedge x} E \wedge E \wedge E \wedge X \xrightarrow{E \wedge m \wedge X} E \wedge E \wedge X$$

Note that the right E_*-module structure on E_*E gets used in the tensor product. In particular we have

$$E_*(E \wedge E) \cong E_*E \otimes_{E_*} E_*E$$

and the map

$$E \wedge \eta \wedge E : E \wedge S^0 \wedge E \to E \wedge E \wedge E$$

defines a diagonal map

$$\Delta : E_*E \longrightarrow E_*E \otimes_{E_*} E_*E.$$

With this map, the two units and the multiplication map

$$E_*E \otimes_{E_*} E_*E \longrightarrow E_*E$$

the pair becomes a *Hopf algebroid*; that is, this pair represents a a functor from graded rings to groupoids. One invariant of this setup is the graded ring R_E of *primitives* in E_* defined by the equalizer diagram

$$R_E \longrightarrow E_* \underset{\eta_L}{\overset{\eta_R}{\rightrightarrows}} E_*E.$$

Note that if $R_E = E_*$, then E_*E is a Hopf algebra over E_*.

For any graded commutative ring, write $\mathrm{Spec}(R)$ for the covariant functor on graded commutative rings it represents. As a shorthand, I am going to say that

$$\mathrm{Spec}(E_*E \otimes_{E_*} E_*E) \begin{smallmatrix}\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\end{smallmatrix} \mathrm{Spec}(E_*E) \begin{smallmatrix}\longleftarrow\\\longrightarrow\\\longleftarrow\end{smallmatrix} \mathrm{Spec}(E_*)$$

is an affine groupoid scheme over R_E. (I leave you to label all the maps and to check they satisfy the simplicial identities.) This is an abuse of nomenclature, as we are working in a grading setting, but it is nonetheless convenient: many Hopf algebroids are best understood by describing the functors they represent. If E_* is 2-periodic it is possible to arrange matters so that this abuse goes away. See Example 4.6.

A *comodule* over this Hopf algebroid is a left E_*-module M equipped with a morphism of graded R_E-modules

$$\psi_M : M \longrightarrow E_*E \otimes_{E_*} M$$

subject to the obvious compatibility maps with the structure maps of the Hopf algebroid. If X is a spectrum, then the inclusion

$$E \wedge \eta \wedge X : E \wedge S^0 \wedge X \longrightarrow E \wedge E \wedge X$$

defines a comodule structure on E_*X. For example, if $X = S^0$, then $E_*S^0 = E^*$ and

$$\psi_{E_*} = \eta_R : E_* \longrightarrow E_*E.$$

For much more on comodules see [27] §A.1 and [12].

Let's write $\mathbf{Comod}_{E_*E}$ for the category of comodules over the Hopf algebroid (E_*, E_*E) and $\mathbf{Mod}_{E_*}$ for the modules over E_*.

3.1 Remark. The forgetful functor

$$\mathbf{Comod}_{E_*E} \to \mathbf{Mod}_{E_*}$$

has a right adjoint $M \mapsto E_*E \otimes_{E_*} M$ – the *extended comodule* functor. Thus $\mathbf{Comod}_{E_*E}$ has enough injectives and we can define right derived functors; in particular,

$$\mathrm{Ext}^s_{E_*E}(M, N)$$

are the right derived functors of $\mathbf{Comod}_{E_*E}(M, -)$. As often happens, we can sometimes replace injectives by appropriate acyclic objects. For example, if M is *projective* as an E_*-module and

$$N \to I^0 \to I^1 \to \cdots$$

is a resolution by modules I^s which are retracts of the extended comodules, then a simple bicomplex argument shows that

$$\mathrm{Ext}^s_{E_*E}(M,N) \cong H^s\mathbf{Comod}_{E_*E}(M,I^\bullet).$$

To be brutally specific, the triple resolution of N obtained from the composite functor $N \mapsto E_*E \otimes_{E_*} N$ yields a resolution with

$$I^s = (E_*E)^{\otimes s+1} \otimes_{E_*} N.$$

3.2 Remark. Of particular interest is the case of

$$\mathrm{Ext}^s_{E_*E}(\Sigma^t E_*, N).$$

Letting t vary we have an equalizer diagram of R_E-modules

$$\mathbf{Comod}_{E_*E}(\Sigma^* E_*, N) \longrightarrow N \underset{d^1}{\overset{d_0}{\rightrightarrows}} E_*E \otimes_{E_*} N$$

where $d^0(x) = 1 \otimes x$ and $d^1(x) = \psi_N(x)$. This is equalizer will be called the module of *primitives* and, hence, the functors $\mathrm{Ext}^s_{E_*E}(\Sigma^* E_*, -)$ are the right derived functors of the primitive element functor. The natural map

$$M \xrightarrow{x \mapsto 1 \otimes x} E_*E \otimes_{E_*E} M$$

defines an isomorphism of M to the primitives in the extended comodule; hence, if we use the triple resolution of the previous remark we have

$$\mathrm{Ext}^s_{E_*E}(\Sigma^* E_*, N) = H^s(C^\bullet(N))$$

where $C^\bullet(N)$ is the cobar complex. (See Remark 2.1 for formulas.) This normalization of this complex is the reduced cobar complex $\bar{C}^\bullet(N)$.

If A is a graded commutative algebra in E_*E-comodules, then $C^\bullet(A)$ is a simplicial graded commutative R_E-algebra and $\bar{C}^\bullet(A)$ is a differential graded commutative algebra.

We now define the Adams-Novikov Spectral Sequence (ANSS) for E_*. Call a spectrum K a *relative E-injective* if K is a retract of a spectrum of the form $E \wedge K_0$ for some K_0. The following begins the process. Compare Lemma 1.1.

3.3 Lemma. *Let X be a spectrum with E_*X projective as an E_*-module. Then for all spectra Y there is a Hurewicz map*

$$[X,Y] \longrightarrow \mathbf{Comod}_{E_*E}(E_*X, E_*Y)$$

which is an isomorphism if Y is a relative E-injective.

Then we have resolutions. Compare Definition 1.2.

3.4 Definition. *Let Y be a spectrum. Then an E_*-Adams resolution for Y is a sequence of spectra*

$$Y \to K^0 \to K^1 \to K^2 \to \cdots$$

where each K^s is a relative E_*-injective and for all relative E_*-injectives, the chain complex of R_E-modules

$$\cdots \to [K^2, K] \to [K^1, K] \to [K^0, K] \to [Y, K] \to 0$$

is exact.

Again such exist and are unique up to an appropriate notion of chain equivalence. Furthermore, out of any E_*-Adams resolution we get – exactly as in Remark 1.3 – an Adams tower and hence the Adams-Novikov Spectral Sequence

$$\mathrm{Ext}^s_{E_*E}(\Sigma^t E_*X, E_*Y) \Longrightarrow [\Sigma^{t-s}X, \lim Y_s].$$

Again the spectrum $\lim Y_s$ is E_*-local and, under favorable circumstances it is the E_*-localization L_EY of Y. Convergence remains an issue. Again see [4]. A sample nice result along these lines is the following. The spectrum E still satisfies the assumptions at the beginning of this section.

3.5 Lemma. *Let E be a spectrum so that*

1. *$\pi_tE = 0$ for $t < 0$ and either $\pi_0E \subseteq \mathbb{Q}$ or $\pi_0E = \mathbb{Z}/n\mathbb{Z}$ for some n;*
2. *E_*E is concentrated in even degrees.*

Suppose Y is also has the property that $\pi_tY = 0$ for $t < 0$. Then

1. *the inverse limit of any E_*-Adams tower for Y is the E_*-localization L_EY;*
2. *$L_EY = L_{H_*(-,\pi_0E)}Y$;*
3. *$Ext^s(\Sigma^tE^*, E_*Y) = 0$ for $s > t - s$ and the ANSS converges.*

4 Complex oriented homology theories

We begin to add to our assumptions on homology theories. In the next definition we are going to use a natural isomorphism

$$E^0 \cong \tilde{E}^2S^2$$

where the target is the reduced cohomology of the two sphere. There is an evident such isomorphism given by

$$x \mapsto S^2 \wedge x : [S^0, E] \longrightarrow [S^2, S^2 \wedge E],$$

but we can always multiply by -1 to get another. We nail down the isomorphism by insisting that for $E = H_*(-,\mathbb{Z})$ the isomorphism should send $1 \in H^0(\mathrm{pt}, \mathbb{Z})$ to the first Chern class of the canonical complex line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$. We do this so that we obtain the usual complex orientation for $H_*(-,\mathbb{Z})$.

4.1 Definition. *Let E_* be a multiplicative homology theory. A complex orientation for E^* is a* **natural** *theory of Thom classes for complex vector bundles. That is, given a complex n-plane bundle over a CW-complex X, there is a Thom class*

$$U_V \in E^{2n}(V, V - \{0\})$$

so that

1. *U_V induces a Thom isomorphism*

$$E^k X \xrightarrow{\cong} E^k V \xrightarrow{U_V \smile} E^{2n+k}(V, V - \{0\});$$

2. *if $f : X' \to X$ is continuous, then $U_{f^*V} = f^* U_V$.*

3. *if γ is the canonical line over $\mathbb{C}\mathrm{P}^1 = S^2$, then the image of U_γ under*

$$E^2(V(\gamma), V(\gamma) - \{0\}) \to E^2(V(\gamma)) \cong E^2(S^2)$$

is the generator of $\tilde{E}^*(S^2)$ given by the image of unit under the natural suspension isomorphism $E^0 \cong \tilde{E}^2 S^2$.

If the homology theory E_* is complex oriented, it will have a good theory of Chern classes. In particular, any complex bundle V has an Euler class $e(V)$ (the top Chern class) given as the image of U_V under the map

$$E^{2n}(V, V - \{0\}) \to E^{2n}(V) \cong E^{2n} X.$$

From this and the Thom isomorphism one gets a Gysin sequence and the usual argument now implies that

$$E^* \mathbb{C}\mathrm{P}^\infty \cong E^*[[e(\gamma)]] \tag{4.1}$$

where γ is the universal 1-plane bundle. In fact,

$$E^*((\mathbb{C}\mathrm{P}^\infty)^n) \cong E^*[[e(\gamma_1), \ldots, e(\gamma_n)]]$$

where $\gamma_i = p_i^* \gamma$ is the pull-back of γ under the projection onto the ith factor. The method of universal examples now implies that there is a power series

$$x +_F y \stackrel{\text{def}}{=} F(x, y) \in E^*[[x, y]]$$

for calculating the Euler class of the tensor product of two line bundles:

$$e(\xi \otimes \zeta) = e(\xi) +_F e(\zeta).$$

The usual properties of the tensor product imply that the power series $x +_F y$ satisfies the following properties

1. (Unit) $x +_F 0 = x$;

2. (Commutativity) $x +_F y = y +_F x$; and

3. (Associativity) $(x +_F y) +_F z = x +_F (y +_F z)$.

A power series satisfying these axioms is called a *formal group law*; thus a choice of complex orientation for E_* yields a formal group law over E^*.[2]

Now suppose we have two different orientations for E_* and hence two Euler classes $e(V)$ and $e'(V)$. An examination of Equation 4.1 shows that there must be a power series $f(x) \in E^*[[x]]$ so that

$$e'(\gamma) = f(e(\gamma)).$$

The normalization condition of Definition 4.1.3 implies that $f'(0) = 1$; thus, f is an element of the group of power series invertible under composition. Again checking universal examples we see that if F and F' are the two resulting formal group laws, then we see that

$$f(x +_F y) = f(x) +_{F'} f(y).$$

Such a power series is called a *strict isomorphism* of formal group laws.

4.2 Example (Complex cobordism). Formal group laws and their isomorphisms assemble into a functor **fgl** to groupoids from the category of graded rings: given a graded ring R the objects of $\mathbf{fgl}(R)$ are the formal group laws over R and the morphisms are the strict isomorphisms of formal group laws. If $g : R \to S$ is a homomorphism of graded commutative rings, we will write

$$f^* : \mathbf{fgl}(R) \longrightarrow \mathbf{fgl}(S)$$

for the resulting morphism of groupoids.

It is easy to see that this is an affine groupoid scheme (in the graded sense) over $\mathbb{Z}$; a theorem of Lazard calculates the rings involved. The key observation – which makes smooth, 1-dimensional formal group laws special – is the symmetric 2-cocycle lemma. This says that given a formal group law F over a graded ring R, there are elements $a_1, a_2, \cdots$ of R and an equality

$$x +_F y = x + y + a_1 C_2(x, y) + a_2 C_3(x, y) + \cdots$$

in $R/(a_1, a_2, \ldots)^2$ where $C_n(x, y)$ is the nth homogeneous symmetric 2-cocycle

$$C_n(x, y) = (1/d)[(x + y)^n - x^n - y^n].$$

Here $d = p$ if n is a power of a prime p; otherwise $d = 1$. It follows easily from this that objects of $\mathbf{fgl}(-)$ are represented by the Lazard ring, which is a polynomial ring

$$L \cong \mathbb{Z}[a_1, a_2, \ldots]$$

[2] This is a smooth one-dimensional formal group law, but this is the only kind we'll have, so the adjectives aren't necessary. Also, if you're into algebraic geometry, you realize that the important object here is not F, but the formal group scheme represented by $E^*\mathbb{C}\mathrm{P}^\infty$ – which does not depend on the choice of orientation. Such formal group schemes have been called by one-parameter formal Lie groups. See [31] for the background on nomenclature. Here we're doing computations here, so we'll stick to the formulas.

Note this isomorphism is canonical only modulo $(a_1, a_2, \ldots)^2$. The morphisms in $\mathbf{fgl}(-)$ are represented by

$$W = L[t_1, t_2, \ldots]$$

where the t_i are the coefficients of the universal strict isomorphism. The elements a_i and t_i have degree $2i$ (in the homology grading). The pair (L, W) is a Hopf algebroid over $\mathbb{Z}$.

A celebrated theorem of Quillen [29] implies that any choice of a complex orientation for the complex cobordism spectrum MU yields an isomorphism of Hopf algebroids

$$(MU_*, MU_*MU) \cong (L, W).$$

4.3 Example (Brown-Peterson theory). The integers $\mathbb{Z}$ has some drawbacks as a ring – most dauntingly, it is not a local ring. We often prefer to work over $\mathbb{Z}_{(p)}$-algebras for some fixed prime p. Then we can cut down the formal group laws in question.

For any integer n and any formal group law over any ring R we can form the n-series

$$[n]_F(x) = [n](x) = x +_F + \cdots +_F x = nx + \cdots$$

where the sum is taken n-times. If R is a $\mathbb{Z}_{(p)}$-algebra and n is prime to p, this power series is invertible (under composition); let $[1/n](x)$ be the inverse. A formal group law is p-typical if for all n prime to p the formal sum

$$[1/n](x +_F \zeta x +_F \zeta^2 x +_F \cdots +_F \zeta^{n-1} x) = 0.$$

Here ζ is a primitive nth root of unity. The sum is *a priori* over $R[\zeta]$; however, since it is invariant under the Galois action, it is actually over R.

This definition, admittedly rather technical, has the following implications (see [27] §A.2 for pointers to original sources):

1. Let F be any formal group law over a $\mathbb{Z}_{(p)}$-algebra R. Then there is a strict isomorphism

   $$e_F : F \to G$$

   from F to a p-typical formal group law G. If F is already p-typical, then e_F is the identify.

2. The idempotent e is natural in F; that is, if $\phi : F \to F'$ is a strict isomorphism of formal group laws, then there is a unique isomorphism e_ϕ making the following diagram commute[3]

   $$\begin{array}{ccc} F & \xrightarrow{e_F} & G \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle e_\phi} \\ F' & \xrightarrow[e_{F'}]{} & G'. \end{array}$$

[3] This statement is obvious, since all the maps are isomorphisms. I added this to belabor the functoriality of the idempotent.

3. For any p-typical formal group law F over a $\mathbb{Z}_{(p)}$-algebra R, there are elements $v_i \in R$ so that

$$[p]_F(x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots . \tag{4.2}$$

4. Notice that we could write $v_n(F)$ for the elements $v_n \in R$ of Equation 4.2 as they depend on the formal group law F. Now we can prove that if F and G are two p-typical formal group laws over R so that $v_n(F) = v_n(G)$ for all n, then $F = G$. We can also prove that given element a_n of degree $2(p^n - 1)$, there is a p-typical formal group F with $v_n(F) = a_n$.

5. If $\phi : F \to F'$ is any isomorphism of p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebra R, then there are elements $t_i \in R$ so that

$$\phi^{-1}(x) = x +_F t_1 x^p +_F t_2 x^{p^2} +_F \cdots .$$

Let **fglp** denote the functor from graded $\mathbb{Z}_{(p)}$-algebras to groupoids which assigns to any R the groupoid of p-typical formal group laws over R and all their isomorphisms. Then points (1) and (2) show that there is a natural equivalence of groupoids (the *Cartier idempotent*)

$$e : \mathbf{fgl} \to \mathbf{fglp}$$

which is idempotent. Points (3), (4), and (5) show that **fglp** is represented by the Hopf algebroid (V, VT) where

$$V = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$$

where the v_i arise from the universal p-typical formal group as in point (3) and

$$VT = V[t_1, t_2, \ldots]$$

where the t_i arise from the universal isomorphism as in point (5). The degrees of v_i and t_i are $2(p^i - 1)$. Note that these degrees grow exponentially in i, which often helps in computations.

Quillen [29] noticed that there is a complex oriented ring spectrum BP (constructed, without its ring structure, by Brown and Peterson) and an isomorphism of Hopf algebroids

$$(BP_*, BP_*BP) = (V, VT).$$

4.4 Example (Landweber exact theories). Let F be a formal group law over a graded ring A classified by map of graded rings $F : L \to A$. We would like the functor

$$X \mapsto A \otimes_L MU_*X$$

to define a homology theory $E(A, F)$. Using Brown representability, this will happen if the functor $A \otimes_L (-)$ is exact on the category of $(MU_*, MU_*MU) \cong$

(L, W)-comodules. The Landweber Exact Functor Theorem (LEFT) (see [15] and [21]) writes down a condition for this to be true.

We can write down a formal requirement fairly easily. Assuming for the moment that $E(A, F)$ exists, we have that

$$E(A, F)_*E(A, F) \cong A \otimes_L MU_*E(A, F)$$

and $MU_*E(A, F) = MU_*MU \otimes_L A$, by switching the factors. Thus

$$E(A, F)_*E(A, F) \cong A \otimes_L W \otimes_L A. \tag{4.3}$$

Gerd Laures has noticed that $A \otimes_L (-)$ is exact on (L, W) comodules if the *right* inclusion

$$\eta_R : L \longrightarrow A \otimes_L W$$

remains flat. For a computational condition, it's easier to work at a prime p and suppose that F is p-typical. Then Landweber's criterion is that the sequence $p, v_1, v_2, \cdots$ is a regular sequence in A. The p-typical requirement can be removed: it turns out that v_i is defined for all formal group laws modulo $p, v_1, \ldots, v_{i-1}$ and we simply require that the resulting sequences be regular for all p. If A is $\mathbb{Z}_{(p)}$-algebra and q is prime to p this is automatic for the prime q.

If the formal group F satisfies these conditions, then we say $E(A, F)$ is a Landweber exact theory. The equation 4.3 now implies that there is an isomorphism of Hopf algebroids

$$(E(A, F)_*, E(A, F)_*E(A, F)) \cong (A, A \otimes_L W \otimes_L A).$$

This represents the (graded) affine groupoid scheme which assigns to any ring R the groupoid with objects the ring homomorphisms $f : A \to R$. An isomorphism $\phi : f \to g$ will be a strict isomorphism

$$\phi : f^*F \to g^*F$$

of formal group laws.

Important first examples of Landweber exact theories are the *Johnson-Wilson* theories $E(n)$ with

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n^{\pm 1}].$$

We normally choose a p-typical formal group law F over $E(n)_*$ with p-series

$$[p](x) = px +_F v_1 x^p +_F \cdots +_F v_n x^{p^n}.$$

Since $E(n)$ is not connected, the localization $L_{E(n)} = L_n$ can be a radical process. Nonetheless we have the Hopkins-Ravenel *chromatic convergence* theorem [28] which says that there are natural transformation of functors $L_n \to L_{n-1}$ and, if X is a finite CW spectrum, then the induced map

$$X \longrightarrow \operatorname{holim} L_n X$$

is localization at $H_*(-, \mathbb{Z}_{(p)})$. See [28].

4.5 Example (K-theory). Let K be complex K-theory. The Euler class of a complex line bundle is normally defined to be

$$e'(\gamma) = \gamma - 1 \in \tilde{K}^0(X).$$

With this definition,

$$e'(\gamma_1 \otimes \gamma_2) = e'(\gamma_1) + e'(\gamma_2) + e'(\gamma_1)e'(\gamma_2).$$

The resulting formal group law is the multiplicative formal group law

$$x +_{\mathbb{G}_m} y = x + y + xy \in K^0[[x, y]].$$

Since we have arranged our definitions so that the Euler class lies in $K^2(X)$, we redefine the Euler class so that

$$e(\gamma) = u^{-1}e'(\gamma) = u^{-1}(\gamma - 1)$$

where $u \in K_2 = K^{-2}$ is the Bott element. The graded formal group law then becomes

$$x +_{F_u} y = x + y + uxy = u^{-1}(ux + \mathbb{G}_m uy).$$

This has p-series $(1 + ux)^p - 1$; hence K-theory is Landweber exact. (But $\mathbb{Z}_{(p)} \otimes K_*(-)$ is not p-typical.) The groupoids scheme represented by the pair (K_*, K_*K) assigns to each graded ring R the groupoid with objects the units $v \in R_2$ and morphisms the strict isomorphisms

$$\phi : F_v \to F_w.$$

By replacing $\phi(x)$ by $w^{-1}\phi(vx)$ we see that this is isomorphic to the groupoid with objects the units v and morphisms the non-strict isomorphisms ϕ of $\mathbb{G}_m$ so that $\phi'(0) = w^{-1}v$.

4.6 Example (2-periodic theories). With the example of K-theory (see 4.5) we first encounter a 2-periodic theory. Other important examples are the Lubin-Tate theories E_n. See 7.6 below.

Let A be a graded ring of the form $A = A_0[u^{\pm 1}]$ where A_0 is in degree 0 and u is a unit in degree 2. Let F be a formal group law over A. Then we can define an ungraded formal group law G over A_0 by the formula

$$x +_G y = u(u^{-1}x +_F u^{-1}y).$$

Let $\Gamma = A \otimes_L W \otimes_L A$. The groupoid scheme represented by (A, Γ) assigns to each graded commutative ring the groupoid with objects $f : A \to R$ and morphism the strict isomorphisms $\phi : f_0^*F \to f_1^*F$. This is isomorphic to the groupoid with objects (g, v) where $g : A_0 \to R_0$ is a ring homomorphism and $v \in R_2$ is a unit. The morphisms are all isomorphisms $\phi : g_0^*G \to g_1^*G$ so that $v_1 = \phi'(0)v_0$. Thus if $a_i \in \Gamma_{2i}$ are the coefficients of the universal strict isomorphism from η_L^*F to η_R^*F, then

$$b_i = \begin{cases} u^{-i}a_i, & i > 0; \\ u^{-1}\eta_R(u), & i = 0 \end{cases}$$

are the coefficients of the universal isomorphism from $\eta_L^* G$ to $\eta_R^* G$.

There is an equivalence of categories between (A,Γ)-comodules concentrated in even degrees[4] and (A_0,Γ_0)-comodules, where $(-)_0$ means the elements in degree 0. The functor one way simply send M to M_0. To get the functor back, let N be an (A_0,Γ_0)-comodule and, define, for i, new (A_0,Γ_0)-comodule $M(i)$ with $M(i) = u^i M$ and

$$\psi_{M(i)}(u^i x) = b_0^i u^i \psi_M(x).$$

If M is an (A,Γ)-comodule in even degrees, then $M = \oplus M_0(i)$, where $M_0(i)$ is now in degree $2i$.

Using this equivalence of categories we have that

$$\mathrm{Ext}_\Gamma(A,M) \cong \mathrm{Ext}_{\Gamma_0}(A_0,M_0)$$

and if M is concentrated in even degrees,

$$\mathrm{Ext}_\Gamma(\Sigma^{2t}A,M) \cong \mathrm{Ext}_\Gamma(A,\Sigma^{-2t}M) \cong \mathrm{Ext}_{\Gamma_0}(A_0,M_0(t)).$$

Also $\mathrm{Ext}_\Gamma(\Sigma^{2t+1}A,M) = 0$.

4.7 Example (The Morava stabilizer group). Let F be a formal group law over a graded commutative ring R; it is possible for the Hopf algebroid to be important even if the pair (R,F) is not Landweber exact. Here is a crucial example. Set

$$K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$$

and let F_n be the unique p-typical formal group law with p-series $[p](x) = v_n x^{p^n}$. This is the *Honda* formal group law. Then the Hopf algebroid

$$(K(n)_*, K(n)_* \otimes_L W \otimes_L K(n)_*)$$

If we write

$$\Sigma(n) \overset{\mathrm{def}}{=} K(n)_* \otimes_L W \otimes_L K(n)_*$$

then $\Sigma(n)$ is a graded Hopf *algebra* over $K(n)_*$ – since v_n is primitive – by looking at the p-series we have

$$\Sigma(n) \cong \mathbb{F}_p[v_n^{-1}][t_1,t_2,\cdots]/(t_i^{p^n} - v_n^{p^i-1}t_i)$$

is a graded Hopf algebra.

This Hopf algebra can be further understood as follows. Define a faithfully flat extension

$$i : \mathbb{F}_p[v_n^{\pm 1}] \longrightarrow \mathbb{F}_p[u^{\pm 1}]$$

where $u^{p^n-1} = v_n$. Because $\mathbb{F}_{p^n}[u^{\pm 1}]$ is now 2-periodic, we can (as in the last example) rewrite the Honda formal group law i^*F_n as an (ungraded) formal group law Γ_n over $\mathbb{F}_p$ with p-series exactly x^{p^n}. Then

$$\mathbb{F}_p[u^{\pm 1}] \otimes_{K(n)_*} (K(n)_*, \Sigma(n)) \cong \mathbb{F}_p[u^{\pm 1}] \otimes (\mathbb{F}_p, S(n))$$

[4]There is a theory that includes odd degrees; this is assumption is for convenience. See [21]

where $S(n)$ is a Hopf algebra concentrated in degree 0 which can be written

$$S(n) = \mathbb{F}_p[x_0^{\pm 1}, x_1, \cdots]/(x_i^{p^n} - x_i).$$

Then $\mathrm{Spec}(S(n))$ is the group scheme which assigns to each commutative $\mathbb{F}_p$-algebra R the (non-strict) automorphisms of Γ_n over R. The $\bar{\mathbb{F}}_p$ points of this group scheme – the automorphisms of Γ_n over the algebraic closure of $\mathbb{F}_p$ – is called the *Morava stabilizer group* S_n. From the form of $S(n)$ or from the fact that the p-series of Γ_n is simply x^{p^n}, we see that all these automorphisms are realized over $\mathbb{F}_{p^n}$. We take this group up below in Section 8. For now I note that it is known and has been heavily studied: see [27]§A2.2 and [10] – and from either of these references you can find many primary sources. If $n = 1$ it is the group $\mathbb{Z}_p^\times$ of units in the p-adic integers; for $n \geq 2$ it is non-abelian. For a thorough review of these groups, their cohomology and many more references, see [10].

5 The height filtration

Let $f : F \to G$ be a homomorphism of formal group laws over a commutative ring R; thus

$$f(F(x, y)) = G(f(x), f(y)).$$

Differentiating with respect to y and setting $y = 0$, we get a formula

(5.1) $$f'(x)F_y(x, 0) = G_y(f(x), 0)f'(0).$$

Here $F_y(x, y)$ is the partial derivative of $F(x, y)$ with respect to y. The power series $F_y(x, 0) = 1 +$ higher terms and is therefore invertible. Thus if $f'(0) = 0$, then $f'(x) = 0$. If R is a $\mathbb{Q}$-algebra, this implies $f(x) = 0$; however, if R is an $\mathbb{F}_p$ algebra, this simply implies that there is a power series $g(x)$ so that

$$f(x) = g(x^p).$$

If R is an $\mathbb{F}_p$ algebra, we let $\sigma : R \to R$ denote the Frobenius, and if F is a formal group law over R, we let

$$F^{(p)} = \sigma^* F$$

be the pull-back of F along the Frobenius. Thus if

$$F(x, y) = \sum a_{ij} x^i y^j,$$

then

$$F^{(p)}(x, y) = \sum a_{ij}^p x^i y^j.$$

The power series $\sigma(x) = x^p$ defines a homomorphism

$$\sigma : F \to F^{(p)}.$$

Combining all these remarks, we have the following:

5.1 Lemma. *Let $f : F \to G$ be a homomorphism of formal group laws over an $\mathbb{F}_p$-algebra R and suppose that $f'(0) = 0$. Then there is a unique homomorphism of formal group laws $g : F^{(p)} \to G$ and a factoring*

$$\begin{array}{c} F \xrightarrow{\sigma} F^{(p)} \xrightarrow{g} G. \\ \underbrace{\qquad\qquad\qquad\qquad}_{f} \end{array}$$

This can be repeated if $g'(0) = 0$ to get a factoring

$$\begin{array}{ccccc} F & \xrightarrow{\sigma} & F^{(p)} & \xrightarrow{\sigma} & F^{(p^2)} \\ & \searrow^{f} & & & \downarrow{\scriptstyle g_2} \\ & & & & G \end{array}$$

5.2 Definition. *Let F be a formal group law over an $\mathbb{F}_p$-algebra R. Then F has a* **height** *at least n if there is a factoring of the p-series*

$$\begin{array}{ccccccc} F & \xrightarrow{\sigma} & F^{(p)} & \xrightarrow{\sigma} & \cdots & \xrightarrow{\sigma} & F^{(p^n)} \\ & \searrow^{[p]} & & & & & \downarrow{\scriptstyle g_n} \\ & & & & & & F \end{array}$$

Every formal group law over an $\mathbb{F}_p$-algebra has height at least 1. The multiplicative formal group law $\mathbb{G}_m(x, y) = x + y + xy$ has height 1 but not height 2; the additive formal group law $\mathbb{G}_a(x, y) = x + y$ has height at least n for all n; thus, we say it has infinite height. The Honda formal group law of 4.7 has height exactly n.

By construction, if F has height at least n there is a power series $g_n(x) = v_n^F x + \cdots$ so that

$$[p]_F(x) = g_n(x^{p^n}) = v_n^F x^{p^n} + \cdots \tag{5.2}$$

and F has height at least $n+1$ if and only if $v_n^F = 0$. The element v_n^F depends only on the strict isomorphism class of F. Thus, to check whether $v_n = 0$, we can assume that F is p-typical, where we already know that

$$[p]_F(x) = px +_F v_1 x^p +_F \cdots +_F v_n x^{p^n} + \cdots$$

This formula implies that the two definitions of v_n-agree. It also implies that there is an ideal

$$I_n \overset{\text{def}}{=} (p, \ldots, v_{n-1}) \subseteq BP_*$$

so that a p-typical formal group law F has height at least n if and only if $0 = FI_n \subseteq R$, where $F : BP_* \to R$ classifies F. The image of I_n in R depends only on the strict isomorphism class of F; from this is follows that the ideal I_n is *invariant* – that is, it is a subcomodule of BP_*. It is a theorem of Landweber's (see [14]) that the I_n and $\{0\}$ are the only prime (and, in fact, radical) invariant ideals of BP_*.

We now have a filtration of the groupoid scheme **fglp** over $\mathbb{Z}_{(p)}$ the groupoid schemes

$$\cdots \subseteq \mathbf{fglp}(n) \subseteq \mathbf{fglp}(n-1) \subseteq \cdots \subseteq \mathbf{fglp}(1) \subseteq \mathbf{fglp}$$

where $\mathbf{fglp}(n)$ assigns to each commutative ring R the groupoid of p-typical formal groups F for which $FI_n = 0$. Since these are defined by the vanishing of an ideal, we call these closed; the open complement of $\mathbf{fglp}(n)$ is the subgroupoid scheme $\mathcal{U}(n-1) \subseteq \mathbf{fglp}$ of p-typical formal groups F for which $FI_n = R$. (Note the shift in numbering.) This gives a filtration

$$\mathcal{U}(0) \subseteq \mathcal{U}(1) \subseteq \cdots \mathcal{U}(n-1) \subseteq \mathcal{U}(n) \subseteq \cdots \subseteq \mathbf{fglp}.$$

Note that neither filtration is exhaustive as

$$\cap \mathbf{fglp}(n) = \mathbf{fglp} - [\cup\, \mathcal{U}(n-1)]$$

contains the additive formal group law over $\mathbb{F}_p$.

5.3 Remark. Here are some basic facts about this filtration:

1. The groupoid scheme $\mathbf{fglp}(n)$ is defined by the Hopf algebroid

$$\begin{aligned}(BP_*/I_n, BP_*/I_n \otimes_{BP_*} BP_*BP)\\ \cong (\mathbb{F}[v_n, v_{n+1}, \ldots], \mathbb{F}[v_n, v_{n+1}, \ldots][t_1, t_2, \ldots]).\end{aligned}$$

2. A p-typical formal group law F over R lies in $\mathcal{U}(0)$ if and only if R is a $\mathbb{Q}$-algebra; F is then isomorphic to the additive formal group $\mathbb{G}_a$ over R (induced from $\mathbb{Q}$). The automorphisms of $\mathbb{G}_a$ over a $\mathbb{Q}$-algebra R is isomorphic to the group of units $R^\times$.

3. The groupoid scheme $\mathcal{U}(n)$, $n \geq 1$, is not isomorphic or even equivalent to an affine groupoid scheme given by a Hopf algebroid. It is, however, equivalent *locally for the flat topology* (see Definition 7.1 for this notion.) to the groupoid scheme of defined by the Hopf algebroid of the Johnson-Wilson theories $(E(n)_*, E(n)_*E(n))$. This is a result of Hovey-Strickland [13] and Naumann [24].

4. The relative open $\mathbf{fglp}(n) \cap \mathcal{U}(n)$ consists of those p-typical formal group laws F over an $\mathbb{F}_p$ algebra R which are of *exact* height n; that is, $FI_n = 0$ and $v_n^F \in R$ is a unit. This groupoid scheme has exactly one "geometric point"; that is, over a algebraically closed field there is a unique isomorphism class of formal groups of height n. Its automorphism group was discussed below in Section 8.

All of this should be treated rigorously with the language of stacks. See [31].

5.4 Remark (Quasi-coherent sheaves over groupoid schemes). Because groupoid scheme $\mathcal{U}(n)$ is not isomorphic to the groupoid scheme given by a Hopf algebroid; thus the scope of inquiry must expand a little at this point. In particular, we need to find a reformulation of the notion of comodule which can be adapted to such groupoid scheme. These are the *quasi-coherent sheaves.* For much more on this topic see [12] or, for the general theory [16]

Let $\mathcal{X}$ be a groupoid scheme. If $\mathcal{X}$ is not affine, we have to take some care about what this means. At the very least it includes the provision that the objects and morphisms form sheaves in the flat topology. Thus, if $\mathcal{X}_0$ is the objects functor of $\mathcal{X}$, then for all faithfully flat extensions $R \to S$ there is an equalizer diagram

$$\mathcal{X}_0(R) \longrightarrow \mathcal{X}_0(S) \rightrightarrows \mathcal{X}_0(S \otimes_R S)$$

There is a similar condition on morphisms. The examples $\mathcal{U}(n)$ above satisfy this conditions, as does an groupoid scheme arising from a Hopf algebroid.

If R is a ring, write $\mathrm{Spec}(R)$ for the functor it represents. (Remember, we are working with graded rings, for the most part, so this is a bit of an abuse of notation.) The functor $\mathrm{Spec}(R)$ is trivially a groupoid scheme, with only identity morphisms. Then the morphisms of groupoid schemes

$$x : \mathrm{Spec}(R) \to \mathcal{X}$$

are in one-to-one correspondence with objects $x \in \mathcal{X}(R)$. A 2-commuting diagram

(5.3)
$$\begin{array}{ccc} \mathrm{Spec}(S) & & \\ {\scriptstyle \mathrm{Spec}(f)}\downarrow & \overset{y}{\searrow} & \\ & & \mathcal{X} \\ & \underset{x}{\nearrow} & \\ \mathrm{Spec}(R) & & \end{array}$$

is a pair (f, ϕ) where $f : R \to S$ is a ring homomorphism and $\phi : y \to f^*x$ is an isomorphism in $\mathcal{X}(S)$.

A quasi-coherent sheaf $\mathcal{F}$ assigns to each morphism of groupoid schemes $x : \mathrm{Spec}(R) \to \mathcal{X}$ an R-module $\mathcal{F}(R, x)$ and to each 2-commuting diagram 5.3 a morphism of R-modules $\mathcal{F}(R, x) \to \mathcal{F}(S, y)$ so that the induced map

$$S \otimes_R \mathcal{F}(R, x) \longrightarrow \mathcal{F}(S, y)$$

is an isomorphism.

If $\mathcal{X}$ is the groupoid scheme arising from the Hopf algebroid (A, Γ), then an object $x \in \mathcal{X}(R)$ is a homomorphism $x : A \to R$. If M is a comodule we set $\mathcal{F}(R, x) = R \otimes_A M$. This is part of an equivalence of categories.

If two groupoid schemes are locally equivalent in the flat topology, they have equivalent categories of quasi-coherent sheaves; this implies they have equivalent sheaf cohomologies for these sheaves (i.e., coherent cohomology). If $\mathcal{X}$ arises from a Hopf algebroid (A, Γ) this cohomology is exactly $\mathrm{Ext}^*_\Gamma(\Sigma^* A, -)$. See Theorem 7.2.

5.5 Remark (The invariant differential). Let $F(x, y)$ be a formal group law over R. The power series $F_y(x, 0)$ introduced at the beginning of this section didn't come from the ether. We define a differential $f(x) \in R[[x]]dx$ to be invariant if

$$f(x +_F y)d(x +_F y) = f(x)dx + f(y)dy.$$

Expanding this formula out and setting $y = 0$, we see that the invariant differentials form a free sub-R-module of rank 1 of $R[[x]]dx$ generated by

$$\omega_F = \frac{dx}{F_y(x, 0)};$$

indeed, if $f(x)dx$ is an invariant differential, then $f(x)dx = f(0)\omega_F$. If $f : F \to G$ is a homomorphism of formal groups then the formula of 5.1 says that

$$f^*\omega_G = f'(0)\omega_F.$$

If R is a $\mathbb{Q}$-algebra, then the integral of ω_F with zero constant term defines an isomorphism of F with the additive formal group law.

6 The chromatic decomposition

The chromatic decomposition attempts to isolate what part of the Ext groups $\mathrm{Ext}^*_{BP_*BP}(\Sigma^* BP_*, BP_*)$ arise from the formal groups of exact height n. The result is a spectral sequence.

To begin, we can easily isolate the part of the Ext groups arising from formal groups over the rational numbers (exact height 0 if you want) by inverting p. Thus we have a short exact sequence of comodules

$$0 \to BP* \to p^{-1}BP_* \to BP_*/(p^\infty) \to 0.$$

The comodule $BP_*/(p^\infty)$, because it p-torsion, lives (in a sense I won't make precise here) on $\mathbf{fglp}(1)$. We'd like something that lives on $\mathbf{fglp}(1) \cap \mathcal{U}(1)$ – on groupoid scheme of formal groups of exact height 1. To do this we'd like to form $v_1^{-1}BP_*/(p^\infty)$. For this we need the following result.

6.1 Lemma. *Let M be (BP_*, BP_*BP)-comodule which is I_n-torsion as a BP_*-module. Then there is a unique comodule structure on $v_n^{-1}M$ so that the natural homomorphism $M \to v_n^{-1}M$ is a comodule map.*

Proof. Write $\eta_R(v_n) = v_n - f$ where

$$f \in I_n \otimes_{BP_*} BP_*BP \cong BP_*BP_* \otimes_{BP_*} I_n.$$

Suppose we have a comodule structure map ψ for $v_n^{-1}M$ extending ψ_M. Then for all $a \in v_n^{-1}M$, there is a k so that $v_n^k a \in M$. Thus we have

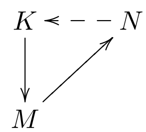
$$\psi_M(v_n^k a) = \eta_R(v_n^k)\psi(a)$$

or

$$\psi(a) = \eta_R(v_n)^{-k}\psi_M(v_n^k a)$$
$$= v_n^{-k}(1 - f/v_n)^{-k}\psi_M(v_n^k a).$$

Expanding $(1 - f/v_n)^{-k}$ as a power series and using that M is I_n-torsion we see that this equation defines ψ and proves its uniqueness. □

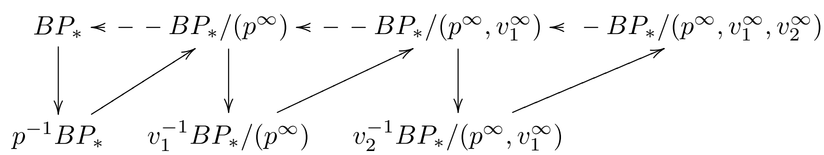
We now form the following diagram, where each of the triangles



represents a short exact sequence

$$0 \to K \to M \to N \to 0$$

in comodules. The dotted arrow indicates that we will have a long exact sequence in Ext-groups. Using Lemma 6.1 we now inductively construct a diagram extending infinitely far to the right:



Let's write BP_*/I_n^∞ for $BP_*/(p^\infty, \ldots, v_{n-1}^\infty)$. We now apply the functor

$$H^{s,t}(-) \cong \mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, -)$$

to these short exact sequences to get an exact couple; the dotted arrows now become maps $H^{s,t}(N) \to H^{s+1,t}(K)$. The resulting spectral sequence – *the chromatic spectral sequence* – then reads

$$E_1^{p,q} = H^{q,*}(v_p^{-1}BP/I_p^\infty) \Longrightarrow H^{p+q}(BP_*) \tag{6.1}$$

with differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

In particular, E_2 is the cohomology of the very interesting chain complex

$$0 \to H^{*,*}(p^{-1}BP_*) \to H^{*,*}(v_1^{-1}BP_*/I_1^\infty) \to H^{*,*}(v_1^{-2}BP_*/I_2^\infty) \to \cdots$$

and there is an edge homomorphism

$$E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \rightarrowtail \mathrm{Ext}^p_{BP_*BP}(\Sigma^* BP_*, BP_*)$$

with $E_2^{p,0}$ a subquotient of

$$Ext^p_{BP_*BP}(\Sigma^* BP_*, v_p^{-1}BP_*/I_p^\infty).$$

6.2 Example (The Greek letter elements). It is easy to produce an interesting family of permanent cycles in this spectral sequence. There is a canonical inclusion of comodules

$$\iota_n : \Sigma^{-f(n)} BP_*/I_n \longrightarrow BP_*/I_n^\infty$$

where $f(1) = 0$ and for $n > 1$

$$f(n) = 2(p-1) + 2(p^2-1) + \cdots + 2(p^{n-1}-1).$$

The morphism ι_n sends the generator of BP_*/I_n to the element

$$\frac{1}{pv_1 \cdots v_{n-1}} \in BP_*/I_n^\infty.$$

This can also be defined inductively by letting $\iota_0 : BP_* \to p^{-1}BP_*$ be the usual inclusion and noting there is a commutative diagram of comodules

$$\begin{array}{ccc} \Sigma^{-f(n)} BP_*/I_n & \xrightarrow{v_n} & \Sigma^{-f(n+1)} BP_*/I_n \\ \downarrow{\scriptstyle \iota_n} & & \downarrow \\ BP_*/I_n^\infty & \longrightarrow & v_n^{-1} BP_*/I_n^\infty \end{array}$$

and taking the induced map on quotients. The element $v_n \in BP_*/I_n$ is primitive; this is a consequence of the formula 5.2 and the fact that if a formal group law F has height at least n, then v_n^F is a strict isomorphism invariant. This yields a sub-ring

$$\mathbb{F}_p[v_n] \subseteq H^{0,*}(BP_*/I_n).$$

Since $H^{0,*}$ is left exact (preserves injections) we get non-zero classes v_n^k in

$$H^{0,2k(p^n-1)-f(n)}(BP_*/I_n^\infty) \subseteq H^{0,2k(p^n-1)-f(n)}(v_n^{-1} BP_*/I_n^\infty).$$

The image of the class v_n^k in

$$H^{0,2k(p^n-1)-f(n)}(BP_*) = \mathrm{Ext}^s_{BP_*BP}(\Sigma^{2k(p^n-1)-f(n)} BP_*, BP_*)$$

is the kth element in nth Greek letter family and is sometime written $\alpha_k^{(n)}$. There is no reason at this point to suppose that these classes are non-zero – although we will see shortly that for low values of n there can be no more differentials in the chromatic spectral sequence.

If n is small we use the actual Greek letter. Thus the cases $n = 1$ and $n = 2$ are written

$$\alpha_k \in \mathrm{Ext}^1_{BP_*BP}(\Sigma^{2k(p-1)} BP_*, BP_*)$$

and

$$\beta_k \in \mathrm{Ext}^2_{BP_*BP}(\Sigma^{2k(p^2-1)-2(p-1)} BP_*, BP_*).$$

This brings us to our first periodic families in the stable homotopy groups of spheres.

6.3 Theorem. *Let $p \geq 3$. Then the class*

$$\alpha_k \in \mathrm{Ext}^1_{BP_*BP}(\Sigma^{2k(p-1)}BP_*, BP_*)$$

is a non-zero permanent cycle in the ANSS spectral sequence and detects an element of order p.

Let $p \geq 5$. Then the class

$$\beta_k \in \mathrm{Ext}^2_{BP_*BP}(\Sigma^{2k(p^2-1)-2(p-1)}BP_*, BP_*).$$

is a non-zero permanent cycle in the ANSS spectral sequence and detects an element of order p.

.

At the prime 2 we know $d_3\alpha_3 \neq 0$; at the prime 3 we know $d_5\beta_4 \neq 0$.

We will see in the next section that these classes are non-zero at E_2. They cannot be boundaries, but to show they are permanent cycles requires some homotopy theory.

6.4 Remark (Smith-Toda complexes). Suppose there exists finite CW spectrum $V(n)$ so that there is an isomorphism of comodules

$$BP_*V(n) \cong BP_*/I_n$$

and so that there is a self map $v : \Sigma^{2(p^n-1)}V(n) \to V(n)$ which induces multiplication by v_n on $BP_*(-)$. A short calculation shows that the top cell of $V(n)$ must be in degree $f(n)+n$ – the degree of the Milnor operation $Q_0 \ldots Q_{n-1}$. Let $i : S^0 \to V(n)$ and $q : V(n) \to S^{f(n)+n}$ be the inclusion of the bottom cell and the projection onto the top cell respectively. Then the composition

$$S^{2k(p^n-1)} \xrightarrow{i} \Sigma^{2k(p^n-1)}V(n) \xrightarrow{v^k} V(n) \xrightarrow{q} S^{f(n)+n}$$

is detected by $\alpha_k^{(n)}$ in the ANSS; thus, this class is a permanent cycle and, if non-zero, detects a non-trivial element in the homotopy groups of spheres. These ideas originated with [30] and [33] and have been explored in great details by many people; see [27] §5.5.

This pleasant story is marred by the fact that the $V(n)$ are known to exist only for small values of n and large values of p. The spectrum $V(0)$ is the Moore spectrum, which exists for all n, but has a v_1-self map only if $p > 2$. (At $p = 2$ one can realize v_1^4, which is of degree 8, a number which should not surprise Bott periodicity fans.) The spectrum $V(1)$ exists if $p > 2$, but has a v_2-self map only if $p > 3$. One can continue for a bit, but the spectrum $V(4)$ is not known to exist at any prime and non-existence results abound. See especially [25].

Note that any v_n-self map must have the property that $v^k \neq 0$ for all k. The existence and uniqueness of maps of finite complexes which realize multiplication by powers of v_n is part of the whole circle of ideas in the nilpotence conjectures. See [26], [9], [11], and [28].

6.5 Remark. The groups

$$\mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, v_n^{-1}BP_*/I_n^\infty)$$

form the E_2-term of an ANSS converging to an appropriate suspension of the fiber of the map between localizations

$$L_nS^0 \longrightarrow L_{n-1}S^0.$$

This fiber, often written M_nS^0 is the "monochromatic" piece of the homotopy theory of the sphere S^0. Again see [28].

7 Change of rings

We now begin trying to calculate E_2-terms of the ANSS spectral sequence based on some complex oriented, Landweber exact homology theory E_* over $\mathbb{Z}_{(p)}$. The primordial examples are BP_* and $\mathbb{Z}_{(p)} \otimes MU_*$; the Cartier idempotent shows that the Hopf algebroids

$$(BP_*, BP_*BP) \qquad \text{and} \qquad \mathbb{Z}_{(p)} \otimes (MU_*, MU_*MU)$$

represent equivalent affine groupoid schemes over $\mathbb{Z}_{(p)}$, from which is follows that

$$\mathbb{Z}_{(p)} \otimes \mathrm{Ext}^s_{MU_*MU}(\Sigma^t MU_*, MU_*) \cong \mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, BP_*).$$

This is an example of a *change of rings* theorem; we give several such theorems in this section.

Recall that a functor $f : G \to H$ of groupoids is an equivalence if it induces a weak equivalence on classifying spaces $BG \to BH$; equivalently, it induces an isomorphism

$$f_* : \pi_0 G \xrightarrow{\cong} \pi_0 H$$

on the sets of isomorphism classes of objects and for each object $x \in G$ we get an isomorphism

$$f_* : \pi_1(G, x) = \mathrm{Aut}_G(x) \xrightarrow{\cong} \mathrm{Aut}_H(f(x)) = \pi_1(H, f(x)).$$

This notion has a weakening.

7.1 Definition. *Suppose G and H are actually functors to groupoids from commutative rings. Then a functor $f : G \to H$ (now a natural transformation) is an* **equivalence locally in the flat topology** *if*

1. *for all rings R and all $x \in H(R)$ there exists a faithfully flat extension $R \to S$ of rings and $y \in G(R)$ so that $f(y) \cong x$ in $H(S)$;*

2. *for all rings R and all $x, y \in G(R)$ and $\phi : f(x) \to f(y)$ in $H(R)$, there is a faithfully flat extension $R \to S$ and a unique $\psi : x \to y$ in $G(S)$ so that $f\psi = \phi$ in $H(S)$.*

It is equivalent to say that the induced morphism $f_* : \pi_i H \to \pi_i G$ $(i = 0, 1)$ of presheaves on affine schemes induces an isomorphism on associated sheaves in the flat topology.

7.2 Theorem (Change of Rings I). *Let* $(A, \Gamma) \to (B, \Lambda)$ *be a morphism of Hopf algebroids. If* M *is an* (A, Γ)*-comodule, then* $B \otimes_A M$ *is a* (B, Λ)*-comodule and there is a homomorphism*

$$\mathrm{Ext}^s_\Gamma(\Sigma^t A, M) \longrightarrow \mathrm{Ext}^s_\Lambda(\Sigma^t B, B \otimes_A M).$$

If the morphism of Hopf algebroids induces an equivalence of affine groupoid schemes locally in the flat topology, then this morphism on Ext*-groups is an isomorphism.*

7.3 Remark. A conceptual proof of this can be constructed using the ideas of Remark 5.4; indeed, we can even relax the requirement that we are working with Hopf algebroids and consider groupoid schemes such as $\mathcal{U}(n)$. See [12].

7.4 Example. For example, let (A, Γ) be a Hopf algebroid and $A \to B$ a faithfully flat extension of rings. Then we get a new Hopf algebroid (B, Λ) with

$$\Lambda = B \otimes_A \Gamma \otimes_A B$$

and we have that for all (A, Γ)-comodules M

$$\mathrm{Ext}^s_\Gamma(\Sigma^t A, M) \cong \mathrm{Ext}^s_\Lambda(\Sigma^t B, B \otimes_A M).$$

To be very concrete let $(A, \Gamma) = (L, W) = (MU_*, MU_*MU)$, the Hopf ring representing the groupoid scheme of graded formal group laws. Let $B = L[u^{\pm 1}]$ where u is a unit in degree 2. Then

$$B = L_0[u^{\pm 1}]$$

where L_0 is now the Lazard ring concentrated in degree 0. We also have

$$B \otimes_L W \otimes_L B = L_0[u^{\pm 1}, b_0^{\pm 1}, b_1, \ldots] = W_0[u^{\pm 1}]$$

where the b_i are the coefficients of the universal isomorphism of (ungraded) formal group laws. This yields

$$\mathrm{Ext}^s_W(\Sigma^t L, M) \cong \mathrm{Ext}^s_{W_0[u^{\pm 1}]}(\Sigma^t L_0[u^{\pm 1}], M[u^{\pm 1}]).$$

This can be further simplified using Example 4.6.

7.5 Example (Morava K-theory to group cohomology). We apply these results to the Hopf algebroid $(K(n)_*, \Sigma(n))$ constructed in Example 4.7. Let

$$K(n)_* = \mathbb{F}_p[v_n^{\pm 1}] \longrightarrow \mathbb{F}_{p^n}[u^{\pm 1}]$$

be the faithfully flat extension with $u^{p^n - 1} = v_n$. The extended Hopf algebroid is

$$\mathbb{F}_{p^n}[u^{\pm 1}] \otimes_{\mathbb{F}_{p^n}} (\mathbb{F}_{p^n}, \mathbb{F}_{p^n} \otimes S(n) \otimes \mathbb{F}_{p^n}).$$

(We must take a little care with this formula, as $\eta_R(u) = b_0 u$, where b_0 is the leading term of the universal isomorphism of ungraded formal groups laws.) By Galois theory, we know

$$\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n} \cong \mathrm{map}(\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p), \mathbb{F}_{p^n})$$

and

$$\mathbb{F}_{p^n} \otimes S(n) \cong \mathrm{map}(\mathrm{Aut}(\Gamma_n), \mathbb{F}_{p^n})$$

by Example 4.7. The group $\mathrm{Aut}(\Gamma_n) = S_n$ is a profinite group and the last set of maps is actually the continuous maps. Putting these two facts together we have

$$\mathbb{F}_{p^n} \otimes S(n) \otimes \mathbb{F}_{p^n} \cong \mathrm{map}(G_n, \mathbb{F}_{p^n})$$

where $G_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \rtimes S_n$ is the semi-direct product. It follows that if M is a $(K(n)_*, \Sigma(n))$ comodule concentrated in even degrees (see Example 4.6) then

$$\begin{aligned}\mathrm{Ext}_{K(n)_* \otimes S(n)}(\Sigma^{2t} K(n)_*, M) &\cong H^s(G_n, \mathbb{F}_{p^n} \otimes M_0(t)) \\ &\cong H^s(S_n, \mathbb{F}_{p^n} \otimes M_0(t))^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_n)}.\end{aligned}$$

Cohomology here is continuous cohomology. If N is continuous G_n module, $\phi \in S_n$ and $u^t x \in N(t)$, then

$$\phi(u^t x) = \phi'(0)^t u^t (\phi x).$$

7.6 Example. The previous example can be extended to a wider class of examples. Let M be a comodule over $(E(n)_*, E(n)_* E(n))$ and suppose n is I_n torsion. Here $I_n = (p, v_1, \cdots, v_{n-1}) \subseteq E(n)_*$ and

$$E(n)_*/I_n \cong K(n)_*.$$

We'd like to relate the Ext groups of M to an appropriate group cohomology. Let $(E_n)_* = E_{n*}$ be the 2-periodic Lubin-Tate homology theory obtained from the deformation theory of the formal group law Γ_n over $\mathbb{F}_{p^n}$. Thus $E_{n*} = R(\mathbb{F}_{p^n}, \Gamma_n)[u^{\pm 1}]$ where u is in degree 2 and the deformation ring

$$R(\mathbb{F}_{p^n}, \Gamma_n) = W(\mathbb{F}_{p^n})[[u_1, \cdots, u_{n-1}]]$$

is in degree 0. This is a local ring with maximal ideal

$$\mathfrak{m} = (p, u_1, \ldots, u_{n-1}).$$

Define a faithfully flat extension $E(n)_* \to E_{n*}$ by

$$v_i \mapsto \begin{cases} u_i u^{p^i - 1}, & i < n; \\ u^{p^n - 1}, & n = 1. \end{cases}$$

Then $\mathfrak{m} = [E_{n*} \otimes_{E(n)_*} I_n]_0$. Let's also write $E_{n*}E_n$ for the completion of

$$E_{n*} \otimes_{E(n)_*} E(n)_* E(n) \otimes_{E(n)_*} E_{n*}$$

at the maximal ideal $\mathfrak{m}$. Then Lubin-Tate theory tells us that we obtain the continuous maps:

$$E_{n*}E_n \cong \mathrm{map}(G_n, E_{n*}).$$

Let M be an $(E(n)_*, E(n)_*E(n))$-comodule which is I_n-torsion as a $E(n)_*$-module. Set $\bar{M} = E_{n*} \otimes_{E(n)_*} M$. Then, because of the torsion condition,

$$\begin{aligned}\mathrm{map}(G_n, \bar{M}) &\cong E_{n*}E_n \otimes_{E_{n*}} \bar{M} \\ &\cong E_{n*} \otimes_{E(n)_*} E(n)_*E(n) \otimes_{E(n)_*} E_{n*} \otimes_{E_{n*}} \bar{M}\end{aligned}$$

Here $\mathrm{map}(G_n, \bar{M})$ is the module of continuous maps, where $\bar{M}$ has the discrete topology. From this, and argument similar to that of the previous example, we have

$$\mathrm{Ext}^s_{E(n)_*E(n)}(\Sigma^{2t}, M) \cong H^s(G_n, \bar{M}_0(t)). \tag{7.1}$$

The final change of rings result is the following – this is Morava's original change of rings result. See [7], [23] and [27].

7.7 Theorem (Change of rings II). *Let M be a (BP_*, BP_*BP)-comodule which is I_n-torsion as BP_*-module. Let*

$$\bar{M} = E_{n*} \otimes_{BP_*} M \cong E_{n*} \otimes_{E(n)_*} E(n)_* \otimes_{BP_*} M.$$

Then if M is concentrated in degrees congruent to 0 modulo $2(p-1)$, there is a natural isomorphism

$$\mathrm{Ext}^s_{BP_*BP}(\Sigma^{2t}BP_*, v_n^{-1}M) \cong H^s(G_n, \bar{M}_0(t)).$$

The odd cohomology groups are zero.

7.8 Remark. Here is a sketch of the proof. It is sufficient, by Example 7.6 to show

$$\mathrm{Ext}^s_{BP_*BP}(\Sigma^{2t}BP_*, v_n^{-1}M) \cong \mathrm{Ext}_{E(n)_*E(n)}(\Sigma^{2t}E(n)_*, E(n)_* \otimes_{BP_*} M).$$

We use the ideas of Remark 5.4. Let $i : \mathcal{U}(n) \to \mathbf{fglp}$ be the inclusion of the open sub-groupoid scheme of formal group laws of height no more than n. The pull-back functor i^* from quasi-coherent sheaves (that is, comodules) over **fglp** has a left exact right adjoint i_*. Then, by Remarks 5.3.3 and 7.3, it is equivalent to prove that

$$H^*(\mathbf{fglp}, v_n^{-1}\mathcal{F}_M) \cong H^*(\mathcal{U}(n), i^*\mathcal{F}_M).$$

I have written $\mathcal{F}_M$ for the quasi-coherent sheaf associated to M and suppressed the internal grading. It turns out that because M is I_n-torsion

$$i_*i^*\mathcal{F}_M \cong v_n^{-1}\mathcal{F}_M.$$

Indeed, this formula is an alternate and conceptual proof of Lemma 6.1. Furthermore,

$$R^q i_*i^*\mathcal{F}_M = 0, \qquad q > 0.$$

The result now follows from the composite functor spectral sequence

$$H^p(\mathbf{fglp}, R^q i_*\mathcal{E}) \Longrightarrow H^{p+q}(\mathcal{U}(n), \mathcal{E}).$$

8 The Morava stabilizer group

The Morava stabilizer group G_n was introduced in Example 4.7 and now we write down a little bit about its structure. Much of this material has classical roots; for more discussion see [27] §A.2.2 and [10].

We have fixed an ungraded p-typical formal group law Γ_n over $\mathbb{F}_p$ with p-series

$$[p](x) = x^{p^n}.$$

(The choice of Γ_n is simply to be definite; there is a general theory which we will not invoke.) Then G_n is the set of pairs (σ, ϕ) where $\sigma : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ is a element of the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ and

$$\phi : \Gamma_n \longrightarrow \sigma^* \Gamma_n$$

is an non-strict isomorphism of formal groups. We see immediately that we have a semi-direct product decomposition

$$G_n \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \rtimes \mathrm{Aut}(\Gamma_n)$$

where we are write $\mathrm{Aut}(\Gamma_n)$ for the automorphisms of Γ_n over $\mathbb{F}_{p^n}$. These automorphisms are the units in the ring $\mathrm{End}(\Gamma_n)$ of endomorphisms and any such endomorphism $f(x)$ can be written

$$f(x) = a_0 x +_{\Gamma_n} a_1 x^p +_{\Gamma_n} a_2 x^{p^2} +_{\Gamma_n} \cdots$$

where $a_i \in \mathbb{F}_{p^2}$. The automorphisms have $a_0 \neq 0$.

Of particular note are the endomorphisms $p = [p](x) = x^{p^n}$ and $S(x) = x^p$. We have, of course, that $S^n = p$.

The subring of the endomorphisms with elements

$$f(x) = a_0 x +_{\Gamma_n} a_n x^{p^n} +_{\Gamma_n} a_{2n} x^{p^{2n}} +_{\Gamma_n} \cdots$$

is a commutative complete local ring with with maximal ideal generated by p and residue field $\mathbb{F}_{p^n}$; thus, from the universal property of Witt vectors, it is canonically isomorphic to $W(\mathbb{F}_{p^n})$. It follows that

$$\mathrm{End}(\Gamma_n) \cong W(\mathbb{F}_{p^2})\langle S \rangle / (S^n - p).$$

The brackets mean that S does not commute the Witt vectors; in fact $Sa = \phi(a)S$ where ϕ is the Frobenius. While we will not use this fact, the endomorphism ring has a classical description as the maximal order in a division algebra over the p-adic rationals. We now have

$$\mathrm{Aut}(\Gamma_n) \cong [W(\mathbb{F}_{p^2})\langle S \rangle / (S^n - p)]^\times.$$

Thus an element $\mathrm{Aut}(\Gamma_n)$ can be expressed as a sum

$$b_0 + b_1 S + \cdots b_{n-1} S^{n-1}$$

where $b_i \in W(\mathbb{F}_{p^n})$ and $b_i \neq 0$ modulo p.

8.1 Remark. Here are a few basic facts about this group.

1. The powers of S define a filtration on $\mathrm{End}(\Gamma_n)$ and $\mathrm{End}(\Gamma_n)$ is complete with respect to this filtration. There is an induced filtration on $\mathrm{Aut}(\Gamma_n)$ with
$$F_{i/n}\,\mathrm{Aut}(\Gamma_n) = \{\ f \mid f \equiv 1 \bmod S^i\ \}.$$
In this filtration the group is a profinite group. The fractional notation is so that p has filtration degree 1.

2. We define $S_n = F_{1/n}\,\mathrm{Aut}(\Gamma_n)$; it is the set of automorphisms $f(x)$ with $f'(0) = 1$ – the strict automorphisms. This is the p-Sylow subgroup (in the profinite sense) of the automorphism group.

3. The elements $a_0 x \in \mathrm{Aut}(\Gamma_n)$ define a subgroup isomorphic to $\mathbb{F}_{p^2}^{\times}$ splitting the projection
$$\begin{aligned}\mathrm{Aut}(\mathbb{F}_{p^n}) &\longrightarrow \mathbb{F}_{p^2}^{\times}\\ f &\mapsto f'(0).\end{aligned}$$
Thus we have a semi-direct product decomposition
$$\mathrm{Aut}(\Gamma_n) \cong \mathbb{F}_{p^n} \rtimes S_n.$$

4. The center of $C \subseteq S_n$ is the subgroup of elements of the form b_0 where $b_0 \in \mathbb{Z}_p^{\times} \subseteq W(\mathbb{F}_{p^n})^{\times}$ and $b_0 \equiv 1$ modulo p. As power series these are of the form
$$f(x) = x +_{\Gamma_n} a_n x^{p^n} +_{\Gamma_n} a_{2n} x^{p^{2n}} +_{\Gamma_n} \cdots$$
where $a_i \in \mathbb{F}_p$. Thus $\mathbb{Z}_p \cong C \subseteq \mathbb{Z}_p^{\times}$.

5. The action of $\mathrm{Aut}(\Gamma_n)$ on $\mathrm{End}(\Gamma_n)$ defines a homomorphism
$$\mathrm{Aut}(\Gamma_n) \to \mathrm{Gl}_n(W(\mathbb{F}_{p^n}));$$
taking the determinant defines a homomorphism
$$S_n \longrightarrow \mathbb{Z}_p^{\times}.$$
(A priori, the determinant lands in $W(\mathbb{F}_{p^n})^{\times}$, but we check it actually lands in the subgroup $\mathbb{Z}_p^{\times}$.) The elements of the subgroup S_n will all map into $\mathbb{Z}_p$ and the composition $\mathbb{Z}_p \cong C \to S_n \to \mathbb{Z}_p$ is multiplication by n; hence, if $(n,p) = 1$, we get a splitting
$$S_n \cong C \times S_n^1$$
where S_n^1 is the kernel of determinant map $\mathrm{Aut}(\Gamma_n) \to \mathbb{Z}_p^{\times}$.

6. A deeper part of the theory studies the finite subgroups of S_n. We note here only that S_n contains elements of order p if and only if $p-1$ divides n. At the prime 2 this is bad news, of course.

We now turn to a discussion of the continuous cohomology of these groups. The first statement of the following – and much, much more – can be found in [17]. The second is follows from the last item of the previous remark. If G a p-profinite group is a Poincaré duality group of dimension m, then $H^*(G, M)$ (with any coefficients) is a Poincaré duality algebra with top class in degree m.

8.2 Proposition. *The group S_n has a finite index subgroups H_n which is a Poincaré duality group of dimension n^2. If $p-1$ does not divide n, we may take H_n to be the entire group.*

Let's now get concrete and try to calculate $H^*(G_2, \mathbb{F}_{p^2}(*))$, $p > 2$. The action of G_2 on $\mathbb{F}_{p^2}(*)$ is through the quotient subgroup

$$\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) \rtimes \mathbb{F}_{p^2}^{\times}.$$

The kernel of quotient map is the p-Sylow subgroup S_2, which (since $p > 2$) can be written as a product

$$C \times S_2^1$$

where $\mathbb{Z}_p \cong C \subset S_2$ is the center.

If G is a profinite group and $\mathbb{F}$ is a field on which G acts trivially, $H^1(G, \mathbb{F})$ is the group of 1-cocyles; furthermore, these are simply the continuous group homomorphisms

$$f : G \longrightarrow \mathbb{F}.$$

Define a 1-cocyle $\zeta : C \to \mathbb{F}_p$ on the center by

$$\zeta(f) = a_2.$$

Here we are using the notation of Remark 8.1.4.

8.3 Lemma. *There natural map*

$$E(\zeta) \longrightarrow H^*(C, \mathbb{F}_{p^2})$$

is an isomorphism. Furthermore as an $\mathbb{F}_{p^2}^{\times}$-module

$$H^1(C, \mathbb{F}_{p^2}) \cong \mathbb{F}_{p^2}(0).$$

Proof. The first part is standard; the second part follows from the fact that C is the center, so conjugation by an element $a_0 x$, $a_0 \in \mathbb{F}_{p^2}$ is trivial. □

Now define a 1-cocycles $\tau_1, \tau_2 : S_2^1 \to \mathbb{F}_{p^2}$ on S_2 by

$$\tau(f) = a_1 \qquad \text{and} \qquad \tau_2(f) = a_1^p.$$

Again again we use the notation of Remark 8.1.

8.4 Lemma. *There is an isomorphism of $\mathbb{F}_{p^2}^\times$-modules*

$$H^1(S_2^1, \mathbb{F}_{p^2}) \cong \mathbb{F}_{p^2}(p-1) \oplus \mathbb{F}_{p^2}(p^2 - p)$$

with summands generated by τ_1 and τ_2 respectively. Furthermore

$$\tau_1\tau_2 = 0.$$

Proof. We check that

$$\tau_1 : S_2^1/[S_2^1, S_2^1] \cong \mathbb{F}_{p^2}$$

is an isomorphism. Then

$$H^1(S_2^1, \mathbb{F}_{p^2}) \cong \mathrm{Hom}(S_2^1/[S_2^1, S_2^1], \mathbb{F}_{p^2}) \cong \mathbb{F}_{p^2} \times \mathbb{F}_{p^2}$$

generated by τ_1 and the Frobenius applied to τ_1 – that is, τ_2. To get the action of $\mathbb{F}_{p^2}$, let a_0x be typical element of that group and $f(x) \in S_2^1$. Then we check

$$\tau_1(a_0^{-1}(f(a_0x)) = a_0^{p-1}\tau_1(f(x)),$$

from which it follows that

$$\tau_2(a_0^{-1}(f(a_0x)) = a_0^{p^2-p}\tau_2(f(x)),$$

as claimed.

To show the product is zero we define $\sigma(f(x)) = a_2$. (This is not the same of ζ, even though it has the same formula, as the domain of definition is S_2^1, not the center.) Then there is an equality of 2-cocyles $\partial\sigma = -\tau_1\tau_2$. □

We can now define elements γ_1 and γ_2 in $H^2(S_2^1, \mathbb{F}_{p^2})$ by

$$\gamma_1 = \langle \tau_1, \tau_2, \tau_1 \rangle$$
$$\gamma_2 = \langle \tau_1, \tau_2, \tau_1 \rangle$$

This next result can be proved using Hopf algebra techniques as in [27] Theorem 6.3.22 or using Lazard's work [17]. See [10] for an introduction to the latter approach. We must restrict to primes $p > 3$ as G_2 at the prime 3 has 3-torsion.

8.5 Proposition. *Let $p > 3$. The cohomology ring $H^*(S_2^1, \mathbb{F}_{p^2})$ has generators τ_1,τ_2,γ_1, and γ_2 with all products zero except*

$$\tau_1\gamma_2 = \gamma_1\tau_2$$

generates $H^3(S_2^1, \mathbb{F}_{p^2})$.

The next result is a matter of taking invariants and Lemma 8.4. Note that

$$H^0(G_2, \mathbb{F}_{p^2}(*)) = H^0(\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p \rtimes \mathbb{F}_{p^2}^\times, \mathbb{F}_{p^2}[u^{\pm 1}]) \cong \mathbb{F}_p[v_2^{\pm 1}]$$

where $v_2 = u^{p^2-1}$.

8.6 Corollary. *If $p \geq 3$, the free $\mathbb{F}_p[v_2^{\pm 1}]$-module*

$$H^1(G_2, \mathbb{F}_{p^2}(*))$$

is generated by elements

$$\begin{aligned} \zeta &\in H^1(G_2, \mathbb{F}_{p^2}(0)) \\ h_0 &\in H^1(G_2, \mathbb{F}_{p^2}(p-1)) \\ h_1 &\in H^1(G_2, \mathbb{F}_{p^2}(p^2-p)) \end{aligned}$$

9 Deeper periodic phenomena

In this last section we calculate (or at least talk about calculating) enough of the chromatic spectral sequence 6.1 in order to understand the 0, 1, and 2-lines of the ANSS. At larger primes (here $p > 3$), this is largely an algebraic calculation. We begin with the easy calculation:

9.1 Lemma. *For all primes p,*

$$H^{s,t}(p^{-1}BP_*) \cong \begin{cases} \mathbb{Q}, & t = 0; \\ 0 & t \neq 0 \end{cases}.$$

Proof. This is a change of rings arguments. We check that

$$H^{*,*}(p^{-1}BP_*) \cong \operatorname{Ext}^*_{\mathbb{Q}\otimes W}(\Sigma^*(\mathbb{Q}\otimes L), \mathbb{Q}\otimes L)$$

and then apply the change of rings theorem 7.2 to the morphism of Hopf algebroids

$$\mathbb{Q}\otimes(L, W) \longrightarrow (\mathbb{Q}, \mathbb{Q}\otimes_L W \otimes_L \mathbb{Q})$$

induced by the map $L \to \mathbb{Q}$ classifying the additive formal group law. Since the additive formal group law over the rational numbers has no non-trivial strict isomorphisms we deduce that

$$\mathbb{Q}\otimes_L W \otimes_L \mathbb{Q} \cong \mathbb{Q}.$$

The result follows. □

For the higher entries in the chromatic spectral sequence we use Morava's change of ring theorem 7.7. Let

$$R_n = R(\mathbb{F}_{p^n}, \Gamma_n) = (E_n)_0 \cong W(\mathbb{F}_{p^n})[[u_1, \cdots, u_{n-1}]]$$

be the Lubin-Tate deformation ring and let

$$\mathfrak{m} = (p, u_1, \cdots, u_{n-1}) \subseteq R(\mathbb{F}_{p^n}, \Gamma_n)$$

be its maximal ideal. Then we have

$$H^{s,2t}(v_n^{-1}BP/I_n^\infty) \cong H^s(G_n, (R_n/\mathfrak{m}^\infty)(t)). \tag{9.1}$$

We now look at $n = 1$. The ring R_1 is isomorphic to the p-adic integers, $\mathfrak{m} = (p)$ and $R_1/m^\infty \cong \mathbb{Z}/p^\infty$. The group $G_1 = \mathbb{Z}_p^\times$, the units in the p-adic integers, and the action G_1 on $\mathbb{Z}/p^\infty(t)$ is given by

$$a(u^t x) = u^t(a^t x).$$

In particular the action on $\mathbb{Z}/p^\infty(0)$ is trivial. The calculation for the 1 line begins with the following result.

9.2 Theorem. *Let $p > 2$. Then $H^{s,2t} = H^s(G_1, \mathbb{Z}/p^\infty(t)) = 0$ unless $s = 0$ or 1 and $2t = 2k(p-1)$ for some k. In the case $k = 0$ we have*

$$H^0(G_1, \mathbb{Z}/p^\infty) = H^1(G_1, \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty$$

and if $k = p^r k_0$ with $(p, k_0) = 1$,

$$H^0(G_1, \mathbb{Z}/p^\infty(t)) \cong \mathbb{Z}/p^{r+1}\mathbb{Z}$$

and $H^1(G_1, \mathbb{Z}/p^\infty(t)) = 0$.

Proof. We can write $\mathbb{Z}_p^\times \cong C_{p-1} \times \mathbb{Z}_p$ where $C_{p-1} \cong \mathbb{F}_p^\times$ is cyclic of order $p-1$. Then

$$H^s(G_1, \mathbb{Z}/p^\infty(t)) \cong H^s(\mathbb{Z}_p, \mathbb{Z}/p^\infty(t))^{C_{p-1}} \tag{9.2}$$

is zero unless $t = k(p-1)$; in this case,

$$H^s(\mathbb{Z}_p, \mathbb{Z}/p^\infty(t))^{C_{p-1}} \cong H^s(\mathbb{Z}_p, \mathbb{Z}/p^\infty(t)).$$

Choose a generator $\gamma \in \mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$; since p is odd, this is a p-adic unit x congruent to 1 modulo p, and so that $x - 1$ is non-zero modulo p^2. Then the cohomology can be calculated using the very short cochain complex

$$\gamma^t - 1 : \mathbb{Z}/p^\infty(t) \longrightarrow \mathbb{Z}/p^\infty(t).$$

□

9.3 Remark. The morphism

$$\mathbb{Q} \cong H^{0,0}(p^{-1}BP_*) \to H^{0,0}(v_1^{-1}BP_*) \cong \mathbb{Z}/p^\infty$$

is surjective as the target group is generated by the elements $1/p^t \in [v_1^{-1}BP_*]_0$. Thus

$$\pi_0 S^0 \cong \mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*) \cong \mathbb{Z}_{(p)},$$

as it better.

9.4 Remark. At the prime 2, we have an isomorphism

$$G_1 \cong \mathbb{Z}_2^\times \cong C_2 \times \mathbb{Z}_2$$

where $C_2 = \{\pm 1\}$ and $\mathbb{Z}_2 \cong \mathbb{Z}_2^\times$ are the 2-adic units with are congruent to 1 modulo 4. The decomposition of 9.2 now becomes a spectral sequence, which collapses but yields $H^{s,*}(v_1^{-1}BP_*) \neq 0$ for all s.

9.5 Remark (The image of J). Let $p > 2$. The elements in non-positive degree in $H^{0,*}(v_1^{-1}BP_*)$ cannot be permanent cycles in the chromatic spectral sequence as we know $H^{1,*}(BP_*) = 0$ in those degrees – see Lemma 3.5. The elements of positive degree are permanent cycles, indeed if $2t = 2k(p-1)$ and $k = p^r k_0$, then

$$v_1^k/p^{r+1} \in H^{0,2k(p-1)}(BP_*/(p^\infty))$$

maps to a generator of $H^{0,2k(p-1)}(v_1^{-1}BP_*/(p^\infty))$. Here we use that

$$\eta_R(v_1) = v_1 + px$$

to get v_1^k/p^{r+1} to be primitive. Thus we have an isomorphism

$$H^{1,2k(p-1)}(BP_*) \cong \mathbb{Z}/p^{r+1}\mathbb{Z}$$

generated by an element $\alpha_{k/r}$. These elements are all permanent cycles on the ANSS, as well. This can be seen either by using a variant of the Smith-Toda argument of Remark 6.4 with the mod p^{r+1} Moore spectra $S^0 \cup_{p^{r+1}} D^1$ or noting that these elements detect the image of the J homomorphism.

9.6 Remark (Hopf invariant one). Let $p > 2$. The evident map of ring spectra $BP \to H\mathbb{Z}/p$ induces a commutative diagram of spectral sequences

$$\begin{array}{ccc}
\mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, BP) & \Longrightarrow & \pi_{t-s}S^0_{(p)} \\
\downarrow & & \downarrow \\
\mathrm{Ext}^s_{A_*}(\Sigma^t \mathbb{F}_p, \mathbb{F}_p) & \Longrightarrow & \pi_{t-s}(S^0)^\wedge_p .
\end{array}$$

The map

$$\mathrm{Ext}^1_{BP_*BP}(\Sigma^t BP_*, BP) \longrightarrow \mathrm{Ext}^1_{A_*}(\Sigma^t \mathbb{F}_p, \mathbb{F}_p)$$

is almost entirely zero: $\alpha_{k/r}$ is in the kernel unless $k = r = 1$. As a result, the Hopf invariant 1 elements

$$h_i \in \mathrm{Ext}^1_{A_*}(\Sigma^{p^i}\mathbb{F}_p, \mathbb{F}_p)$$

must support differentials if $i > 0$ – if they were permanent cycles there would detect a class of filtration 0 or 1 in the ANSS. A similar argument works at the prime 2. Since this argument uses only information from chromatic level 1, it could be done just as well with K-theory; therefore, this is a reformulation of Atiyah's proof of the non-existence of elements of Hopf invariant 1.

While I've not yet talked about the 2-line, there is a very large kernel at that line as well. Indeed, using the generators of 9.8 below, only β_2, β_{p^i/p^i}, and β_{p^i/p^i-1} do not map to zero.

9.7 Remark. In Remark 9.5 we wrote down specific generators v_1^k/p^{r+1} (with $k = p^r k_0$) for $H^{0,*}(BP_*/(p^\infty))$. These have the property that as $r \to \infty$, they generate a torsion group of higher and higher order and approach the case of $k = 0$, which yields the infinite divisible group $\mathbb{Z}/p^\infty$. The requirement that $r \to \infty$ is the same as the requirement that $k \to 0$ in the p-adic topology on the integers. Thus we regard the case $k = 0$ as a limiting case, even though it never appears in the homotopy groups of spheres. This phenomenon will recur many times below.

9.8 Remark (Bocksteins). Here is the another approach to the calculation of Theorem 9.2. There is a short exact sequence of G_1-modules

$$0 \to \mathbb{Z}/p(t) \xrightarrow{\times p} \mathbb{Z}/p^\infty(t) \longrightarrow \mathbb{Z}/p^\infty(t) \to 0 \tag{9.3}$$

which yields a long exact sequence in cohomology. There is also split short exact sequence of groups (for $p > 2$)

$$0 \to \mathbb{Z}_p \to G_1 \to \mathbb{F}_p^\times \to 0$$

and the action of $G_1 = \mathbb{Z}_p^\times$ on $\mathbb{Z}/p(t) = \mathbb{F}_p(t)$ is through the powers of the evident action of $\mathbb{F}_p^\times = C_{p-1}$ on $\mathbb{F}_p$. Thus

$$H^*(G_1, \mathbb{F}_p(*)) \cong \mathbb{F}_p[v_1^{\pm 1}] \otimes H^*(\mathbb{Z}_p, \mathbb{F}_p)$$

where $\mathbb{Z}_p$ acts trivially on $\mathbb{F}_p$. Thus

$$H^*(G_1, \mathbb{F}_p(*)) \cong \mathbb{F}_p[v_1^{\pm 1}] \otimes E(\zeta)$$

where $\zeta \in H^1(G_1, \mathbb{F}_p(0))$. In [27] Theorem 6.3.21, $v_1\zeta$ is called h_{10}.

To get the calculation of $H^*(G_1, \mathbb{Z}/p^\infty(*))$ we need to calculate the higher Bocksteins inherent in the long exact sequence we have in cohomology. This computation requires essentially the same information at the proof of Theorem 9.2.

At the prime 2, one can run the same argument, although now the action of G_1 on $\mathbb{F}_2(t)$ is trivial for all t and one has

$$H^*(G_1, \mathbb{F}_2(*)) \cong \mathbb{F}_2[v_1^{\pm 1}] \otimes \mathbb{F}_2[\eta] \otimes E(\zeta)$$

where η and ζ are the evident generators of

$$H^1(\{\pm 1\}, \mathbb{F}_2) \subseteq H^1(G_1, \mathbb{F}_2(1))$$

and

$$H^1(\mathbb{Z}_2, \mathbb{F}_2) \subseteq H^1(G_1, \mathbb{F}_2(0))).$$

These elements are variations of the elements h_{10} and ρ_1 of [27] Theorem 6.3.21.

We now turn to the calculation of the 2-line of the ANSS, at least for primes $p > 3$. I'll get more sketchy than ever at this point, as the calculations are beginning to get quite involved.

Because of Lemma 9.1 and Theorem 9.2 and the chromatic spectral sequence we see that there is an isomorphism (in positive degrees)

$$H^{2,*}(BP_*) \cong \mathrm{Ker}\{H^{0,*}(v_2^{-1}BP_*/I_2^\infty) \to H^{0,*}(v_3^{-1}BP_*/I_3^\infty)\}.$$

Thus the first task is to compute

$$H^{0,*}(v_2^{-1}BP_*/I_2^\infty) \cong H^0(G_2, R_2/\mathfrak{m}^\infty(t)).$$

The module we wish to compute the cohomology of is very complicated (already here at the $n = 2$ case!). Some formulas can be found in [8]; however, they aren't an immediate help. Thus we use the methods of Bocksteins which I mentioned in Remark 9.8, except we are now two Bocksteins away from the answer we want. Rewrite

$$R_2/\mathfrak{m}^\infty(t) = u^t R_2/\mathfrak{m}^\infty = u^t W(\mathbb{F}_{p^2})[[u_1]]/(p^\infty, u_1^\infty).$$

Then there are short exact sequences of G_2-modules

$$0 \to u^t\mathbb{F}_{p^2}[[u_1]]/(u_1^\infty) \to u^t R_2/\mathfrak{m}^\infty \xrightarrow{p} u^t R_2/\mathfrak{m}^\infty \to 0$$

and

$$0 \to u^t\mathbb{F}_{p^2} \xrightarrow{v_1^{-1}} u^{t-(p-1)}\mathbb{F}_{p^2}[[u_1]]/(u_1^\infty) \xrightarrow{v_1} u^t\mathbb{F}_{p^2}[[u_1]]/(u_1^\infty) \to 0.$$

In the second of these, the element $v_1 = u^{p-1}u_1$ is invariant under the G_2 action, so that we actually get G_2-module homomorphisms. The action of G_2 on $u^t\mathbb{F}_{p^2} = \mathbb{F}_{p^2}(t)$ is through the quotient $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) \rtimes \mathbb{F}_{p^2}^\times$; the cohomology we need was calculated in Corollary 8.6. In particular we have identified two classes $h_0, h_1 \in H^1(G_2, \mathbb{F}_{p^2}(*))$.

With a little thought we see that h_i is represented by $t_1^{p^i} \in \Sigma(n)$. Because

$$t_1^{p^2} = v_2^{p-1}t_1 \tag{9.4}$$

(see Example 4.7) h_i, $i > 1$, can rewritten as a power of v_2 times h_0 or h_1 and, in particular, is non-zero. For example

$$h_2 = v_2^{p-1}h_0 \qquad \text{and} \qquad h_3 = v_2^{p^2-p}h_1. \tag{9.5}$$

These facts, and the formula

$$\eta_R(v_2) = v_2 + v_1t_1^p - v_1^pt_1 \qquad \text{modulo} \qquad (p) \tag{9.6}$$

allows us to complete the Bockstein calculation to decide how divisible the class v_2^k becomes in the cohomology of $\mathbb{F}_{p^n}[[u_1]]/(u_1^\infty)$. One way to phrase the following result is that $1 = v_2^0$ is infinitely v_1 divisible and if $k = p^ik_0$ with $(p, k_0) = 1$, then v_2^k is $v_1^{a_i}$ divisible where $a_i \to \infty$ as $i \to \infty$. This is analogous to the p-divisibility in Theorem 9.2: the case $k = 0$ is the limiting case achieved as $k \to 0$ in the p-adic topology. See Remark 9.7. Note that since $p = 0$ in $\mathbb{F}_{p^n}[[u_1]]/(u_1^\infty)$, the cohomology groups of this module are modules over $\mathbb{F}_p[v_1]$.

In the following result, the element x_i^s/v_1^j is the corrected version of the element $v_2^{p^is}/v_1^j$ needed to make the Bocksteins work out. See Remark 9.10.

9.9 Theorem. *In the* $\mathbb{F}_2[v_1]$*-module*

$$H^0(G_2, \mathbb{F}_{p^n}[[u_1]]/(u_1^\infty)(*))$$

is spanned by the following sets of non-zero elements, with the evident v_1 *multiplication:*

1. $1/v_1^j$, $j \geq 1$;

2. *if* $s = p^i s_0$ *with* $(p, s_0) = 1$, *there is an element* $x_i \in E(2)_*$ *so that* $x_i \equiv v_2^{p^i}$ *modulo* (p, v_1) *and we have the classes*

$$x_i^{s_0}/v_1^j$$

where

$$1 \leq j \leq a_i = \begin{cases} 1, & i = 0; \\ p^i + p^{i-1} - 1, & i \geq 1. \end{cases}$$

9.10 Remark. This is Theorem 5.3 of [22]. I won't give a proof, but it's worth pointing out how the number a_i arises. Write d^n for the nth v_1-Bockstein. Then the formula 9.6 immediately implies that

$$d^1(v_2^s) = s v_2^{s-1} h_1.$$

Thus, if s is divisible by p, $d_1(v_2^s) = 0$ and we compute also using 9.6 and 9.5 that the next Bockstein is

$$d^p(v_2^s) = (p/s) v_2^{p(s-1)} h_2 = (p/s) v_2^{s-1} h_0.$$

Thus, if p^2 divides s we continue and we have, now again using 9.5

$$\begin{aligned} d^{p^2}(v_2^s) &= (s/p^2) v_2^{s-p^2} h_3 \\ &= (s/p^2) v_2^{s-p} h_1 = d^1((s/p^2) v_2^{s-p+1}). \end{aligned}$$

Thus $d^{p^2}(v_2^s) = 0$ in the Bockstein spectral sequence and, making the correction dictated by this equation we calculate that $d^{p^2+p-1}(v_2^s) \neq 0$.

The next step is to calculate the groups

$$H^0(G_2, R_2/\mathfrak{m}^\infty(*))$$

using the the Bockstein based on multiplication by p. Put another way, we must decide how p-divisible the elements of Theorem 9.9 become. Since this requires knowledge of $H^1(G_2, \mathbb{F}_{p^2}[[u_1]]/(u_1^\infty)(*))$, I won't even pretend to outline the calculation. However, here are two comments.

1. The first differential in the chromatic spectral sequence

$$H^{0,*}(v_1^{-1}BP_*/(p^\infty)) \to H^{0,*}(v_2^{-1}BP_*/I_2^\infty)$$

must be an injection in negative degrees, this forces a maximum p divisibility on the elements $1/v_1^j$ and we get a "reverse" image of J pattern working downward from $1/v_1$.

2. The case when $s = p^i s_0$ – that is, of the elements $x_i^{s_0}/v_1^j$ – must approach the case of $1/v_1^j$ as $i \to \infty$. Thus we would expect the beginning of a reverse image of J pattern working downward from $x_i^{s_0}/v_1$. This pattern is not complete, however, by which I mean we get the image of J divisibility if j is small, but as j becomes large, the divisibility becomes constrained.

One reason for the constraints in (2) was suggested to me by Mark Mahowald. In the clASS, multiplication by v_1 and multiplication by p raise filtration. Thus, after dividing by v_1 a certain of times, we have arrived at a low filtration in the clASS. Since we expect image of J type divisibility to occur in relatively high filtration in the clASS, we don't expect unconstrained divisibility by p at this point. The exact result is Theorem 6.1 of [22]:

9.11 Theorem. *Let $p \geq 3$. The groups*

$$H^0(G_2, R_2/\mathfrak{m}^\infty(*))$$

are spanned by the non-zero classes, with evident multiplication by p,

1. *$1/p^{k+1}v_1^j$ where $j > 0$ and p^k divides j; and*
2. *if $s = p^i s_0$ with $(p, s_0) = 1$, we have the elements*

$$x_i^{s_0}/p^{k+1}v_1^j$$

with $1 \leq j \leq a_i$ and with k subject to the twin requirements that

$$p^k | j \qquad \text{and} \qquad a_{i-k-1} < j \leq a_{i-k}.$$

The first of these contraints is the image of J divisibility; when j is small, the second contraint won't apply. On the other hand, if j is large, the second constraint dominates. For example, if

$$p^{n-1} + p^{n-2} - 1 = a_{n-1} < j \leq a_n = p^n + p^{n-1} - 1$$

then we only have elements of order p.

With the result in hand, the question then remains how many of the elements are non-zero permanent cycles in the chromatic spectral sequence. None of the elements in negative degree can be; thus the limiting case of the reverse image of J pattern cannot survive, even though it has avatars throughout the rest of the groups. There is another class of elements that cannot survive as well; here is heuristic argument for this case.

Consider the family of elements

$$x_i/pv_1^j, \qquad i \geq 2.$$

These are roughly $v_2^{p^i}/pv_1^j$. The element $x_i/pv_1^{p^i}$ does survive to an element

$$\beta_{p^i/p^i} \in \mathrm{Ext}^2_{BP_*BP}(\Sigma^{2p^i(p-1)}BP_*, BP_*)$$

which reduces to a non-zero class

$$b_i \in \mathrm{Ext}^2_{A_*}(\Sigma^{2p^i(p-1)}\mathbb{F}_p, \mathbb{F}_p) \tag{9.7}$$

in the clASS – the odd primary analog of the Kervaire invariant class.[5] On the general principal, alluded to above, that dividing by v_1 cannot raise the cohomological degree s, we see immediately that we've run out of room – there's no place to put x_i/pv_1^j, $j > p^i$. The following result follows from Lemma 7.2 of [22].

9.12 Theorem. *Suppose $s = p^i s_0 > 0$ with $(p, s_0) = 1$. Then all of the classes $x_i^{s_0}/p^{k+1}v_1^j$ survive to E_∞ in the chromatic spectral sequence* **except**

$$x_i/pv_1^j, \qquad 2 \leq i, \ p^i < j \leq a_i.$$

If $s < 0$, the elements $x_i^{s_0}/p^{k+1}v_1^j$ do not survive.

We write

$$\beta_{s/k,j} \in \mathrm{Ext}^2_{BP_*BP}(\Sigma^* BP_*, BP_*) \tag{9.8}$$

for the unique class detected by $x_i^{s_0}/p^{k+1}v_1^j$. If $k = 0$ we write $\beta_{s/j}$; if $j = 1$ as well, then we recover β_s of Theorem 6.3.

9.13 Remark. Now, of course, one can ask which of these extended β-elements are infinite cycles in the ANSS. A survey of known results is in [27] §5.5. For example, it is a result of Oka's that enough of the generalized βs do detect homotopy class to show that for all n there is a k so that $\pi_k S^0$ contains a subgroup isomorphic to $(\mathbb{Z}/p)^n$.

There are also some spectacular non-survival results. For example, Toda's differential reads for primes $p \geq 3$

$$d_{2p-1}\beta_{p/p} = \alpha_1 \beta_1^{p-1}.$$

up to non-zero unit. Ravenel used an induction argument to show

$$d_{2p-1}\beta_{p^{i+1}/p^{i+1}} = \alpha_1 \beta_{p^i/p^i}^{p-1} \qquad \text{modulo} \qquad \mathrm{Ker}\ \beta_1^{c_i}$$

up to non-zero unit. Here $c_i = p(p^i - 1)/(p-1)$. This result has as a corollary that if $p > 3$, the elements b_i of 9.7 cannot be permanent cycles in the clASS.

[5] We would expect such elements to be primordial – and certainly not v_1-divisible.

At the prime 3 the divided βs can gang up to produce infinite cycles that map to b_i – some some b_i survive and some do not. Assuming the Kervaire invariant elements at the prime 2 all do exist, this is an example of the Mahowald axiom that "something has to be sick before it dies".

That the differential is as stated is fairly formal consequence of Toda's differential; the subtle part is to show that the target is non-zero modulo the listed indeterminacy. To do this, Ravenel uses the chromatic spectral sequence and detects the necessary elements in

$$H^*(G_n, \mathbb{F}_{p^n}(*)).$$

with $n = p-1$. In this case, the Morava stabilizer group $S_{p-1} \subseteq G_{p-1} = G_n$ has an element of order p; choosing such an element gives a map

$$H^*(G_n, \mathbb{F}_{p^n}(*)) \to H^*(\mathbb{Z}/p, \mathbb{F}_{p^n}(*))$$

and the necessary element is detected in the (highly-non-trivial) target. A variation on this technique was used in [25] to get non-existence results for the Smith-Toda complexes $V(p-1)$.

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