RESEARCH STATEMENT

PIOTR PSTRĄGOWSKI

1. Introduction

I am motivated by the various interactions between homotopy theory and algebraic geometry. In particular, I focus on

1. the relation between stable homotopy and the moduli of formal groups,
2. homotopy theory of schemes and
3. applications of derived methods to classical algebraic geometry.

These three subjects, namely chromatic homotopy theory, motivic homotopy and derived algebraic geometry, are entangled in many surprising ways, which I explore in my work.

One common thread through much of my research is given by the theory of synthetic spectra discussed in §3.2, which deforms ordinary stable homotopy into a derived category of quasi-coherent sheaves. This construction found several applications, such as in my solution to the long-standing problem of algebraicity of chromatic homotopy theory, see §3.1, or the breakthrough work of Burklund, Hahn and Senger on classification of highly-connected manifolds [BHS19].

From a very different perspective, I have shown that a variant of synthetic spectra coincides with the complex cellular motivic category after completion, providing a deep link with algebraic geometry, see §3.3. Nevertheless, many important questions are still open, and some of those I would like to focus on in particular are outlined in §4.1, §4.2 and §4.5.

As an interaction moving in the other direction, together with Dhyan Aranha we have recently used higher algebraic stacks to generalize the construction of the intrinsic normal cone of Behrend-Fantechi, deducing the existence of virtual fundamental classes for a large class of Artin stacks, as described in §3.4. This opens up new possibilities, such as the generalization of Gromov-Witten invariants to the setting of twisted stable maps, see §4.3, or of new wall crossing formulas for Mochizuki’s Donaldson-type invariants of algebraic surfaces, see §4.4.

2. Background

Chromatic homotopy theory

The chromatic viewpoint arose in the 1960s in the work of Quillen [Ada95]. It is the most successful approach to the structure of stable homotopy, both from a conceptual and a calculational standpoint, connecting topology of finite complexes with deep patterns in number theory.

The starting point is the observation that any cohomology theory for which we have Chern classes gives rise to a formal group defined over its ring of coefficients. A formal group is an arithmetic object which can be thought of as a refinement of the concept of a Lie algebra, one which behaves well also in positive characteristic. Surprisingly, the formal group associated to the complex bordism cohomology theory $MU$ turns out to be the universal one, setting up a dictionary between topology and the theory of formal groups.

This relationship was outlined in Morava’s influential Annals paper [Mor85], and breakthrough calculations followed in the work of Miller, Ravenel and Wilson [MRW+77]. Ten years later, Devinatz, Hopkins and Smith proved the Ravenel conjectures, showing that nilpotence phenomena in homotopy theory are completely controlled by the chromatic picture [DHS88], [HS98].
One way to approach these results is to observe that for any space \( X \), its complex bordism homology \( MU_*X \) has a canonical descent datum to a quasi-coherent sheaf over the moduli stack \( \mathcal{M}_{fg} \) of formal groups. The cohomology of that sheaf forms the second page of the Adams-Novikov spectral sequence computing the stable homotopy groups of \( X \), and the geometry of the stack is then reflected at the level of homotopy groups. In particular, after localizing at a prime, the \textit{height filtration} gives rise to a filtration on the category of spectra.

**Motivic homotopy theory**

Informally, motivic homotopy theory is the homotopy theory of smooth schemes, where the role of the unit interval is played by the affine line \( \mathbb{A}^1 \). Introduced Morel and Voevodsky, it has been referred to as a "vogue blend of algebra and topology" [VR07].

The definition of an appropriate category of \textit{motivic spaces} is technically involved, but the theory has spectacular applications, like the construction of the derived category of mixed motives and the solution to Bloch-Kato conjectures, due to Voevodsky [SV00] and building on many ideas of Rost.

The motivic category over a scheme \( S \) has a stable variant usually denoted by \( \text{SH}(S) \), which behaves like a category of "noncommutative motives" [Rob12]. This homotopy theory has many interesting aspects, but what I would like to focus on are the deeper connections with the chromatic picture, a story that is perhaps not yet fully understood.

To begin with, the complex bordism spectrum has a motivic analogue \( MGL \), which represents \textit{algebraic cobordism}. Away from the characteristic of the field, \( MGL \) behaves very much like its topological counterpart by the Hopkins-Morel-Hoyois theorem [Hoy15], and it was shown by Levine that as a consequence the slices of the motivic sphere, defined purely in terms of geometry of smooth schemes, can be described using the classical Adams-Novikov spectral sequence from chromatic homotopy theory [Lev14].

After \( p \)-completion at a prime, the connection becomes even stronger, and as a consequence of the work of Gheorghe, Isaksen, Wang, and Xu the cellular part of the complex motivic category is a deformation of the usual stable homotopy theory whose special fibre is the derived category of quasi-coherent sheaves over \( \mathcal{M}_{fg} \) [GWX]. This observation has led to spectacular advances in the knowledge of stable homotopy groups of spheres, advancing our knowledge from the 61st to roughly the first hundred stems [IWX].

**Virtual fundamental classes**

Enumerative geometry is a classical subject concerned with counting geometric objects subject to given constraints. Starting with Kontsevich's striking solution to the problem of counting planar curves of given degree passing through a general collection of points [Kon95a], this is often accomplished by integrating over the moduli spaces of objects or maps of the needed type.

One important incarnation is the theory of \textit{Gromov-Witten invariants}, which counts the number of stable curves intersecting a given collection of subvarieties, and which also has important applications to symplectic topology and mirror symmetry [TW00], [Kon95b].

One recurrent feature of the theory is that the moduli spaces in enumerative geometry often fail to be smooth, or are smooth of the "wrong" dimension, and so integration against the usual fundamental class fails to yield invariants with the needed properties. In the Deligne-Mumford case, a general procedure due to Behrend and Fantechi produces a better-behaved replacement, the \textit{virtual fundamental class} [BF97].

These virtual fundamental classes are compatible with each other, which yields formulas between the different integrals which allows one to evaluate them by reducing to simpler cases. This often allows one to answer enumerative questions by giving explicit, recursive formulas.
3. Past work

3.1. Algebraicity of chromatic homotopy theory

In the chromatic picture, the moduli stack of formal groups of height at most $n$ corresponds to the category of $E$-local spectra, where the latter is the Morava $E$-theory at height $n$. By the chromatic convergence theorem of Hopkins and Ravenel, if we take all of the heights together, this accurately reflects the homotopy theory of finite complexes; in particular, it is possible to recover the homotopy of the sphere from knowledge of the $E$-local ones.

On the other hand, the same result also implies that as the height grows, $\pi_*E^n S^0$ needs to get as complicated as $\pi_* S^0$, where by "complicated" I mean fundamentally non-algebraic, by the Mahowald uncertainty principle. This is in stark contrast with what happens when the height is small compared to the prime, where the Adams-Novikov spectral sequence for the sphere collapses for degree reasons, giving a purely algebraic description of $\pi_*E^n S^0$. In my work, I give a precise sense in which the whole $E$-local category is algebraic.

**Theorem 3.1** (P., [Pst18a]). Let $p > n^2 + n + 1$. Then, there exists an equivalence of homotopy categories $h\text{Sp}_E \simeq h\text{D}(E,E)$ between $E$-local spectra and differential $E_*E := \pi_* (E \wedge E)$-comodules.

At $n = 1$, **Theorem 3.1** is a classical result of Bousfield [Bou85]. In a range where it holds it yields, in particular, a complete classification of homotopy types of $E$-local spectra and of homotopy classes of maps between them. The result I prove in my paper is in fact more quantitative, showing that if $p > n^2 + n + \frac{5}{2}$, then we have an equivalence of homotopy $k$-categories.

It is natural to ask whether **Theorem 3.1** has computational consequences for more familiar invariants; this is indeed the case. A particularly important invariant is given by the Picard group, the group of equivalence classes of invertible objects in a given symmetric monoidal category [HMS94]. In the $E$-local case, it is known that at large primes $\text{Pic}(\text{Sp}_E) \simeq \mathbb{Z}$ by the work of Hovey and Sadofsky [HS99], the isomorphism given by taking rational homology.

In the $K$-local setting, where $K := E/m$ is the residue field of $E$, a more subtle algebraic invariant is needed, and so one works with the completed $E$-homology $E^\wedge_0 X := \pi_* LK(E \wedge X)$. One can show that if $X$ is $K$-locally invertible, then $E^\wedge_0 X$ is an invertible $\text{Murava module}$; that is, a complete $E_*$-module equipped with a continuous action of the Morava stabilizer group $G_n$.

**Theorem 3.2** (P., [Pst18b]). If $2p - 2 > n^2 + n$, then $\text{Pic}(\text{Sp}_K) \to \text{Pic}(\text{Mod}_{E_*}^{\wedge})$ is an isomorphism.

The proof of this result rests on chromatic algebraicity of **Theorem 3.1**. In fact, one only needs a weaker statement that any $E_*$-comodule can be realized as a homology of a canonical $E$-local spectrum, and so the isomorphism between Picard groups holds in a slightly large range of primes.

One can also ask about algebraic $K$-theory, and it is proven in the recent work of Raptis that the latter is determined in a range by the homotopy $k$-category of a given stable $\infty$-category [Rap19]. Combined with the strong form of **Theorem 3.1** this immediately yields the following.

**Theorem 3.3** (P., Raptis). If $p > n^2 + n + \frac{5}{2}$, then in the range $0 \leq i < k$ there is an isomorphism $K_i(S^0_E) \simeq K_i(E,E)$ between the $K$-theory of the $E$-local sphere and the $K$-theory of $E_*E$.

A particularly interesting aspect to chromatic algebraicity is that both $E(n)$-local spectra and differential comodules can be naturally assembled into stable $\infty$-categories, and these are not equivalent at any prime [BSS17]. Thus, **Theorem 3.1** gives examples of what is classically known as an exotic equivalence; to my knowledge, these are first such where the homological dimension is unbounded. To give an idea how this is obtained, I will briefly sketch the proof.

We consider the Goerss-Hopkins tower $\text{Sp}_{E(n)} \to \ldots \to M_{1,1}^{\text{top}} \to M_{0,0}^{\text{top}}$ of $\infty$-categories associated to $E$-homology, as constructed in [GH]. This tower was classically used to obtain obstructions to constructing a spectrum with prescribed homology, and its properties imply that if $l$ is larger then the homological dimension of $E_*E$, then $h\text{Sp}_E \simeq hM_{1,1}^{\text{top}}$, as all possible obstructions vanish.

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1* Any algebraic approximation to the homotopy of the sphere is infinitely far away from the actual answer.*
One can also write down an algebraic Goerss-Hopkins tower $D(E,E) \to \ldots \to \mathcal{M}^{alg}_0$ with the same formal properties. The same argument then implies that $hD(E,E) \simeq h\mathcal{M}^{alg}_l$ for large $l$ and so to prove Theorem 3.1 it is enough to construct an equivalence $\mathcal{M}^{alg}_l \simeq \mathcal{M}^{alg}_l$. I do this by using sparsity arguments to write down a partial splitting of the $E$-homology functor and showing that it induces an equivalence between topological and algebraic towers in a certain range.

### 3.2. Synthetic spectra

As explained above, the key ingredient in the algebraicity result for the $E$-local categories is the Goerss-Hopkins tower, which was also famously used to prove the existence of an $E_{\infty}$-ring structure on $E$ [GH, [CH03]. The construction of the tower given in the original papers is rather involved, taking place in an unusual homotopy theory underlying an exotic model structure on simplicial spectra. My work on synthetic spectra was motivated by the following natural question.

**Question 3.4.** What is the homotopy theory underlying Goerss-Hopkins theory, and what are the properties that make it suitable?

In my research, I answer this question by showing that:

1. it is a homotopy theory of certain sheaves of spectra and that
2. it is suitable because it is in a precise sense a deformation of the homotopy theory of spectra whose special fibre is the derived category of comodules.

To make this precise, I will need to get a little technical, and so let $E$ be fixed a Adams-type homology theory, not necessarily Morava $E$-theory. We say a finite spectrum $P$ is projective if $E,P$ is projective as an $E_{\ast}$-module. The $\infty$-category $S^p_{alg}$ of finite projective spectra can be made into a Grothendieck site by declaring coverings to be $E_{\ast}$-surjections.

**Definition 3.5.** A synthetic spectrum based on $E$ is a sheaf $X : (S^p_{alg})^{op} \to Sp$ of spectra which is product-preserving in the sense that $X(P \oplus Q) \simeq X(P) \times X(Q)$.

If $X$ is a spectrum, then its synthetic analogue, denoted by $\nu X$, is the unique connective sheaf of spectra on $S^p_{alg}$ such that $\Omega^\nu(\nu X) \simeq y(X)$, where the latter is the Yoneda embedding. This construction can be used to define the family $S^{p,q} := \Sigma^{p-q} \nu S^q$ of bigraded spheres, the choice of which fixes a notion of homotopy and homology groups of synthetic spectra.

**Theorem 3.6** (P, [Pst18c]). The $\infty$-category $S_{yn}$ admits a t-structure in which a synthetic spectrum $X$ is connective if and only if $\nu_{E_{p,q}}X = 0$ if $p - q < 0$. Moreover, there is a canonical equivalence $S_{yn}^\heartsuit \simeq \text{Comod}_{E_{\ast}}$ between the heart of this t-structure and the abelian category of $E_{\ast}$-$E$-comodules.

The above result is a first indication that $S_{yn}$ has something to do with comodules, but that connection is in fact stronger. Namely, for any sheaf on $S_{alg}^p$, functoriality gives a natural map $X(\Sigma P) \to \Omega X(P)$ which measures the degree to which $X$ preserves suspensions. In the particular case of bigraded spheres, this turns out to yield a map $S^{0,-1} \to S^{0,0}$ which we denote by $\tau$.

**Theorem 3.7** (P, [Pst18c]). The cofibre $C_{\tau} := S^{0,0}/\tau$ admits a canonical structure of a commutative algebra in synthetic spectra. Moreover, there is a symmetric monoidal, fully faithful embedding $\text{Mod}_{C_{\tau}}(S_{yn}) \hookrightarrow \text{Stable}_{E_{\ast}}$ of the $\infty$-category of $C_{\tau}$-modules into Hovey’s stable $\infty$-category of comodules. If $E$ is Landweber exact, this is an equivalence.

Here, $\text{Stable}_{E_{\ast}}$ is a better-behaved thickening of the derived $\infty$-category of $E_{\ast}$-comodules, and it is of purely algebraic origin [Hov, [HIV15]. Thus, intuitively the above result shows that after we mod out by $\tau$, there is no "topology" left, only homological algebra of comodules.

To prove the above results, I describe $\text{Stable}_{E_{\ast}}$ as an $\infty$-category of sheaves of spectra on the Grothendieck site of dualizable comodules. The latter admits a map from $S_{alg}^p$ by taking $E$-homology, and I show that the resulting adjunction is monadic. An identification of that monad...
with \( \otimes \) is the most difficult part of the proof, as it requires one to build a kind of a partial Adams resolution in \( Sp^{fp} \).

Note that the bigraded nature of \( \text{Syn} \) and the existence of the natural transformation \( \tau \) suggest that we can think of it as a suitable formal deformation, **Theorem 3.7** can be then thought of as identifying the special fibre. The next result does the same for the general fibre.

**Theorem 3.8** (P., [Pst18c]). We say that a synthetic spectrum \( X \) is \( \tau \)-invertible if \( \tau : \Sigma^{0,-1} \to X \) is an equivalence. Then, there exists a symmetric monoidal equivalence \( \text{Syn}^{\tau^{-1}} \to Sp \) between the \( \infty \)-categories of \( \tau \)-invertible synthetic spectra and spectra.

One interesting consequence of these results is that in the synthetic setting, the \( E \)-based Adams spectral sequence converging to \([X, L_E Y]\) can be reinterpreted as the spectral sequence coming from the *Postnikov tower* in synthetic spectra. This makes \( \text{Syn} \) a convenient place to argue how Toda brackes interact with the Adams filtration; this is a crucial ingredient in the breakthrough work of Burklund, Hahn and Senger on classification of highly-connected manifolds [BHS19].

Going back to the original motivation, using synthetic spectra and can also give a short and clean account of Goerss-Hopkins theory, as was done in my joint work with Paul VanKoughnett [PVI9].

### 3.3. Description of the complex motivic category

The results of [Pst22] imply that the morphism \( \tau \) controls the degree to which a given synthetic spectrum is algebraic or topological. The choice of the letter \( \tau \) is not accidental, but rather ties to the \( \text{C}_\tau \)-philosophy of Gheorghe, Isaksen, Wang and Xu, which I will now recall.

Working in the motivic category over \( \text{Spec} (C) \), one shows that after \( p \)-completion, there exists a map \( \tau : S^0_p \to S^0_{p,0} \) topologically realizing to the identity [Isa18]. By results of Gheorghe, Wang and Xu, the cofibre \( \text{C}_\tau \) admits a structure of a commutative algebra and \( \text{C}_\tau \)-modules can be identified with an appropriate derived category of even \( BP, BP \)-comodules [Ghe17], [GWX18].

This yields a span of Adams spectral sequences

\[
\begin{array}{c}
topological \text{ASS} \quad \text{Betti realization} \quad \text{motivic ASS} \quad S^{0,0}_p \to \text{C}_\tau \quad \text{motivic ASS} \quad \text{computing} \quad \text{computing} \\
\text{computing} \quad \pi_\ast S^0_p \quad \text{computing} \quad \pi_\ast S^0_{p,0} \quad \text{computing} \quad \pi_\ast C_\tau
\end{array}
\]

and the description of \( \text{C}_\tau \)-modules identifies the right one with a purely algebraic Cartan-Eilenberg spectral sequence. Extensive calculations have been made with these spectral sequences by Isaksen, Wang and Xu, allowing them to compute \( \pi_\ast S^0 \) up to stem 90 and above [Isa18], [IWX].

The similar behaviour of motivic and synthetic \( \tau \) led me to investigate whether the two are related; in fact, they turn out to be extremely close. To fix terminology, let me say that an \( MU \)-based synthetic spectrum is *even* if it belongs to the localizing subcategory generated by the synthetic analogues \( pF \) with \( p \) finite, projective with \( MU \)-concentrated in even degree.

**Theorem 3.9** (P., [Pst18c]). There exists an adjunction \( \text{SH}(C)^{\text{cell}} \rightleftarrows \text{Syn}^{even} \) between the cellular motivic category and the even synthetic category based on \( MU \) which restricts to an adjoint equivalence between the \( \infty \)-categories of \( p \)-complete objects for each prime \( p \).

In particular, **Theorem 3.9** gives a purely topological description of the \( p \)-complete cellular motivic category, but what is perhaps even more surprising is that it is a description in terms of \( MU \), which plays a crucial role in chromatic homotopy theory.

To prove it, I show that \( \text{SH}(C)^{\text{cell}} \) itself can be described as an \( \infty \)-category of product-preserving sheaves over certain kinds of finite motivic spectra with projective \( MGL_\ast \)-homology, the adjunction is then induced by the Betti realization. To verify that it is an equivalence after \( p \)-completion, I use Gheorghe-Isaksen results on the structure of motivic homotopy groups.

One could make a case that as a background for the \( \text{C}_\tau \)-philosophy of Isaksen, Gheorghe, Wang and Xu, the synthetic category is better behaved than the motivic one. For one thing, it displays the needed behaviour integrally, before \( p \)-completion, and has a canonical synthetic analogue construction \( \nu : Sp \to \text{Syn} \).
More importantly, the use of synthetic methods avoids resting topological calculations on difficult algebro-geometric results. As an example, the calculation of the homotopy of the motivic mod \( p \) cohomology spectrum alone is a form of Bloch-Kato conjectures, and the calculation of the motivic Steenrod algebra is a difficult theorem of Voevodsky [Voe03]. On the other hand, synthetically both results can be derived in just a couple of pages.

3.4. Virtual fundamental classes for Artin stacks

If \( X \) is a projective variety, then the key ingredient in the construction of the Gromov-Witten invariants is that of virtual fundamental classes for the moduli stacks \( \overline{M}_{g,n}(X) \) of stable maps from \( n \)-pointed, genus \( g \) curves into \( X \). Taken together, these virtual fundamental classes satisfy a number of compatibility properties [Beh97], these then translate to axioms of Gromov-Witten theory which can be used to evaluate the invariants by reducing to simpler cases.

In the relative Deligne-Mumford case, such as that of the map \( \overline{M}_{g,n}(X) \rightarrow \overline{M}_{g,n} \), these virtual fundamental classes have been defined by Behrend and Fantechi [BF97]. In joint work with Dhyan Aranha, we extend this construction to a large class of Artin stacks.

**Theorem 3.10** (Aranha, P., [AP19]). Let \( X \rightarrow Y \) be a morphism of finite type Artin stacks. Suppose that \( Y \) is of pure dimension \( r \) and that we have a global perfect obstruction theory \( E \rightarrow L_{X/Y} \). Then, there exists an associated virtual fundamental class \([X \rightarrow Y, E] \in A^r(X)\) in the Chow group of Kresch.

Examples of stacks to which the above result applies include the moduli of twisted stable maps whose target is an Artin stack discussed below in §4.3, the moduli of canonical surfaces in positive characteristic, and truncations of quasi-smooth morphisms of derived stacks. In the last case and rationally, the required classes were also constructed in the recent work of Khan using the formalism of six operations for the motivic category [Kha19], who shows that the result agrees with ours.

The key notion in the construction of Behrend and Fantechi is that of an intrinsic normal cone, which even in the Deligne-Mumford case is in general only an Artin stack. To define the intrinsic normal cone in the general case, one has to work with higher algebraic stacks, that is, certain étale sheaves of spaces on the site of schemes. In this context, Aranha and I in fact uniquely characterize it through a short list of natural axioms.

**Theorem 3.11** (Aranha, P., [AP19]). There exists a unique functor \( \mathcal{C} : \text{RelArt} \rightarrow \text{Art} \) on the \( \infty \)-category of relative higher Artin stacks, called the normal cone, satisfying the following properties:

1. If \( U \hookrightarrow V \) is a closed embedding of schemes, then \( \mathcal{C}_U V \) coincides with the classical normal cone, that is, \( \mathcal{C}_U V \simeq_U (\bigoplus I^k/I^{k+1}) \), where \( I \) is the ideal sheaf.
2. \( \mathcal{C} \) preserves coproducts.
3. \( \mathcal{C} \) preserves smooth and smoothly surjective maps.
4. \( \mathcal{C} \) commutes with pullbacks along smooth morphisms of relative Artin stacks.

Moreover, if \( X \rightarrow Y \) is relatively Deligne-Mumford, then \( \mathcal{C}_X Y \) coincides with the relative intrinsic normal cone of Behrend and Fantechi.

One advantage of using the axiomatic approach to defining the intrinsic normal cone is that many related stacks can be shown to exist in a similarly formal manner; in particular, we are able to construct the deformation to the normal cone.

4. Future work

4.1. Synthetic spectra and derived categories

As explained in §3.1, the result on algebraicity of chromatic homotopy theory crucially uses Goerss-Hopkins towers, which in joint work with Paul VanKoughnett we have shown are associated to suitable "deformations" of homotopy theories, such as the \( \infty \)-category of synthetic spectra of
§3.2. The latter informally plays a role of the "derived category of spectra", and it is natural to ask to what extent this heuristic can be made precise.

**Question 4.1.** Can one characterize the $\infty$-category $\text{Syn}$ of synthetic spectra based on $E$ through a universal property related to the homology functor $E_* : Sp_E \to \text{Comod}_{E,E}$?

Note that asking for a universal property is related to asking for a relation with derived categories, as the latter can be uniquely characterized as certain stable $\infty$-categories equipped with a $t$-structure with a prescribed heart [Lur]. The goal of this joint project with Tobias Barthel is to extend this characterization to cover cases like that of $\text{Syn}$, which is naturally associated to a certain homology functor rather than to an abelian category alone. One conjecture for which I have some evidence is that the connective part of $\text{Syn}$ is the universal locally graded prestable $\infty$-category equipped with an appropriate lift of $E$-homology.

The motivation for this project is that it would allow us

(1) to recognize an a priori unrelated $\infty$-category as one of "synthetic" origin (as in the case of the complex motivic category described in §3.3, see also §4.2 below) and

(2) to formally prove a general existence existence result for $\infty$-categories satisfying the given universal property, allowing one to generalize results obtained using $\text{Syn}$.

As one consequence of (2) in particular, I expect algebraic descriptions of homotopy categories of $R$-modules where $R$ is a ring spectrum with nice homotopy groups, analogous to chromatic algebraicity; this is joint work with Irakli Patchkoria extending his results at lower heights [Pat13].

Another direction I want to pursue is the open problem of constructing residue fields for tensor stable $\infty$-categories, which in the work of Balmer, Krause and Stevenson is shown to be always possible at the level of abelian categories [BKS19]. Thus, what is missing is a way to non-trivially lift a given abelian category to a stable one without losing the given homology theory, which is precisely what this project aims to address.

4.2. **Real cellular motivic spectra**

In my previous research, I have given a topological description of the cellular motivic category over $\text{Spec}(\mathbb{C})$ by identifying it with a suitable $\infty$-category of synthetic spectra, see §3.3. This makes it natural to ask about what happens over other fields, in particular the reals.

**Question 4.2.** Can we give a topological description of cellular motivic spectra over $\text{Spec}(\mathbb{R})$?

In the complex case, the equivalence is lifted from the Betti realization $SH(\mathbb{C})^{\text{cell}} \to Sp$, which over the reals is more naturally thought of as a functor $SH(\mathbb{R})^{\text{cell}} \to Sp^{C_2}$ valued in the $C_2$-equivariant stable homotopy theory, where $C_2 \simeq \text{Gal}(\mathbb{C}/\mathbb{R})$. This functor has been extensively studied [Bac18], [BS19], in particular in the work of Behrens-Shah it is shown to be a localization after $p$-completion by explicitly recovering each piece of the isotropy separation square.

A quick count shows that we shouldn’t expect $SH(\mathbb{R})^{\text{cell}}$ and appropriate $\infty$-category of synthetic $C_2$-spectra to be equivalent: the latter should be trigraded, one grading above that of $Sp^{C_2}$, while $SH(\mathbb{R})^{\text{cell}}$ is only bigraded. I expect that the right object to consider here is not the cellular part, but the larger "Galois-cellular" category where we also allow objects built from the image of the Galois correspondence functor $c_{\mathbb{C}/\mathbb{R}} : Sp^{C_2} \to SH(\mathbb{R})$ of Heller and Ormsby [HO16].

One natural question which lines up very well with the program outlined in §4.1 is whether this subcategory of $SH(\mathbb{R})$ admits, after $p$-completion, a universal lift of the real complex bordism functor $MU_* : Sp^{C_2} \to \text{Comod}_{MU_* \text{, } MU_*}$, expressing it in terms of real chromatic homotopy.

4.3. **Gromov-Witten invariants for twisted stable maps**

The construction of virtual fundamental classes which I described in §3.4 allows one to define an appropriate "integration" map for a large class of Artin stacks, in particular enabling a construction of higher analogues of Gromov-Witten invariants.
One important example is given by the moduli space \( K_{g,n}(X, \beta) \) of twisted stable maps into a tame Artin stack \( X \), as constructed by Abramovich-Ollson-Vistoli [AOV08]. In positive characteristic, this moduli space fails to be Deligne-Mumford already in the case of the classifying stack \( B\mu_p \), whose points are suitable \( p \)-fold covering maps of twisted curves, leaving the associated Gromov-Witten invariants inaccessible by classical methods.

The goal of this joint project with Dhyan Aranha and Adeel Khan is to construct a quasi-smooth derived enhancement to the moduli of twisted curves, following the work of Schürg, Toën and Vezzosi in the stable case [STV15], and verify that the resulting virtual fundamental classes satisfy the axioms of Gromov-Witten theory, as outlined by Kontsevich and Manin [KM94]. Once this is done, we will pursue a computation of the resulting Gromov-Witten invariants, and of the form and properties of their generating functions.

4.4. Wall crossing for Mochizuki’s invariants

In [Moc72], Mochizuki defines and investigates Donaldson-type invariants for algebraic surfaces obtained by integrating over the moduli space of semi-stable sheaves. The latter usually fails to be Deligne-Mumford, and so to define the needed integrals Mochizuki equips sheaves with auxiliary data such as an orientation or a map from a fixed line bundle with its own stability conditions.

The goal of this joint project with Dhyan Aranha and Yuuji Tanaka is to use the formalism of virtual fundamental classes for Artin stacks to define integration directly on the moduli of semi-stable sheaves, and to prove wall crossing formulas expressing the dependence of the resulting invariants on the polarization. As is already known in many cases [EG94], [EG95], [Moc72], one expects that the difference between invariants as the polarization varies should be expressible as an integral over products of moduli spaces of lower-dimensional objects.

4.5. The edge of algebraicity

The statement of Theorem 3.1 gives a precise sense in which \( S^p_E \) is algebraic at large primes, extending the observation that at large primes the \( E \)-local Adams-Novikov collapses.

On the other hand, sometimes even when the relevant spectral sequence does not collapse, the differentials turn out to be uniform, at least at all primes large enough. The prime example here is the Toda differential \( d_2 p^{-1}(\beta_p/p) = \alpha_1 \beta_1^p \), the first non-zero differential of the Adams-Novikov spectral sequence at all odd primes [Tod68].

**Question 4.3.** Is there a systematic way to talk about non-algebraic phenomena which happen uniformly throughout all primes large enough? In the range of heights in which the \( E \)-local ANSS has only one non-zero differential, are those differentials uniform?

One doesn’t expect an answer to Question 4.3 to be obtained by brute force computation, the \( E_2 \)-term alone seems all but hopeless to calculate completely for \( n > 2 \). Instead, recall that chromatic algebraicity is proven by studying the Goerss-Hopkins tower of \( S^p_E \), which exists at all height and primes. The main result then follows from observing that

1. the lower stages of the tower can be given a purely algebraic description and
2. the higher stages have homotopy categories equivalent to \( h(S^p_E) \) itself,

because when \( p \) is large enough, then the two above classes of layers have a non-zero intersection. An interesting question is thus the following.

**Question 4.4.** Can we pinpoint the place in the Goerss-Hopkins tower of \( S^p_E \) where algebraicity was broken? Can we understand the nature of that shift?

One possible approach to the above problem would go back to the observation that \( S^p_E \) itself cannot be equivalent to any derived \( \infty \)-category because it cannot be made \( H\mathbb{Z} \)-linear. I would like to investigate at precisely which stage along the tower is this linearity lost, as well as what is the exact nature of the obstruction.