A CONJECTURE OF GRAY
AND THE p-TH POWER MAP ON $\Omega^2 S^{2np+1}$

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Abstract. For $p \geq 2$, the $p$-th power map $[p]$ on $\Omega^2 S^{2np+1}$ is homotopic to a composite $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1} E^2 \xrightarrow{\pi_n} \Omega^2 S^{2np+1}$, where the fiber of $\phi_n$ is $BW_n$.

1. INTRODUCTION

Localizing spaces and maps at any prime $p \geq 2$, we prove a conjecture of Gray.

Theorem 1.1. For $p \geq 2$, the $p$-th power map $\Omega^2 S^{2np+1} \xrightarrow{[p]} \Omega^2 S^{2np+1}$ is homotopic to a composite $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1} E^2 \xrightarrow{\pi_n} \Omega^2 S^{2np+1}$, where the fiber of $\phi_n$ is $BW_n$.

Gray found a serious gap in Theriault’s proof of Gray’s conjecture for $p \geq 5$, which we fill in [11]. Our proof combines [12] with Gray’s spectacular construction of $BW_n$. Using ideas of Barratt, Boardman and Steer, we show (Theorem 2.6) that $p$ times the unstable $p$th James-Hopf invariant is a cup product. Our factorization Theorem 3.3 applies such cup product information to a map $p$ similar to a map of Gray’s, but defined in a more combinatorial way. This proves Theorem 1.1, the factorization $E^2 \phi_n = [p]$. A result of Gray’s (Theorem 4.5), which uses only the Serre spectral sequence, shows that the fiber of $\phi_n$ is $BW_n$, as $\phi_n$ kills the Hopf invariant $\Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}$ and is degree $p$ on the bottom cell.

Theorem 1.1 relates to [4], where Cohen, Moore and Neisendorfer showed that $\pi_*(S^{2n+1})$ has exponent $p^\alpha$ for $p$ odd, by showing that $[p]$ on $\Omega^2 S^{2n+1}$ factors, for some $\pi_n$, as the composite $\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} E^2 \xrightarrow{\pi_n} \Omega^2 S^{2n+1}$. Anick [1] solved a conjecture of [4], in 270 pages, constructing a fibration sequence $\Omega^2 S^{2n+1} \xrightarrow{\alpha_n} S^{2n-1} \to T_n \to \Omega^2 S^{2n+1}$, for $p \geq 5$. Gray and Theriault [9] gave a shorter construction of Anick’s fibrations, for $p \geq 3$, and showed that $\alpha_n$ was essentially $\pi_n$ of [4]. Gray noted that Theorem 1.1 gives evidence that $\phi_n = \pi_{np}$, as $E^2 \phi_n = [p] = E^2 \pi_{np}$, and that this would imply that $BW_n$ is the loop space $\Omega T_{np}$. See [15] for 2-primary Anick fibration analogues. Theriault [16] constructed the odd-primary Anick fibrations not using [4] (thus re-proving the [4] exponent theorem), but using Gray’s conjecture. As Theriault observes [17], with our proof of Theorem 1.1 [16] is now correct.
2. Combinatorial James-Hopf invariants and cup products

We follow [19] and work in the category TOP* of pointed compactly generated weak Hausdorff spaces. Whitehead largely follows Strom [13], who shows that TOP* satisfies all the axioms of a proper model category (cf. [5]), except the limit and colimit axioms, using the model structure of homotopy equivalences, Hurewicz fibrations, and NDR pairs. So cofibration (→) will mean an NDR pair, and equivalence (∼) will mean homotopy equivalence. Assume all spaces are well-pointed and have the homotopy type of a CW-complex.

We use the conventions of [12] and [2]. Suspension is given by smashing on the right with \( S^1 = I/\{0,1\} \), so \( \Sigma X = X \wedge S^1 \). There are adjoint functors given by the evaluation map \( \sigma: \Sigma \Omega B \to B \) and the suspension map \( E: B \to \Omega \Sigma B \). Given maps \( f: \Sigma A \to X \) and \( g: A \to \Omega X \), we call their adjoints \( f^\vee: A \to \Omega X \) and \( g^\wedge: \Sigma A \to X \). A composite \( A \xrightarrow{f} \Omega Y \xrightarrow{\Omega E} \Omega^2 \Sigma Y \) is adjoint to the suspension of \( f^\vee: \Sigma A \to Y \). Recall the shuffle

\[
\text{shuffle: } \Sigma^{n+m}(A \wedge B) = A \wedge B \wedge S^n \wedge S^m \xrightarrow{1_A \wedge T \wedge 1_B} A \wedge S^n \wedge B \wedge S^m = \Sigma^n A \wedge \Sigma^m B
\]

and the permutation group \( \Sigma_r \) action on \( X^{[r]} \). Given \( f: A \to X \) and \( g: A \to Y \), define the cup product \( f \cdot g: A \to X \wedge Y \) to be the composite \( A \xrightarrow{\Delta} A \wedge A \xrightarrow{f \cdot g} X \wedge Y \). The cup product is compatible with permutations: given \( \rho \in \Sigma_r \) and maps \( f_i: A \to X \), we have

\[
(2.1) \quad \rho(f_1 \cdot \ldots \cdot f_r) = f_{\rho^{-1}(1)} \cdot \ldots \cdot f_{\rho^{-1}(r)}: A \to X^{[r]}.\]

Let \( \bar{k} = \{1, \ldots, k\} \), and call \( \binom{\bar{k}}{r} \) the set of subsets \( S \subset \bar{k} \) of size \( r \). For each \( S \), let \( (s_1, \ldots, s_r) \) be its ordered sequence of elements. Order \( \binom{\bar{k}}{r} \) using the left-lexicographical (left-lex) order of the \( (s_1, \ldots, s_r) \). Given \( S \in \binom{\bar{k}}{r} \), define the map \( \pi_S: X^k \to X^{[r]} \) by \( \pi_S(x_1, \ldots, x_k) = x_{s_1} \wedge \ldots \wedge x_{s_r} \), so \( \pi_S = \pi_{s_1} \cdot \ldots \cdot \pi_{s_r} \) is an iterated cup product. Then

Lemma 2.1. Let \( X \) be a co-H space. Then for any \( x \in \bar{k} \) and any subset \( T \in \binom{\bar{k}}{r-1} \), the cup product \( \pi_x \cdot \pi_T: X^k \to X^{[r]} \) is nullhomotopic if \( x \in T \).

Proof. Write \( \pi_T = \pi_{t_1} \cdot \pi_{T'} \), where \( T' \) is the complement of the smallest element \( t_1 \in T \). By (2.1), it suffices to consider the case \( x = t_1 \). Then \( \pi_x \cdot \pi_T = (\pi_x \cdot \pi_{s_1}) \cdot \pi_{T'} \). But \( \pi_x \cdot \pi_{s_1} \), the composite \( X^k \xrightarrow{\pi_{s_1}} X \xrightarrow{\Delta} X \wedge X \), is nullhomotopic, since \( X \) is a co-H space. \( \square \)

The James construction \( J(X) \) (cf. [19] §VII(2), [2] §3) has subspaces \( J_k(X) \). The identification map \( \iota_k: X^k \to J_k(X) \) induces a cofibration \( \iota_k: X^k \to J(X)/J_{k-1}(X) \) and a homeomorphism \( \overline{i}_k: X^k \cong J_k(X) / J_{k-1}(X) \) defined by \( \overline{i}_k(x_1 \wedge \ldots \wedge x_k) = [x_1, \ldots, x_k] \), which gives a cofibration sequence \( J_{k-1}(X) \to J_k(X) \xrightarrow{\partial} X^k \). For \( X \) connected, there is an equivalence \( J(X) \cong \Omega \Sigma X \). We define the \( r \)th combinatorial James-Hopf invariant \( j_r: J(X) \to J(X^{[r]}) \) using the left-lex order, and call \( H_r \) the composite \( J(X) \xrightarrow{j_r} J(X^{[r]}) \xrightarrow{\Omega \Sigma} \Omega \Sigma X^{[r]} \). \( H_1: J(X) \xrightarrow{\Omega \Sigma} \Omega \Sigma X \) is the equivalence, as \( j_1: J(X) \to J(X) \) is the identity. \( H_k: J(X) \to \Omega \Sigma X^k \) factors through a map \( \overline{H}_k: J(X)/J_{k-1}(X) \to \Omega \Sigma X^k \), by construction, for \( X \) connected.
The composite $\Sigma X^k \xrightarrow{\Sigma \iota_k} \Sigma J(X) \xrightarrow{H_r^*} \Sigma X^r$ satisfies the crucial property ($\prec$ means the left-lex order)

\[(2.2)\quad H_r^* \Sigma \iota_k = \sum_{S \in \binom{r}{k}} \Sigma \pi_S \in [\Sigma X^k, \Sigma X^r].\]

Note that $H_1 \Sigma \iota_k = \sum_{x \in k} \pi_x \in [\Sigma X^k, \Sigma X]$. By the proof of \cite[Lem. 3.7]{2}, we have

**Lemma 2.2.** Given two maps $f, g: J(X) \to \Omega Y$, suppose for each $k$ that the composites $f \cdot \iota_k, g \cdot \iota_k: X^k \to \Omega Y$ are homotopic. Then $f$ and $g$ are homotopic.

Define $\Omega \Sigma X \wedge \Omega \Sigma Y \xrightarrow{\otimes} \Omega \Sigma (X \wedge Y)$ to be the adjoint of the composite

$$\Sigma \Omega \Sigma X \wedge \Omega \Sigma Y \xrightarrow{\sigma \otimes 1} \Sigma X \wedge \Omega \Sigma Y \xrightarrow{1 \otimes \sigma} \Sigma X \wedge Y.$$  

Given $f: \Sigma A \to \Sigma X$ and $g: \Sigma A \to \Sigma Y$, define $f^\sharp g: \Sigma A \to \Sigma X \wedge Y$ as the adjoint of the composite $A \xrightarrow{\Delta} A \wedge A \xrightarrow{f \wedge g} \Omega \Sigma X \wedge \Omega \Sigma Y \xrightarrow{\otimes} \Omega \Sigma X \wedge Y$. Then $f^\sharp g$ is the composite

\[(2.3)\quad f^\sharp g: \Sigma A \xrightarrow{\Delta} \Sigma A \wedge A \xrightarrow{f \wedge g} \Sigma X \wedge A \xrightarrow{1 \wedge g} \Sigma X \wedge Y.
\]

Note that $f^\sharp g$ is a desuspension of what Boardman and Steer \cite{2} call the cup product $f \cdot g$.

**Lemma 2.3.** Given maps $f_i: A \to X$ and $g_j: A \to Y$, let $f = \sum_i \Sigma f_i: \Sigma A \to \Sigma X$ and $g = \sum_j \Sigma g_j: \Sigma A \to \Sigma Y$. Then $f^\sharp g = \sum_{i,j} \Sigma f_i \cdot g_j \in [\Sigma A, \Sigma X \wedge Y]$.

**Proof.** Composition is left-distributive, and composition is right-distributive if the right map is a suspension. So $(1 \wedge g)(f \wedge 1) = \sum_i (\sum_j 1 \wedge g) \Sigma f_i \wedge 1 = \sum_{i,j} \Sigma f_i \wedge g_j$.

Let $\theta_i = (12 \ldots i)$ be the cyclic permutation of length $i$. Then

**Proposition 2.4.** For any co-H space $X$ and any $r$, the diagram homotopy commutes:

$$\begin{array}{ccc}
J(X) & \xrightarrow{H_r} & \Omega \Sigma X^r \\
\Delta & \downarrow & \Omega \Sigma X^r \\
J(X)^{[2]} \xrightarrow{\Sigma \iota_k \wedge H_r} & \Omega \Sigma X \wedge \Omega \Sigma X^{r-1} & \xrightarrow{\otimes} \Omega \Sigma X^r \\
& \Omega \Sigma X^r & \xrightarrow{\Omega E} \Omega^2 \Sigma^2 X^r
\end{array}$$

**Proof.** The lower and upper composites of the diagram composed with $\iota_k: X^k \to J(X)$ are adjoint to maps which we will call $L, U \in [\Sigma^2 X^k, \Sigma^2 X^r]$. Note this group is abelian. By Lemma 2.2 it suffices to show that $L = U$. $L$ is the suspension of the composite

$$\Sigma X^k \xrightarrow{\Sigma \iota_k} \Sigma J(X) \xrightarrow{H_r^* \Sigma \iota_k} \Sigma X^r.$$  

By naturality of the $\sharp$ product, $L$ is the suspension of $(H_r^* \Sigma \iota_k)^\sharp (H_r^* \Sigma \iota_k)^{-1}$. Then

\[(2.4)\quad L = \sum_{x \in k, T \in \binom{T}{r_x}} \Sigma^2 \pi_x \cdot \pi_T \in [\Sigma^2 X^k, \Sigma^2 X^r].
\]

By \cite[2.2]{2} and Lemma 2.3, $U$ is the suspension of the sum $\sum_{i=1}^r \theta_i H_r^* \Sigma \iota_k$. By \cite[2.2]{2}, this sum equals $\sum_{i=1}^r \sum_{S \in \binom{S}{r}} \Sigma \theta_i \pi_S$. But $\theta_i \pi_S = \pi_x \cdot \pi_T$, where $x$ is the $i$th element...
of $S$ and $T = S - \{x\}$, by (2.1). We have a bijection $\tilde{r} \times \binom{k}{i} \cong \{(x, T) \in \tilde{k} \times \binom{k}{r-1} : x \not\in T\}$ given by $(i, S) \mapsto (s_i, S - \{s_i\})$. Hence, after suspending, 

$$U = \sum_{x \in \tilde{k}, T \in \binom{k}{r-1}, x \not\in T} \Sigma^2 \pi_x \cdot \pi_T \in [\Sigma^2 X^k, \Sigma^2 X^{(r)}].$$

By Lemma 2.1 the condition $x \not\in T$ is unnecessary, as the terms with $x \in T$ are nullhomotopic. By comparing with (2.3), we have $L = U \in [\Sigma^2 X^k, \Sigma^2 X^{(r)}]$. \hfill \Box

We specialize soon to $X = S^{2n}$. Let $\Phi : \Omega P \wedge Q \to \Omega(P \wedge Q)$ be the adjoint of $\Sigma \Omega P \wedge Q \xrightarrow{\sigma \wedge 1} P \wedge Q$, so $\Phi(\alpha \wedge q)(t) = \alpha(t) \wedge q$, for spaces $P$ and $Q$. Then

**Lemma 2.5.** For $e$ even, the map $\Phi : \Omega U \wedge S^e \to \Omega(U \wedge S^e)$ is homotopic to the composite

$$(2.5) \quad \Omega U \wedge S^e \xrightarrow{\Sigma^{e-1} \sigma} U \wedge S^{e-1} \xrightarrow{E} \Omega(U \wedge S^e),$$

for any space $U$. Take also a space $V$ and a map $f : U \to \Omega \Sigma V$. Then the composite $z : \Sigma^e U = U \wedge S^e \xrightarrow{f \wedge E} \Omega \Sigma V \wedge \Omega S^{e+1} \xrightarrow{\sigma \wedge 1} \Omega(V \wedge S^{e+1}) = \Omega(\Sigma^{e+1} V)$ is homotopic to the composite $\Sigma^e U \xrightarrow{\Sigma^{e-1} f \wedge 1} \Sigma^e V \xrightarrow{E} \Omega(\Sigma^{e+1} V).

**Proof.** The shuffle $\tau : S^e \wedge S^1 \to S^1 \wedge S^e$ is homotopic to the identity on $S^{e+1}$, as it is defined by a permutation with even sign, since $e$ is even. The adjoint of $\Phi$ is the composite

$$\Omega U \wedge S^{e+1} = \Omega U \wedge S^e \wedge S^1 \xrightarrow{\Sigma^e \wedge 1} \Omega U \wedge S^1 \wedge S^e \xrightarrow{\sigma \wedge 1 \wedge 1} U \wedge S^e.$$ 

Thus $\Phi^\wedge : \Omega U \wedge S^{e+1} \to U \wedge S^e$ is homotopic to $\sigma \wedge 1_{S^e} = \Sigma^e \sigma$, since $\tau$ is homotopic to the identity. As the adjoint of (2.5) is $\Sigma(\Sigma^{e-1} \sigma) = \Sigma^e \sigma$ as well, this proves the first part.

The second part is similar. By the smash product version of the cup product formula (2.3), the adjoint of the map $z : \Sigma^e U \to \Omega(\Sigma^{e+1} V)$ is the composite

$$z^\wedge : U \wedge S^e \wedge S^1 \xrightarrow{\Sigma^e \wedge 1} U \wedge S^1 \wedge S^e \xrightarrow{f \wedge 1 \wedge 1} V \wedge S^1 \wedge S^e \wedge S^1,$$

since the adjoint of $E : S^e \to \Omega S^{e+1}$ is the identity map on $S^e \wedge S^1 = S^{e+1}$. Since $\tau$ is homotopic to the identity, $z^\wedge : \Sigma^{e+1} U \to \Sigma^{e+1} V$ is homotopic to $f \wedge 1_{S^e} = \Sigma^e f^\wedge$. But $\Sigma^e f^\wedge$ is also the adjoint of $\Sigma^e U \xrightarrow{\Sigma^{e-1} f} \Sigma^e V \xrightarrow{E} \Omega(\Sigma^{e+1} V)$. \hfill \Box

The composite $J(X) \xrightarrow{\Delta} J(X) \wedge J(X)/J_{r-2}(X) \xrightarrow{H_1 \wedge \Pi_{r-1}} \Omega \Sigma X \wedge \Omega \Sigma X^{(r-1)}$ is homotopic to the composite $(H_1 \wedge H_{r-1}) \circ \Delta$ of Proposition 2.4 and the suspension $\pi \circ \Delta$ $E$ is homotopic to the composite $X^k \xrightarrow{i} J(X)/J_{k-1}(X) \xrightarrow{\Pi \circ \Delta} \Omega \Sigma X^k$.

Now we specialize Proposition 2.4 to $X = S^{2n}$. Let $m = 2n(p-1)$. Then

**Theorem 2.6.** Localize at prime $p \geq 2$. The diagram is homotopy commutative:

$$\begin{array}{ccc}
J(S^{2n}) \xrightarrow{\Delta} & J(S^{2n}) \wedge J(S^{2n}/J_{p-2}(S^{2n})) \xrightarrow{H_1 \wedge H_{p-1}} & \Omega S^{2n+1} \wedge \Omega S^{m+1} \xrightarrow{\sigma} \Omega S^{2np+1} \\
& \downarrow & \downarrow[p] \\
J(S^{2n}) \wedge S^m & \xrightarrow{\Sigma^{m-1} H_1^\wedge} & S^{2np}
\end{array}$$
3. Cup products and a factorization result for a map \( \rho \)

For a co-\( H \)-space \( X \), we construct a map \( \rho: \Omega(J(X), J_{p-1}(X)) \to \Omega J(X)^+ \wedge X^{[p-1]} \) analogous to Gray’s clutching construction collapse map \([7]\) and prove a factorization result involving \( \rho \) (Theorem 3.3) that we combine in \([4]\) with Theorem \([2.6]\).

Given a cofibration \( K \to L \), call \( \Omega(L, K) = \{ \lambda \in PL : \lambda(1) \in K \} \) its homotopy fiber. The natural map \( v: \Omega(L, K) \to \Omega(L/K) \), adjoint to the natural map from the homotopy fiber to the cofiber, sends a path \( \lambda \) to the loop \([\lambda]\).

Let \( p_1, p_2: X \to X \) be the factors of the comultiplication map \( X \to X \vee X \) of the co-\( H \)-space \( X \). The self-maps \( J(p_1) \) and \( J(p_2) \) of \( J(X) \) are homotopic to the identity. Then

**Lemma 3.1.** Let \( k = a + b + 1 \) for \( a, b \geq 0 \). The map \( J(p_1) \times J(p_2) \) defines a factorization

\[
J_k(X) \to J_a(X) \times J_k(X) \cup J_k(X) \times J_b(X)
\]

up to homotopy of the diagonal map on \( J_k(X) \). Furthermore \( J(p_1) \times J(p_2) \) induces a map

\[
\rho: \Omega(J(X), J_k(X)) \to \Omega(J(X)/J_a(X))^+ \wedge J_k(X)/J_b(X),
\]

defined explicitly by the formula \( \rho(\lambda) = J(p_1)\lambda^+ \wedge J(p_2)\lambda(1) \), and also the map \( \hat{\Delta} \) below which makes the following diagram homotopy commutative:

\[
\begin{array}{ccc}
J(X) & \xrightarrow{\Delta} & J(X) \wedge J(X) \\
\downarrow & & \downarrow \\
J(X)/J_k(X) & \xrightarrow{\hat{\Delta}} & J(X)/J_a(X) \wedge J(X)/J_b(X)
\end{array}
\]

**Proof.** The map \( J(p_1) \times J(p_2) \) is homotopic to the diagonal map, as is its restriction \( J_k(X) \to J_k(X) \times J_k(X) \). Any \( \alpha \in J_k(X) \) is represented by some \((y_1, \ldots, y_k) \in X^k\). Let \( N_1 \) be the number of indices \( i \) for which \( p_1(y_i) = * \), and \( N_2 \) be the number of indices \( i \) for which \( p_2(y_i) = * \). For any index \( i \), either \( p_1(y_i) = * \) or \( p_2(y_i) = * \), so \( N_1 + N_2 \geq k \). Thus either \( N_1 > b \) or \( N_2 > a \), since \( k > a + b \). Use \( k = a + b + 1 \) to rewrite this as either \( N_1 \geq k - a \) or \( N_2 \geq k - b \). Thus either \( J(p_1)(\alpha) \in J_a(X) \) or \( J(p_2)(\alpha) \in J_b(X) \).

This proves the first claim, which then implies the second and third claims. \( \square \)

Let \( E = J(X) \), \( M = J_{p-1}(X) \), and \( Y = J_{p-2}(X) \). Lemma 3.1 in the case \( k = p - 1 \) and \( \alpha = 0 \) gives the maps \( \hat{\Delta}: E/M \to E \wedge E/Y \) and \( \rho: \Omega(E, M) \to \Omega E^+ \wedge X^{[p-1]} \), defined explicitly by \( \rho(\lambda) = J(p_1)\lambda^+ \wedge \partial J(p_2)\lambda(1) \). Then:
Proposition 3.2. The composite \( \Omega(E) \to \Omega(E, M) \xrightarrow{\rho} \Omega E^+ \land X^{[p-1]} \) is trivial, and the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
\Omega(E, M) & \xrightarrow{\nu} & \Omega(E/M) \\
\rho \downarrow & & \downarrow \Omega \Delta \\
\Omega E^+ \land X^{[p-1]} & \xrightarrow{\text{id} \land \tilde{r}_{p-1}} & \Omega E \land E/Y \xrightarrow{\Phi} \Omega(E \land E/Y)
\end{array}
\]

Proof. The first assertion is immediate from the definition of \( \rho \), as \( \lambda(1) = \ast \) if \( \lambda \in \Omega E \).

The composite \( M \to E \to E/Y \) factors as \( M \xrightarrow{\partial} X^{[p-1]} \xrightarrow{\tilde{r}_{p-1}} E/Y \). The straight line homotopy of \( [\lambda(s)] \in E/Y \) to \( [\lambda(1)] = \tilde{r}_{p-1} \partial \lambda(1) \in M \land E/Y \cong X^{[p-1]} \) gives a homotopy of the diagram, as \( (\Omega \Delta \cdot v)(\lambda)(s) = J(p_1)\lambda(s) \land [J(p_2)\lambda(s)] \), for any \( \lambda \in \Omega(E, M) \), and \( (\Phi \cdot (\text{id} \land \tilde{r}_{p-1}) \land \rho)(\lambda)(s) = J(p_1)\lambda(s) \land \tilde{r}_{p-1} \partial J(p_2)\lambda(1) \). \( \square \)

Let \( p: E \to B \) be a map with \( p(M) = \ast \). Let \( F \) be the homotopy fiber of \( p \), and suppose that the natural lift of \( M \to E \) is an equivalence \( \tilde{M} \xrightarrow{\sim} F \). Call \( \tilde{p}: E/M \to B \) the map induced by \( p \). Then the following composite is an equivalence:

\[
(3.1) \quad \Omega(E, M) \xrightarrow{\nu} \Omega(E/M) \xrightarrow{\Omega \tilde{p}} \Omega B.
\]

Call \( \rho': \Omega B \to \Omega E^+ \land X^{[k]} \) the composite of \( \rho \) with the inverse of equivalence (3.1). We prove an analogue of [12] Thm. 4.1.

Theorem 3.3. Let \( Z = \Omega Z_0 \) be a loop space, and assume that \( \Sigma E \to \Sigma E/M \) has a right homotopy inverse. Take maps \( f: B \to Z \) and \( \alpha: E \land E/Y \to Z \) making the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
\Delta \downarrow & & \downarrow f \\
E \land E/Y & \xrightarrow{\alpha} & Z
\end{array}
\]

homotopy commute. Then \( \Omega f: \Omega B \to \Omega Z \) is homotopic to the composite

\[
\Omega B \xrightarrow{\rho'} \Omega E^+ \land X^{[p-1]} \xrightarrow{\Phi} \Omega(E \land E^{[p-1]}) \xrightarrow{\Omega \text{id} \land \tilde{r}_{p-1}} \Omega(E \land E/Y) \xrightarrow{\Omega \alpha} \Omega Z.
\]

If \( X = S^{2n} \), let \( m = 2n(p-1) \). Then \( \Omega f: \Omega B \to \Omega Z \) is homotopic to the composite

\[
(3.3) \quad \Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\Sigma^{m-1} \alpha} \Sigma^{m-1} E \xrightarrow{\zeta^\vee} \Omega Z,
\]

where \( \zeta^\vee \) is the adjoint of the composite \( \Sigma^m E \xrightarrow{\text{id} \land \tilde{r}_{p-1}} E \land E/Y \xrightarrow{\alpha} Z \).

Proof. The triangle of the following diagram homotopy commutes by Lemma 3.1

\[
\begin{array}{ccc}
E & \xrightarrow{\partial} & E/M \xrightarrow{\tilde{\rho}} B \\
\Delta \downarrow & & \downarrow \tilde{\Delta} \\
E \land E/Y & \xrightarrow{\alpha} & Z
\end{array}
\]

By (3.2), the outer polygon homotopy commutes. We see that the square homotopy commutes by adjoining with \( Z = \Omega Z_0 \) and using the right homotopy inverse.
By looping the right square and using Proposition 3.2 we see that the diagram

\[
\begin{array}{ccc}
\Omega(E, M) & \xrightarrow{v} & \Omega(E/M) \\
\downarrow \rho & & \downarrow \rho \phi \\
\Omega E^+ \wedge X^{[p-1]} & \xrightarrow{\text{id} \wedge \tau p^{-1}} & \Omega E \wedge E/Y \\
\end{array}
\]

homotopy commutes. Now invert the top horizontal composite using the equivalence (3.1).

In the case \(X = S^{2n}\), we showed that \(\Omega f: \Omega B \to \Omega Z\) is homotopic to the composite

\[
\Omega B \xrightarrow{\rho'} \Omega E^+ \wedge S^m \xrightarrow{\Phi} \Omega(E \wedge S^m) \xrightarrow{\Omega(\text{id} \wedge \tau p^{-1})} \Omega(E \wedge E/Y) \xrightarrow{\Omega \phi} \Omega Z.
\]

Lemma 2.5 and the fact that \(\Omega(\zeta) \cdot E = \zeta^\vee: \Sigma^{m-1}E \to \Omega Z\) establishes (3.3). \(\square\)

An obvious result follows from the first part of Proposition 3.2 as the composite of equivalence (3.1) with \(\Omega E \xrightarrow{\Omega \phi} \Omega(E, M)\) is \(\Omega p: \Omega E \to \Omega B\).

**Lemma 3.4.** The composite \(\Omega E \xrightarrow{\Omega \phi} \Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+\) is nullhomotopic.

### 4. Proof of Gray’s conjecture

Specialize to \(X = S^{2n}\), so \(E = J(S^{2n})\). Let \(B = \Omega S^{2np+1}\) and let \(p: E \to B\) be the \(p^\text{th}\) James-Hopf invariant \(H_p\). The EHP sequences of James [10] for \(p = 2\) and Toda [18] for odd primes show that the hypothesis of (3.3) is satisfied, so we have the equivalence \(\Omega(E, M) \xrightarrow{\sim} \Omega B\) of (3.1), where \(M = J_{p-1}(S^{2n})\) and \(Y = J_{p-2}(S^{2n})\). Thus we have the map \(\rho': \Omega B \to \Omega E^+ \wedge X^{[k]} = \Sigma^m \Omega E^+\). Let \(Z = B\) and \(f = [p]: B \to B\), the \(p\)-th power map on \(B\). The loop of \([p]\) on \(B\) is \([p]\) on \(\Omega B\). Define \(\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}\) as the composite \(\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{s} S^{2np-1}\), where \(s\) is the composite

\[
(4.1) \quad s: \Sigma^m \Omega E^+ \to \Sigma^m \Omega E \xrightarrow{\text{id} \wedge \tau p^{-1}} \Sigma^m \Omega E \xrightarrow{\zeta^\vee} \Omega B,
\]

where as usual \(m = 2n(p - 1)\). Now we state our main result.

**Theorem 4.1.** \(E^2 \phi_n = [p] \in [\Omega^2 S^{2np+1}, \Omega^2 S^{2np+1}]\).

*Proof.* \(\Sigma E \to \Sigma E/M\) has a right homotopy inverse by the James splitting, so we can apply Theorem 3.3 and Theorem 2.6. Thus \([p]: \Omega B \to \Omega B\) is the homotopy class of the composite \(\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\zeta^\vee} \Omega B\), using the adjoint of the composite \(\zeta: \Sigma^m E \xrightarrow{\text{id} \wedge \tau p^{-1}} \Sigma^m E \xrightarrow{\alpha} B\). By Theorem 2.6 \(\alpha\) is the composite

\[
\alpha: J(S^{2n}) \wedge J(S^{2n})/J_{p-2}(S^{2n}) \xrightarrow{H_1 \wedge H_1} \Omega S^{2n+1} \wedge \Omega S^{m+1} \xrightarrow{\zeta^\vee} \Omega S^{2np+1}
\]

and \(\zeta\) is the composite \(\Sigma^m J(X) \xrightarrow{\zeta^\vee} \Sigma^m X \xrightarrow{E^2} \Omega S^{m+1} X\). So \(\zeta^\vee\) is the composite \(\Sigma^m J(X) \xrightarrow{\zeta^\vee} \Sigma^m X \xrightarrow{E^2} \Omega^2(\Sigma^m+1 X)\). Thus \([p]\) is the composite

\[
\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\zeta^\vee} \Sigma^m X \xrightarrow{E^2} \Omega^2(\Sigma^m+1 X)
\]
or \(\Omega B \xrightarrow{\phi_n} S^{2np-1} X \xrightarrow{E^2} \Omega^2 S^{2np+1}\). Thus \(E^2 \phi_n = [p] \in [\Omega^2 S^{2np+1}, \Omega^2 S^{2np+1}]\). \(\square\)
We show the homotopy fiber of $\phi_n$ is $BW_n$ using an argument essentially due to Gray and largely contained in [S], an earlier version of [7]. Lemma 3.3 immediately implies

**Lemma 4.2.** The composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$ is nullhomotopic.

The next result is due to James and Toda [10, 18].

**Lemma 4.3.** Let $h: \Omega J_{p-1}(S^{2n}) \to \Omega S^{2np-1}$ be a map which gives an isomorphism in $(2np - 2)$-dimensional integral cohomology. Then the homotopy fiber of $h$ is $S^{2n-1}$.

**Remark 4.4.** Gray’s proof of Lemma 4.3 for odd primes [9 Thm. 1(a)] using the $\mathbb{Z}$ cohomology Serre $ss$ also works for $p = 2$. Letting $F$ be the homotopy fiber of $h$, the cohomology of the total space $\Omega J_{p-1}(S^{2n})$ is the tensor product of the cohomologies of the base and fiber. The other 2-primary EHP sequence $S^{2n} \to \Omega S^{2n+1} \to \Omega S^{4n+1}$ has a harder proof [3 Thm. 3.1], using the binomial coefficient identity $\binom{2n}{2k} \equiv \binom{n}{k} \pmod{2}.

Note that $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$ is degree $p$ on the bottom cell by Theorem 4.3. We prove

**Theorem 4.5.** A map $\Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$ has homotopy fiber $BW_n$ if $f$ is degree $p$ on the bottom cell and the composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$ is nullhomotopic.

**Proof.** Looping the EHP fibration [10, 18] gives the homotopy fibration sequence

$$
\Omega J_{p-1}(S^{2n}) \xrightarrow{\Omega E} \Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1}.
$$

Take the fibration sequence $\Omega S^{2np-1} \xrightarrow{\partial} L \xrightarrow{\pi} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$. A nullhomotopy of the composite $\partial \circ \Omega H$ gives a lift $\nu: \Omega^2 S^{2n+1} \to L$ of $H$ and a map of homotopy fibers $h: \Omega J_{p-1}(S^{2n}) \to \Omega S^{2np-1}$, yielding the homotopy commutative diagram

$$
\begin{array}{ccc}
\Omega S^{2np-1} & \xrightarrow{\partial} & L \\
\uparrow h & & \downarrow \pi \\
\Omega J_{p-1}(S^{2n}) & \xrightarrow{\Omega E} & \Omega^2 S^{2n+1} \\
\end{array}
$$

$$
\begin{array}{ccc}
& & \downarrow \Omega H \\
& & \uparrow f \\
& & \Omega S^{2np-1} \\
\end{array}
$$

where $\mathcal{F}$ is the homotopy fiber of both $h$ and $\nu$. The $\mathbb{Z}$ cohomology Serre $ss$ of the path fibration proves $H^{2np-1}(\Omega^2 S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}/p$. The $\mathbb{Z}/p$ cohomology Serre $ss$ proves $H^*(\Omega^2 S^{2n+1}, \mathbb{Z}) = 0$ in dimensions $2np - 2$ and $2np - 1$, connected by a Bockstein by the integral result. Since $H^{2np-1}(\Omega J_{p-1}(S^{2n}, \mathbb{Z}/p) = 0$, the $\mathbb{Z}/p$ cohomology Serre exact sequence of [12] shows that $(\Omega H)^*$ is an isomorphism in dimension $2np - 1$ and $(\Omega E)^*$ is an isomorphism in dimension $2np - 2$. The $\mathbb{Z}/p$ cohomology of $L$ is also $\mathbb{Z}/p$ in dimensions $2np - 2$ and $2np - 1$, connected by a Bockstein. By naturality of the Bockstein, $\nu^*$ is an isomorphism in dimension $2np - 2$, so $h^*$ is an isomorphism in dimension $2np - 2$. Thus $h$ is degree $r$ on the
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Theorem 1.1 now follows from Theorem 4.1, Lemma 4.2 and Theorem 4.5.

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