FORMALIZING RIGOROUS HILBERT AXIOMATIC GEOMETRY PROOFS IN THE PROOF ASSISTANT HOL LIGHT

WILLIAM RICHTER

Abstract. This note is intended to be useful to good high school students wanting a rigorous treatment of Hilbert's axioms for plane geometry.

1. Introduction

Euclid's Elements [5] is very creative geometry, which e.g. proves the triangle inequality (Prop. I.20) without the Pythagorean theorem or the parallel axiom. However Euclid's work contains errors involving angle addition, fixed by Hilbert [13], who added betweenness axioms. It seems that Euclid's errors are made in every widely used high school Geometry text in the US today. Hartshorne [11] clearly explains how Hilbert's axioms rigorize Euclid's work. However Hartshorne's book contains gaps that should trouble a high school student. In §7 we fill in needed details to his proofs of Euclid's Propositions I.7, I.9, I.21, I.27, I.29 and I.32, and work his overly difficult parallelogram and circle convexity exercises [11, Ex. 10.10].

We hope to teach Hilbert axiomatic geometry in high school Geometry using proof assistants (e.g. HOL Light, HOL4, Coq, Isabelle and Mizar) to write formal proofs, computer programs that verify (or formalize) theorems and proofs. High school students might enjoy writing rigorous axiomatic geometry proofs which they can formalize, gaining programming experience and a concrete understanding of their proofs. We formalized §2–7 (and most of Euclid's book I up Prop. I.28) in 3300 lines of HOL Light code, found in `http://www.math.northwestern.edu/~richter/RichterHilbertAxiomGeometry.tar.gz`, or the HOL Light subversion in `http://www.cl.cam.ac.uk/~jrh13/hol-light`. Proof assistants gained prominence with Hales's project [9] to formalize his proof of the Kepler sphere-packing conjecture [8].

Many of our proofs seem minor improvements over those in the literature, and some may be new. Our proof of the Crossbar Theorem 3.10, a tool to show two lines intersect, is simpler than those of [6, 16, 17], as we do not use their plane separation axiom (Prop. 4.14). Without the parallel axiom, we prove (Prop. 7.13 and 7.14) that a quadrilateral with opposite sides (or angles) congruent is a parallelogram. We can not find our proof of the precursor Lemma 7.16 of the triangle sum theorem (the sum of the angles of a triangle is 180°) in the literature. Our texts [16, 6, 11, 17] concentrate on the much harder Saccheri-Legendre Theorem, which requires a new axiom (Archimedes's), that without the parallel axiom, the sum of angles in a triangle is less than or equal to 180°. Some of our work on quadrilaterals in §5 may be new, and we use it §7 to explain Hartshorne's work.

A good high school student could read Moise's or Venema's college geometry texts [16, 17], which have rigorous, elegant proofs. We cite these texts freely. Venema lists the minimal version of Hilbert's axioms [17, App. B1], but uses stronger axioms, which obviate much of §3–4. The graduate geometry texts [6, 11, 14] also use stronger axioms. We prove the minimal Hilbert's axioms suffice. This result was proved by Hilbert, Moore, Wylie, and Greenberg [13, 10, 18, 6], but our proofs seem elegant, and are collected in one source.

Venema discusses the high school Geometry text [2], but not its lack of rigor. In §10 we discuss its inadequate proof of the triangle sum theorem. Venema, following Moise, also uses Birkhoff's axioms [4] that the real line $\mathbb{R}$ measures segments and angles, but he does not explain Birkhoff's work. In §9 we discuss Birkhoff's work, explained by MacLane [15], which gives an alternative to Hilbert's work by using directed angles and taking the Crossbar Theorem 3.10 and its converse (Lemma 3.6) as an axiom.

Hilbert's axioms [13] are actually for 3-dimensional space. In §8 we discuss a minimal version of Hilbert's full set of axioms, and prove various 3-dimensional results of Euclid [5], profiting from Moise's [16] work, which he taught rigorously to high school students [1].

Greenberg's text [6] seems too difficult for high school students, but it is a fundamental text on Hilbert's work, and a big influence on our paper (and [11, 14]). We cite his results freely, and reprove some of his results. Greenberg's survey [7] gives reasons to use Hilbert's axioms instead of assuming Birkhoff's $\mathbb{R}$, and contains interesting mathematical logic, including this outstanding quote in §3.2 about undecidability, "Thus elementary Euclidean geometry is genuinely creative, not mechanical."

Thanks to Bjørn Jahren, who recommended [6, 11], found [18] and discovered a proof of Prop. 4.1 independently. Thanks to Benjamin Kordesh, my high school Geometry student "test pilot", Miguel Lerma, who found [10], Takuo Matsuoka, and Stephen Wilson, who recommended [17], for helpful conversations. Thanks to John Harrison, who wrote HOL Light, Freek Wiedijk, who wrote the HOL Light program miz3, a Mizar emulator for HOL Light in which my code was originally written, and everyone else who taught me HOL Light, especially Mark Adams, Rob Arthan, Vincent Aravantinos Marco Maggesi, Michael Norrish, Petros Papapanagiotou, and Vladimir Voevodsky.

2. Hilbert's simplest axioms and HOL

Following Venema, we shall give a set-theoretic version of Hilbert's axioms. We are given a set which is called a *plane*. Elements of this set are called *points*. We are given certain subsets of a plane called *lines*. There are *axioms* that a plane and its subset lines must satisfy, explained here and also §3 and §6. In §8 we will consider many different planes, but for now we discuss one plane, referred to as 'the plane'. Hilbert's simplest axioms I1, I2 and I3 are called *incidence* axioms. Incidence refers to the question of set membership, whether points belong to lines or the plane.

A line l is then the set of all point P in the plane so that $P \in l$. Two lines l and m are then equal if $P \in l$ iff $P \in m$, for all points P. As usual we write 'iff' for 'if and only if'. We have the usual intersection $l \cap m$ of two lines l and m, the set of points P in the plane such that $P \in l$ and $P \in m$. We will call points *collinear* if there is a line in the plane containing them. Hilbert's incidence axioms are

- **I1.** Given distinct points A and B in the plane, there is a unique line l in the plane such that $A \in l$ and $B \in l$.
- **I2.** Given a line l in the plane, there exists two distinct points A and B in the plane such that $A \in l$ and $B \in l$.
- **I3.** There exist three distinct non-collinear points in the plane.

We will call $\overline{AB}$ the unique line in the plane containing A and B. We will need

Lemma 2.1. *Given two distinct lines l and m, if $X \in l \cap m$, then $l \cap m = \{X\}$.*

Proof. Suppose that $A \neq X$ and $A \in l \cap m$. By axiom I1, $l = \overline{AX} = m$. This contradicts our assumption, so there can not exist any such A. ☐

We will use the next two simple result often, usually without mentioning them.

Lemma 2.2. *Take non-collinear points A, B and C. Then A, B and C are distinct.*

Proof. Suppose (for a contradiction) the set $\{A, B, C\}$ has size 1, so $A = B = C$. By axiom I3, there exists a point $Q \neq A$. By axiom I1, there exists a line l containing A and Q. Thus $A, B, C \in l$, so A, B and C are collinear. This is a contradiction, so $\{A, B, C\}$ has size 2 or 3. Assume (for a contradiction) $\{A, B, C\}$ has size 2. By axiom I1 there exists a line containing the two distinct points. Thus A, B and C are collinear. This is a contradiction, so $\{A, B, C\}$ has size 3, and we are done. □

Lemma 2.3. *Take non-collinear points A, O and B. Take a point $X \neq O$ with O, B and X collinear. Then A, O and X are non-collinear.*

Proof. A, O and B are distinct by Lemma 2.2. Let $b = \overline{OB}$. Then $A \notin b$, since A, O and B are non-collinear. By axiom I1, $X \in b$. Since $X \neq O$, we can write $b = \overline{OX}$. Since $A \notin b$, A, O and X are non-collinear, by axiom I1. □

It would help a student to read the logic chapters of Greenberg's book [6], and to know something about first order logic (FOL), discussed in [7]. FOL, roughly speaking, is what you get from variables, the logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\Rightarrow$ (implies), $\Leftrightarrow$ (if and only if), the quantifiers $\forall$ (for all) and $\exists$ (there exists), axioms (such as Hilbert's axioms used here), and obvious-looking rules of inference. Here are two simple FOL results. $x \Rightarrow y$ is equivalent to its contrapositive $\neg y \Rightarrow \neg x$, and $x \wedge y \Rightarrow z$ is equivalent to $x \wedge \neg z \Rightarrow \neg y$. These two FOL equivalences are often what is meant by a proof by contradiction. A feature of FOL is that proofs can be checked with a computer, and a simple computer program can print out the (usually infinite number of) logical consequences of a list of axioms. So using the FOL ZFC set theory axioms, a program can print out the infinite number of all theorems that mathematicians will ever prove, unless we adopt new axioms. However, ZFC/FOL has not proved to be very easy to actually use in practice.

Verifying a proof with a computer is called giving a formal proof. A proof assistant is a programming language in which one can write formal proofs. The foremost interest in proof assistants is that they can often be used to debug software. Partly this is because a computer program is much like a mathematical proof, but it also because hardware (e.g. microprocessor chips) is often built based on complicated mathematical theorems which humans find difficult to check. Today there are a number of proof assistants gaining widespread use based on HOL, called higher order logic, which might be roughly described as an extension of FOL in which set theory is built in. A student interested in a career in this new field of debugging software and hardware with proof assistants should perhaps learn both rigorous axiomatic geometry and to formalize their proofs in as we have done.

Euclid's Elements [5] is a great treasure of Western Civilization, but whenever angle addition arises, Euclid seems to be giving a "picture proof." However, with Hilbert's axioms (over 100 years old), one can rigorously prove all of Euclid's results, as Hartshorne's explains in his beautiful book [11]. With the advent of proof assistants, it is now reasonable for high school students to learn rigorous axiomatic geometry proofs.

Artificial Intelligence (AI) highlights an advantage of rigorous axiomatic geometry proofs. Euclid's picture proofs only lack rigor because they purport to be axiomatic proofs. Mathematicians indeed often give picture proofs. So one asks if humans possess special geometric visualization skills which are necessary to understand Euclid's results. The AI point is that a robot (i.e. a computer program) can prove everything about points, lines and planes in 3-dimensional space that humans can, in the sense that a robot supplied with Hilbert's axioms can prove all the logical consequences of them.

3. The betweenness relation $*$ and the crossbar theorem

We have an undefined *betweenness* relation $*$. For all points A, B and C we have the statement $A * B * C$, which is supposed to capture the idea that B lies between A and C on some line. The betweenness axioms are

B1. If $A * B * C$, then $C * B * A$, and A, B and C are distinct points on a line.

B2. Given two distinct points A and B, there exists a point C such that $A * B * C$.

B3. If A, B and C are distinct points on a line, then exactly one of the statements $A * B * C$, $A * C * B$ and $B * A * C$ is true.

B4. Let A, B and C be non-collinear points not on line l. If there exists $D \in l$ so that $A * D * C$, there exists an $X \in l$ such that $A * X * B$ or $B * X * C$.

Using $*$, we define the *open interval* (A, B) as the subset of the plane

$$(A, B) = \{C \mid A * C * B\} \subset \overline{AB}.$$

Then $A * C * B$ iff $C \in (A, B)$. Note that $(A, A) = \emptyset$. All we will ever use of B1 is

B1'. For any points A and C, there is a subset (A, C) of the plane. If $A \neq C$, then $(A, C) \subset \overline{AC} - \{A, C\}$ and $(A, C) = (C, A)$. Furthermore $(A, A) = \emptyset$.

We use axiom B1' quite often without mentioning it. We often refer to the equality of sets $(A, C) = (C, A)$ as symmetry.

Using open intervals, we have equivalent versions of axioms B2 and B3.

B2'. Given distinct points A and B, there exists a point C such that $B \in (A, C)$.

B3'. Given distinct collinear points A, B and C, exactly one of the statements $A \in (B, C)$, $B \in (A, C)$ and $C \in (A, B)$ is true.

Axiom B4 is Hilbert's axiom II.5. Hilbert draws a picture of a triangle with vertices A, B and C with a line l intersecting the open intervals (A, C) and (A, B).

Take a line l. We define the relation $\sim_l$ on the complement of l. Take two points $A, B \notin l$. We write $A \sim_l B$ if $(A, B) \cap l = \emptyset$. We write $A \nsim_l B$ if $(A, B) \cap l \neq \emptyset$.

We have two equivalent versions of B4.

B4'. Take a line l and three non-collinear points $A, B, C \notin l$. If $A \nsim_l C$, then $A \nsim_l B$ or $B \nsim_l C$.

B4". Take a line l and three non-collinear points A, B and C not on l. If $A \sim_l B$ and $B \sim_l C$, then $A \sim_l C$.

B4' is a restatement of B4 using open intervals. B4' is equivalent to B4" because B4" follows from B4' by a short proof by contradiction, and vice versa.

In Proposition 4.14 we prove a stronger version of axiom B4" that Greenberg [6, p. 64] essentially takes as an axiom. A simple result using the relation $\sim_l$ is

Lemma 3.1. *Take a point on a line $X \in m$ and a line l with $m \cap l = \{X\}$. Take points $A, B \in m - \{X\}$, and suppose that $X \notin (A, B)$. Then $A, B \notin l$ and $A \sim_l B$.*

Proof. $A \notin l$, since $A \neq X$ and $m \cap l = \{X\}$. Similarly $B \notin l$. Assume (for a contradiction) that $A \nsim_l B$. Then there exists a point $G \in (A, B) \cap l$. By axiom B1', $(A, B) \subset m$. So $G \in m \cap l = \{X\}$. Thus $G = X$. But this contradicts $X \notin (A, B)$. Hence $A \nsim_l B$ is false, so $A \sim_l B$. □

A relation is an *equivalence relation* if it is reflexive, symmetric and transitive.

Lemma 3.2. *Take a line l. Then the following statements are true.*

(1) *The relation $\sim_l$ is an equivalence relation on the complement of l.*
(2) *Take points A, B and C not on l. If $A \sim_l B$ and $B \sim_l C$, then $A \sim_l C$.*

Proof. $\sim_l$ is obviously reflexive and symmetric, as $(A, A) = \emptyset$ and $(A, B) = (B, A)$. Transitivity is the second assertion (2), which we now prove. So take points A, B and C not on l with $A \sim_l B$ and $B \sim_l C$. We must prove $A \sim_l C$.

By axiom B4", $A \sim_l C$ if A, B and C are not collinear. Since $\sim_l$ is reflexive, $A \sim_l C$ if A, B and C are not distinct. So we can assume that A, B and C are distinct collinear points contained in a line m. By axiom I1 we can write $m = \overline{AC}$.

If the lines l and m do not intersect, then clearly $A \sim_l C$. So assume that l and m intersect. The lines l and m are distinct, because our three points belong to m but not l. Therefore, $m \cap l = \{X\}$, for some point X, by Lemma 2.1.

By axiom I2, there exists a point $E \neq X$ on l. Thus $E \notin m$. There exists a point B' with $B \in (E, B')$, by axiom B2', since $B \neq E$, since $E \in l$ and $B \notin l$. Then $B' \neq B$, by axiom B1'. Let $n = \overline{EB'}$, so $B \in n$, by axiom B1'. By Lemma 2.1, $n \cap l = \{E\}$, because $B \notin l$. Therefore $B' \notin l$. By axiom B3', $E \notin (B, B')$. So $B \sim_l B'$, by Lemma 3.1. The lines m and n are distinct, because $E \in n$. Thus by Lemma 2.1, $m \cap n = \{B\}$. Thus $A \notin n$, and so A, B and B' are not collinear. Since $A \sim_l B$ and $B \sim_l B'$, axiom B4" implies that $A \sim_l B'$.

By an argument similar to the previous paragraph, C, B and B' are not collinear, and $C \sim_l B'$. Since $\sim_l$ is symmetric, $B' \sim_l C$.

But A, B' and C are also not collinear, since $m \cap n = \{B\}$. Thus by axiom B4", $A \sim_l C$. This proves (2). Since this shows $\sim_l$ is transitive, this proves (1). □

We use Moise's definition of a ray [16, p. 55], which is simpler than Greenberg's.

Definition 3.3. For distinct points O and P, the *ray* $\overrightarrow{OP}$ is the subset of the plane

$$\overrightarrow{OP} = \{X \in \overline{OP} \mid O \notin (X, P)\}.$$

Let O, A and B be non-collinear points, and let $a = \overline{OA}$ and $b = \overline{OB}$. Define the *angle* $\measuredangle AOB$ as the (unordered) set consisting of the two rays

$$\measuredangle AOB = \{\overrightarrow{OA}, \overrightarrow{OB}\}.$$

We define int $\measuredangle AOB$ as the subset of the plane

$$\operatorname{int} \measuredangle AOB = \{P \mid P \notin a, P \notin b, P \sim_a B \text{ and } P \sim_b A\}.$$

One can prove that int $\measuredangle AOB$ only depends on the angle (a subset of the plane) and not the points O, A and B, but we will not prove or use this result. Instead, we will think of int $\measuredangle AOB$ as a function of three non-collinear points O, A and B which ought to be written int-angle(A, O, B). There are two properties we use related to the idea that interior is a function of the angle.

Lemma 3.4. *If $P \in \operatorname{int} \measuredangle AOB$, then $P \in \operatorname{int} \measuredangle BOA$.*

This follows immediately from Definition 3.3. Note that it does not follow from $\measuredangle AOB = \measuredangle BOA$, as it would if int was actually a function defined on angles.

Lemma 3.5. *If $P \in \operatorname{int} \measuredangle AOB$ and $X \in \overrightarrow{OB} - \{O\}$, then $P \in \operatorname{int} \measuredangle AOX$.*

Proof. Since points O, A and B are non-collinear, they are distinct. Let $a = \overline{OA}$ and $b = \overline{OB}$. Then $P \notin a$, $P \notin b$, $P \sim_a B$ and $P \sim_b A$. By Lemma 2.1, $a \cap b = \{O\}$, since $B \notin a$, by non-collinearity. $O \notin (X, B)$ and $X \in b - \{O\}$, since $X \in \overrightarrow{OB} - \{O\}$. Thus $X \notin a$, so O, A and X are non-collinear. $B \sim_a X$ by Lemma 3.1, since $B \in b - \{O\}$. Thus $P \sim_a X$, by Lemma 3.2. Hence $P \in \operatorname{int} \measuredangle AOX$. □

We prove a result of Greenberg's [6, Prop. 3.7] that is in a sense a converse to the Crossbar Theorem 3.10 we prove below.

Lemma 3.6. *Take non-collinear points O, A and B in the plane and a point $G \in (A, B)$. Then $G \in \operatorname{int} \measuredangle AOB$.*

Proof. Let $a = \overline{OA}$, $b = \overline{OB}$, and $l = \overline{AB}$. Then $a \cap l = \{A\}$, by Lemma 2.1, since $B \notin a$. Hence $G \notin a$, as $G \in l$ and $G \neq A$, by axiom B1'. $A \notin (G, B)$, by axiom B3', since $G \in (A, B)$. Thus $G \sim_a B$, by Lemma 3.1, since $a \cap l = \{A\}$ and $A \notin (G, B)$. Similarly $G \notin b$ and $G \sim_b A$, as $B \notin (G, A)$. Thus $G \in \operatorname{int} \measuredangle AOB$. □

We prove Greenberg's result [6, Prop. 3.8(a)].

Lemma 3.7. *If $X \in \operatorname{int} \measuredangle AOB$, then $\overrightarrow{OX} - \{O\} \subset \operatorname{int} \measuredangle AOB$.*

Proof. Take $P \in \overrightarrow{OX} - \{O\}$. Let $a = \overline{OA}$, $b = \overline{OX}$ and $x = \overline{OX}$. $x \cap a = \{O\}$, by Lemma 2.1, since $X \notin a$. Similarly $x \cap b = \{O\}$. Since $X \in \operatorname{int}\measuredangle AOB$, $X \sim_a B$ and $X \sim_b A$. Then $P \notin a$ and $P \sim_a X$, by Lemma 3.1, since $O \notin (P, X)$. Similarly $P \notin b$ and $P \sim_b X$. By Lemma 3.2, $P \sim_a B$ and $P \sim_b A$. Thus $P \in \operatorname{int}\measuredangle AOB$. □

We prove an easy trichotomy law for interiors of angles, inspired by [6, Prop. 3.21].

Lemma 3.8. *Take distinct points O and A. Let $a = \overline{OA}$. Take points $P, Q \notin a$ with $P \sim_a Q$ and P, Q and O non-collinear. Then $P \in \operatorname{int}\measuredangle QOA$ or $Q \in \operatorname{int}\measuredangle POA$.*

Proof. Let $p = \overline{OP}$ and $q = \overline{OQ}$. Then $q \cap a = \{O\}$, by Lemma 2.1, since $Q \notin a$. Thus $A \notin q$, since $A \neq O$. Then $p \cap q = \{O\}$, by Lemma 2.1, since $P \notin q$.

Assume $P \notin \operatorname{int}\measuredangle QOA$. Then $P \nsim_q A$, since $P \notin q$, $P \notin a$ and $P \sim_a Q$. Take a point $G \in (P, A) \cap q$. Then $G \in \operatorname{int}\measuredangle POA$ by Lemma 3.6, so $G \sim_a P$, $G \notin a$, and $G \neq O$. Since $P \sim_a Q$, Lemma 3.2 implies $Q \sim_a G$. Thus $O \notin (Q, G)$, and then $Q \in \overrightarrow{OG} - \{O\}$. Hence $Q \in \operatorname{int}\measuredangle POA$, by Lemma 3.7. □

We give a simpler proof of Greenberg's result [6, Prop. 3.8(c)].

Lemma 3.9. *Take non-collinear points O, A and B, a point $D \in \operatorname{int}\measuredangle AOB$ and a point A' with $O \in (A, A')$. Then $B \in \operatorname{int}\measuredangle DOA'$.*

Proof. Let $a = \overline{OA}$ and $b = \overline{OB}$. By axiom I1, $a = \overline{OA'}$. Then $b \cap a = \{O\}$, by Lemma 2.1, since $B \notin a$. Then $A' \notin b$ and $A \nsim_b A'$, since $A' \in a - \{O\}$. Since $D \in \operatorname{int}\measuredangle AOB$, $D \notin a$, $D \notin b$, $D \sim_a B$, and $D \sim_b A$. Thus $D \nsim_b A'$, by Lemma 3.2 and a short proof by contradiction. Hence $D \notin \operatorname{int}\measuredangle BOA'$. Thus $B \in \operatorname{int}\measuredangle DOA'$, by Lemma 3.8, since $B, D \notin a$, $D \sim_a B$, and $D \notin b$. □

We give a simpler proof of Greenberg's Crossbar Theorem [6, p. 69].

Theorem 3.10. *Take non-collinear points O, A and B, and a point $D \in \operatorname{int}\measuredangle AOB$. Then there exists a point $G \in (A, B)$ with $G \in \overrightarrow{OD} - \{O\}$.*

Proof. By axiom I1 there exists lines $a = \overline{OA}$ and $b = \overline{OB}$, since O, A and B are distinct, since they are non-collinear. $B \notin a$ by non-collinearity. $D \in \operatorname{int}\measuredangle AOB$ implies $D \sim_a B$, $D \notin a$, $D \notin b$, and $D \neq O$. Let $l = \overline{OD}$. Then $a \cap l = \{O\} = b \cap l$, by Lemma 2.1. Thus $A, B \notin l$, since $A \neq O$ and $B \neq O$. By axiom B2' there is a point $A' \in a$ such that $O \in (A, A')$. Then $A' \notin l$, since $A' \neq O$, and $A \nsim_l A'$.

$B \in \operatorname{int}\measuredangle DOA'$, by Lemma 3.9, so $B \sim_l A'$. Then $B \nsim_l A$, since $A \nsim_l A'$, by Lemma 3.2 and a proof by contradiction. Thus there is a point $G \in (A, B) \cap l$. $G \in \operatorname{int}\measuredangle AOB$ by Lemma 3.6, so $G \notin a$ and $G \sim_a B$, so $G \neq O$. Thus $G \sim_a D$ by Lemma 3.2, since $D \sim_a B$. Then $O \notin (G, D)$, so $G \in \overrightarrow{OD}$, hence $G \in \overrightarrow{OD} - \{O\}$. □

Our proof of the Crossbar Theorem 3.10 is simpler than those of Greenberg [6, p. 69], Venema [17, Thm. 5.7.15] and Moise [16, p. 69], which use Proposition 4.14, but only really need Lemma 4.12, based on Lemmas 4.5–4.9 which order four points.

We prove the rest of Greenberg's result [6, Prop. 3.7].

Lemma 3.11. *Take non-collinear points O, A, B, and a point $G \in \operatorname{int}\measuredangle AOB$ with A, G and B collinear. Then $G \in (A, B)$.*

Proof. $A \neq B$, $O \neq A$ and $O \neq B$ by non-collinearity. Let $a = \overline{OA}$ and $b = \overline{OB}$. Then $G \sim_a B$ and $G \sim_b A$, so $A \notin (G, B)$ and $B \notin (G, A)$. Furthermore $G \notin b$ and $G \notin a$, so $G \neq B$ and $A \neq G$. Thus $G \in (A, B)$ by axiom B3'. □

Here are three simple results that we will use later.

Lemma 3.12. *Take non-collinear points A, O and B and a point $P \in \operatorname{int}\measuredangle AOB$. Let $p = \overline{OP}$. Then $A \nsim_p B$.*

Proof. By the Crossbar Theorem 3.10, there exists a point G such that $G \in (A, B)$ and $G \in \overrightarrow{OP}$. Thus $A \nsim_p B$, since $G \in p$. □

Lemma 3.13. *Take a line l and points $O \in l$ and $A \notin l$. Assume $P \in \overrightarrow{OA} - \{O\}$. Then $P \notin l$ and $P \sim_l A$.*

Proof. $O \neq A$, so let $d = \overline{OA}$. Then $A, P \in d - \{O\}$. Thus $l \neq d$, and $l \cap d = \{O\}$ by Lemma 2.1. Hence $P \notin l$. Since $O \notin (P, A)$, $P \sim_l A$ by Lemma 3.1. □

Lemma 3.14. *Take $B \in (A, C)$. Then $B \in \overrightarrow{AC} - \{A\}$ and $C \in \overrightarrow{AB} - \{A\}$.*

Proof. By axiom B1', A, B and C are distinct and collinear. By axiom B3', $A \notin (B, C)$ and $A \notin (C, B)$. This proves our result. □

4. PLANE SEPARATION

We prove below the plane separation axiom of Moise and Venema (Prop. 4.14), Greenberg's axiom B4. Our proof Prop. 4.11 is essentially Moore's proof [10]. See Wylie [18, §6] for an interesting and different proof. Our first result was proved by Wylie [18, §4] with a slick proof by contradiction. We give a longer direct argument that seems more intuitive.

Proposition 4.1. *Take non-collinear points $A, B, C \notin l$ and let $m = \overline{AC}$. Assume there exists a point $Y \in l \cap m$, and that $A \nsim_l B$ and $B \nsim_l C$. Then $A \sim_l C$.*

Proof. There exist points $X, Z \in l$ so that $X \in (A, B)$ and $Z \in (C, B)$. Let $p = \overline{BA}$ and $q = \overline{BC}$. Then $X \notin q$ and $X \sim_q A$, by Lemmas 3.14 and 3.13, since $X \in (A, B)$ and $A \notin q$. Similarly $Z \notin p$ and $Z \sim_p C$, so $X \neq Z$. By Lemma 2.1, $p \cap m = \{A\}$ and $q \cap m = \{C\}$, since $B \notin m$. Then $Y \notin p$ and $Y \notin q$, since $Y \neq A$ and $Y \neq C$, since $A, C \notin l$. Now $X \in \overrightarrow{AB} - \{A\}$ and $Z \in \overrightarrow{CB} - \{C\}$, by Lemma 3.14. Thus $X \notin m$, $X \sim_m B$, $Z \notin m$ and $B \sim_m Z$, by Lemma 3.13, since $B \notin m$. Then $Y \neq X$ and $Y \neq Z$. By Lemma 3.2, $X \sim_m Z$, so $Y \notin (X, Z)$. By axiom B3', $X \in (Z, Y)$ or $Z \in (X, Y)$ since X, Y and Z are distinct and collinear. We study the two cases.

Suppose $X \in (Z, Y)$. Then $Z \nsim_p Y$. Thus $C \nsim_p Y$, by Lemma 3.2 and a short proof by contradiction, since $Z \sim_p C$ and $Z, C, Y \notin p$. Hence $A \in (C, Y)$, by Lemma 3.1 and a short proof by contradiction, since $C, Y \in m - \{A\}$ and $p \cap m = \{A\}$. Then $A \in \overrightarrow{YC} - \{Y\}$, by Lemma 3.14. Thus $A \sim_l C$, by Lemma 3.13.

Suppose $Z \in (X, Y)$. A similar proof shows that $C \in (Y, A)$ and $A \sim_l C$. □

Now we must prove a version of Proposition 4.1 where the points A, B and C are collinear. We first specialize Lemma 3.2 to a fixed line.

Lemma 4.2. *Take a point on a line $O \in m$ and points $P, Q, R \in m - \{O\}$. If $O \notin (P, Q)$ and $O \notin (Q, R)$, then $O \notin (P, R)$.*

Proof. By axiom I3, there exists $E \notin m$. Let $l = \overline{OE}$. Then $m \cap l = \{O\}$, by Lemma 2.1, because $E \notin m$. Then $P \sim_l Q$ and $Q \sim_l R$, by Lemma 3.1 and the hypothesis. Thus $P \sim_l R$, by Lemma 3.2. Therefore $O \notin (P, R)$. □

We now show that the ray $\overrightarrow{OP}$ is in a sense independent of P.

Lemma 4.3. *Take points O, P and $Q \neq O$, with $P \in \overrightarrow{OQ} - \{O\}$. Then $\overrightarrow{OP} = \overrightarrow{OQ}$. If $R \in (O, Q)$, then $\overrightarrow{OR} = \overrightarrow{OQ}$.*

Proof. Let $m = \overline{OQ}$, so $P, Q \in m - \{O\}$. By axiom I1, $m = \overline{OP}$, so $\overrightarrow{OP} \subset m$.

Take $X \in \overrightarrow{OP} - \{O\}$. Then $O \notin (X, Q)$, by Lemma 4.2, since $P, Q, X \in m - \{O\}$, $O \notin (X, P)$, and $O \notin (P, Q)$. Hence $X \in \overrightarrow{OQ}$. Thus $\overrightarrow{OP} \subset \overrightarrow{OQ}$. But $O \notin (Q, P)$, so $Q \in \overrightarrow{OP} - \{O\}$. The same argument shows that $\overrightarrow{OQ} \subset \overrightarrow{OP}$. Thus $\overrightarrow{OP} = \overrightarrow{OQ}$.

Now assume $R \in (O, Q)$. Then O, R and Q are distinct collinear points, and $O \notin (R, Q)$, by axiom B3'. Thus $R \in \overrightarrow{OQ} - \{O\}$. By the first part, $\overrightarrow{OR} = \overrightarrow{OQ}$. □

We will prove Hilbert's axiom II.4 is redundant using only the following result.

Lemma 4.4. *Take points A, B and $O \in (A, B)$, and assume $X \in \overrightarrow{OB} - \{O\}$. Then $X \notin \overrightarrow{OA}$ and $O \in (X, A)$.*

Proof. A, O and B are distinct and collinear. Let $m = \overline{AB}$. Since $X \in \overrightarrow{OB} - \{O\}$, $O, X \in m$, by axiom I1, $O \notin (X, B)$, and $A, X, B \in m - \{O\}$. Hence $O \in (A, X)$ by Lemma 4.2 and a short proof by contradiction, so $O \in (X, A)$. Thus $X \notin \overrightarrow{OA}$. □

The next result follows immediately from the above result.

Lemma 4.5. *Take points A, B and $O \in (A, B)$. Then $\overrightarrow{OA} \cap \overrightarrow{OB} = \{O\}$.*

Given distinct collinear points A, B, C and D, we will say that the 4-tuple $\langle A, B, C, D\rangle$ is an *ordered sequence* if $B \in (A, C)$, $C \in (B, D)$, $B \in (A, D)$ and $C \in (A, D)$. Note that by symmetry, if $\langle A, B, C, D\rangle$ is an ordered sequence, then $\langle D, A, B, C\rangle$ is also an ordered sequence. An easy corollary of Lemma 4.4 is

Lemma 4.6. *Take points A, B, C and D with $B \in (A, C)$ and $C \in (B, D)$. Then $B \in (A, D)$, and $\langle A, B, C, D\rangle$ is an ordered sequence.*

Proof. $D \in \overrightarrow{BC} - \{B\}$ by Lemma 3.14, since $C \in (B, D)$. Then $B \in (D, A)$ by Lemma 4.4, since $B \in (A, C)$, so $B \in (A, D)$. This proves the first statement.

$C \in (D, B)$ and $B \in (C, A)$, by symmetry. By the first statement, $C \in (D, A)$. Thus $C \in (A, D)$. This proves the second statement. □

With our plane separation Proposition 4.14 in mind, we now prove

Lemma 4.7. *Take points $B \in (A, C)$. Then $(A, B) \subset (A, C)$.*

Proof. Take $X \in (A, B)$. Then $X \in \overrightarrow{BA} - \{B\}$, by Lemma 3.14. $B \in (X, C)$ by Lemma 4.4, since $B \in (C, A)$. Hence $X \in (A, C)$, by Lemma 4.6. □

By using the full strength of Lemma 4.6, the above proof shows that

Lemma 4.8. *Let A, X, B and C be distinct points on a line, and suppose that $X \in (A, B)$ and $B \in (A, C)$. Then $\langle A, X, B, C\rangle$ is an ordered sequence.*

This implies

Lemma 4.9. *Let A, B, C and X be distinct collinear points. Then at least one of the statements $X \notin (A, B)$, $X \notin (B, C)$ and $X \notin (A, C)$ is true.*

Proof. By axiom B3', one of the distinct points A, B and C is between the other two. We may assume that $B \in (A, C)$, as our result is symmetric in A, B and C. If $X \notin (A, B)$, we are done, so assume $X \in (A, B)$. Then $\langle A, X, B, C\rangle$ is an ordered sequence by Lemma 4.8, so $B \in (X, C)$. Thus $X \notin (B, C)$, by axiom B3'. □

Now we have the 4-tuple ordered sequence result we have been aiming for.

Lemma 4.10. *Let A, B, C and X be distinct collinear points, with $B \in (A, C)$.*

1. *Either $A \in (X, B)$ or $X \in (A, B)$ or $X \in (B, C)$ or $C \in (B, X)$.*
2. *One of the 4-tuples $\langle X, A, B, C\rangle$, $\langle A, X, B, C\rangle$, $\langle A, B, X, C\rangle$ or $\langle A, B, C, X\rangle$ is an ordered sequence.*

Proof. $A \in (X,B)$ or $X \in (A,B)$ or $B \in (A,X)$, by axiom B3' applied to A, B and X. Assume $B \in (A,X)$. Since $B \in (A,C)$, we have $B \notin (C,X)$ by Lemma 4.9. Then by axiom B3', $X \in (B,C)$ or $C \in (B,X)$. This proves (1).

To prove (2), we consider four cases. If $A \in (X,B)$, then $\langle X,A,B,C\rangle$ is an ordered sequence by Lemma 4.6, since $B \in (A,C)$. If $X \in (A,B)$, then $\langle A,X,B,C\rangle$ is an ordered sequence by Lemma 4.8. If $X \in (B,C)$, then $X \in (C,B)$ and $B \in (C,A)$, by symmetry. Thus $\langle C,X,B,A\rangle$ is an ordered sequence by Lemma 4.8, and so by symmetry, $\langle A,B,X,C\rangle$ is also an ordered sequence. If $C \in (B,X)$, then $\langle A,B,C,X\rangle$ is an ordered sequence by Lemma 4.6. □

Hilbert's axiom II.4 is now an easy corollary of Lemma 4.10.

Proposition 4.11 (Moore)**.** *Any four distinct collinear points can be renamed P_1, P_2, P_3 and P_4 so that $\langle P_1,P_2,P_3,P_4\rangle$ is an ordered sequence.*

Proof. Call three of the points A, B and C and the fourth point X. By axiom B3', one of the points A, B and C must be between the other two. Since we are allowed to rename the points, we can assume that $B \in (A,C)$. By Lemma 4.10, we have one of four possible ordered 4-tuples. In all four cases we can rename the points P_1, P_2, P_3 and P_4 to obtain our ordered sequence $\langle P_1,P_2,P_3,P_4\rangle$. □

Lemma 4.9 also implies a result suggested by Lemma 4.4.

Lemma 4.12. *Take points $O \in (A,B)$ on a line l. Then $l = \overrightarrow{OA} \cup \overrightarrow{OB}$.*

Proof. We prove each set is a subset of the other. $\overrightarrow{OA} \subset l$, since $\overline{OA} = l$, by axioms B1' and I1. Similarly $\overrightarrow{OB} \subset l$. Thus $\overrightarrow{OA} \cup \overrightarrow{OB} \subset l$. For the reverse inclusion, take $X \in l$ with $X \notin \overrightarrow{OB}$. We must show that $X \in \overrightarrow{OA}$. If $X = A$ we are done, so assume that $X \neq A$. Then O, X, A and B are distinct points in l, since $O \in (X,B)$ and $O \in (A,B)$. Thus $O \notin (X,A)$, by Lemma 4.9, so $X \in \overrightarrow{OA}$. Hence $l \subset \overrightarrow{OA} \cup \overrightarrow{OB}$. Therefore $l = \overrightarrow{OA} \cup \overrightarrow{OB}$. □

$\overrightarrow{OA}$ is called the *opposite ray* of $\overrightarrow{OB}$ in l. Proposition 4.1 and Lemma 4.9 imply

Corollary 4.13. *For points $A,B,C \notin l$, either $A \sim_l B$, $A \sim_l C$ or $B \sim_l C$.*

Proof. If A, B and C are non-collinear, we are done by Proposition 4.1. If two of A, B and C are equal, we are done since $\sim_l$ is reflexive. So assume A, B and C are distinct points in a line m. If $l \cap m = \emptyset$, then we are done, as all three statements are true. So assume $l \cap m \neq \emptyset$. Then $l \cap m = \{X\}$, for some point X, by Lemma 2.1, since $A \notin l$. Then either $X \notin (A,B)$, $X \notin (A,C)$ or $X \notin (B,C)$, by Lemma 4.9. Thus either $A \sim_l B$, $A \sim_l C$ or $B \sim_l C$, by Lemma 3.1. □

We call a set U *convex* if $A,B \in U$ implies $(A,B) \subset U$. We prove what Moise [16, PS-1, p. 62] and Venema [17, Axiom 5.5.2] call the plane separation axiom.

Proposition 4.14. *The complement in the plane of a line l is a disjoint union of two nonempty convex sets H_1 and H_2. If $P \in H_1$ and $Q \in H_2$, then $(P,Q) \cap l \neq \emptyset$.*

Proof. By axiom I3, there is a point $A \notin l$. By axiom I2, there is a point $E \in l$, and let $m = \overline{AE}$. Then $m \cap l = \{E\}$ by Lemma 2.1, since $A \notin l$. By axiom B2', there exists a point $B \in m$ such that $E \in (A,B)$. Then $B \notin l$, since $B \neq E$, and $A \nsim_l B$. Define $H_1 = \{X \notin l \mid X \sim_l A\}$ and $H_2 = \{X \notin l \mid X \sim_l B\}$. H_1 and H_2 are disjoint nonempty subsets of the complement of l, containing A and B respectively, by Lemma 3.2. If $C \notin l$, then by Corollary 4.13, either $A \sim_l C$ or $B \sim_l C$, as $A \nsim_l B$. Thus the complement of l is the disjoint union $H_1 \cup H_2$. If $P \in H_1$ and $Q \in H_2$, then $P \nsim_l Q$, by Lemma 3.2. Thus $(P,Q) \cap l \neq \emptyset$.

Take $P,Q \in H_1$, so $P \notin l$, $Q \notin l$, $P \sim_l A$, and $Q \sim_l A$. By Lemma 3.2, $P \sim_l Q$. Let $X \in (P,Q)$. Then $X \notin l$, since $P \sim_l Q$. $(P,X) \subset (P,Q)$ by Lemma 4.7. Take $Y \in (P,X)$.

Then $Y \in (P, Q)$, so $Y \notin l$, since $P \sim_l Q$. Thus $P \sim_l X$. Hence $X \sim_l A$, by Lemma 3.2. Thus $X \in H_1$. Hence $(P, Q) \subset H_1$. Thus H_1 is convex. Similarly H_2 is convex. $\square$

$H_1 = \{X \notin l \mid X \sim_l A\}$ and $H_2 = \{X \notin l \mid X \sim_l B\}$ are called the *half-planes bounded by* l. By Lemma 3.2, H_1 and H_2 are independent of the choice of the points $A \in H_1$ and $B \in H_2$.

Take non-collinear points O, A and B and a point D. The ray $\overrightarrow{OD}$ is *between* $\overrightarrow{OA}$ and $\overrightarrow{OB}$ if $D \in \operatorname{int} \measuredangle AOB$. By Lemma 4.3, the betweenness relation of rays does not depend on the points A, B and D, but only the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OD}$.

A *triangle* $\triangle ABC$ is an ordered triple $\langle A, B, C\rangle$ of three non-collinear points.

Lemma 4.15. *Take a triangle $\triangle AOB$, and points $G \in \operatorname{int} \measuredangle AOB$ and $F \in \measuredangle AOG$. Then $F \in \operatorname{int} \measuredangle AOB$.*

Proof. By the Crossbar Theorem 3.10, there exists $G_0 \in (A, B)$ with $G_0 \in \overrightarrow{OG} - \{O\}$. Then $G_0 \in \operatorname{int} \measuredangle AOB$ and $F \in \measuredangle AOG_0$, by Lemmas 3.7 and 3.5. Replace G by G_0. Then $G \in (A, B)$. By the Crossbar Theorem 3.10, there exists $F_0 \in (A, G)$ with $F_0 \in \overrightarrow{OF} - \{O\}$. By Lemma 4.3, $\overrightarrow{OF_0} = \overrightarrow{OF}$. Then $F_0 \in (A, B)$, since $(A, G) \subset (A, B)$, by Lemma 4.7. Thus $F_0 \in \operatorname{int} \measuredangle AOB$, by Lemma 3.6. Hence $F \in \operatorname{int} \measuredangle AOB$, by Lemma 3.7, since $F \in \overrightarrow{OF_0} - \{O\}$. $\square$

Following Greenberg, we define [6, p. 69, p. 77] triangle interiors and angle order.

Definition 4.16. Given non-collinear points P, Q and R, we define the *interior* of the triangle as the set of points in the plane

$$\operatorname{int} \triangle PQR := \{X \mid X \in \operatorname{int} \measuredangle PQR \text{ and } X \in \operatorname{int} \measuredangle QRP \text{ and } X \in \operatorname{int} \measuredangle RPQ\}.$$

We say $\measuredangle ABC < \measuredangle PQR$ if there exists $G \in \operatorname{int} \measuredangle PQR$ so that $\measuredangle ABC \cong \measuredangle PQG$.

5. Quadrilaterals

Moise has a nice treatment on quadrilaterals [16, §4.4] which we expand on. With Proposition 7.6 in mind, we first generalize the notion of a quadrilateral.

We define a *tetralateral* to be an ordered 4-tuple $\square ABCD$ of distinct points in the plane, no three of which are collinear. Take a tetralateral $\square ABCD$. We define (for this section) the lines $a = \overline{AB}$, $b = \overline{BC}$, $c = \overline{CD}$, $d = \overline{DA}$, $l = \overline{AC}$, and $m = \overline{BD}$. Since no three points of the tetralateral $\square ABCD$ are collinear, we have $C, D \notin a$, $A, D \notin b$, $A, B \notin c$, $B, C \notin d$, $B, D \notin l$, and $A, C \notin m$. There are six intersections of the four open intervals (A, B), (B, C), (C, D) and (D, A). Then

Lemma 5.1. *For a tetralateral $\square ABCD$, we have four empty intersections:*

$$(A, B) \cap (B, C) = (B, C) \cap (C, D) = (C, D) \cap (D, A) = (D, A) \cap (A, B) = \emptyset.$$

Proof. $(A, B) \cap (B, C) \subset a \cap b = \{B\}$, by Lemma 2.1, since $A \notin b$. Since $B \notin (A, B)$, by axiom B1', $(A, B) \cap (B, C) = \emptyset$. The other intersections are similarly empty. $\square$

We define a *quadrilateral* $\square ABCD$ [16, p. 69] to be a tetralateral $\square ABCD$ where $(A, B) \cap (C, D) = \emptyset$ and $(B, C) \cap (A, D) = \emptyset$. Given a quadrilateral $\square ABCD$, all six intersections of the four open intervals (A, B), (B, C), (C, D) and (D, A) are empty, by Lemma 5.1. A quadrilateral $\square ABCD$ is *convex* [16, p. 69] if $A \in \operatorname{int} \measuredangle BCD$, $B \in \operatorname{int} \measuredangle CDA$, $C \in \operatorname{int} \measuredangle DAB$, and $D \in \operatorname{int} \measuredangle ABC$. The term convex quadrilateral indicates that the "inside" of the quadrilateral is a convex set, but we will not pursue this. Moise explains the term convex quadrilateral is inconsistent: the union of four non-collinear line segments (or four points) cannot be a convex set.

We begin by proving an exercise of Moise [16, Ex. 4, §4.4].

Lemma 5.2. *Take a quadrilateral $\square ABCD$. Then either $A \sim_c B$ or $C \sim_a D$.*

Proof. Assume that $C \not\sim_a D$. Then there exists a point $G \in a \cap (C, D)$. Then by Lemma 2.1, $a \cap c = \{G\}$, since $A \notin c$. But $G \notin (A, B)$, since $(A, B) \cap (C, D) = \emptyset$. Furthermore $A, B \in a - \{G\}$, since $A, B \notin c$. Thus $A \sim_c B$, by Lemma 3.1. □

We generalize Moise's [16, Thm. 1, §4.4] slightly, with the same proof.

Lemma 5.3. *Take a tetralateral $\square ABCD$ with $B \not\sim_l D$ and $A \not\sim_m C$. Then there is a point $G \in (A, C) \cap (B, D)$, and we have a convex quadrilateral $\square ABCD$.*

Proof. Since $A \not\sim_m C$, there exists a point $G \in (A, C) \cap m$. So $G \in l \cap m$. Then $l \cap m = \{G\}$, by Lemma 2.1, since $A \notin m$. Thus $B, D \in m - \{G\}$, since $B, D \notin l$. Hence $G \in (B, D)$, since $B \not\sim_l D$, by Lemma 3.1, and a short proof by contradiction. Thus $G \in (A, C) \cap (B, D)$.

Then $B \notin (D, G)$, by axiom B3', so $D \in \overrightarrow{BG} - \{B\}$. Now $G \in \operatorname{int} \measuredangle ABC$, by Lemma 3.6. Thus $D \in \operatorname{int} \measuredangle ABC$, by Lemma 3.7. A similar argument shows that $A \in \operatorname{int} \measuredangle BCD$, $B \in \operatorname{int} \measuredangle CDA$, and $C \in \operatorname{int} \measuredangle DAB$. □

We prove Moise's result on the diagonals of a convex quadrilateral.

Lemma 5.4. *Take a convex quadrilateral $\square ABCD$. Then $B \not\sim_l D$ and $A \not\sim_m C$. There is a point $G \in (A, C) \cap (B, D)$, and $\square ABDC$ is not a quadrilateral.*

Proof. By the Crossbar Theorem 3.10, $B \not\sim_l D$, since $A \in \operatorname{int} \measuredangle BCD$. Similarly $A \not\sim_m C$. Thus there exists a point $G \in (A, C) \cap (B, D)$, by Lemma 5.3.

$\square ABDC$ is not a quadrilateral, because $(C, A) \cap (B, D) \neq \emptyset$. □

We need this tetralateral result for Proposition 7.6.

Lemma 5.5. *Take a tetralateral $\square ABCD$ with $C \sim_a D$. Then either $\square ABCD$ is a convex quadrilateral, $\square ABDC$ is a convex quadrilateral, $D \in \operatorname{int} \triangle ABC$, or $C \in \operatorname{int} \triangle DAB$.*

Proof. Either $C \in \operatorname{int} \measuredangle DAB$ or $D \in \operatorname{int} \measuredangle CAB$, by Lemma 3.8, since $C, D \notin a$ and $C \sim_a D$. We study these two cases.

The case $C \in \operatorname{int} \measuredangle DAB$ follows from Lemma 5.3. So suppose $D \in \operatorname{int} \measuredangle CAB$. By the proof of case 1, either $\square ABDC$ is a convex quadrilateral or $D \in \operatorname{int} \triangle ABC$. We can obtain the proof by switching C and D in the above proof, because our lemma is symmetric in C and D. □

We specialize Lemma 5.5 to quadrilaterals.

Lemma 5.6. *Take a quadrilateral $\square ABCD$. Then either $\square ABCD$ is a convex quadrilateral, $A \in \operatorname{int} \triangle BCD$, $B \in \operatorname{int} \triangle CDA$, $C \in \operatorname{int} \triangle DAB$ or $D \in \operatorname{int} \triangle ABC$. Furthermore either $B \not\sim_l D$ or $A \not\sim_m C$.*

Proof. Either $A \sim_c B$ or $C \sim_a D$, by Lemma 5.2. We will apply to Lemma 5.5 to the quadrilaterals $\square ABCD$ and $\square CDAB$.

$\square CDAB$ is a quadrilateral. $\square ABDC$ is not a convex quadrilateral, by Lemma 5.4 and a short proof by contradiction, since $\square ABCD$ is a quadrilateral. Similarly, $\square CDBA$ is a not a convex quadrilateral, since $\square CDAB$ is a quadrilateral.

If $C \sim_a D$, then either $\square ABCD$ is a convex quadrilateral, $D \in \operatorname{int} \triangle ABC$, or $C \in \operatorname{int} \triangle DAB$, by Lemma 5.5. Similarly, if $A \sim_c B$, then either $\square CDAB$ is a convex quadrilateral, $B \in \operatorname{int} \triangle CDA$, or $A \in \operatorname{int} \triangle BCD$. But $\square CDAB$ is convex iff $\square ABCD$ is convex. This proves our first assertion.

Thus either $\square ABCD$ is a convex quadrilateral, $A \in \operatorname{int} \measuredangle BCD$, $B \in \operatorname{int} \measuredangle ADC$, $C \in \operatorname{int} \measuredangle BAD$, or $D \in \operatorname{int} \measuredangle ABC$. For each of these five possibilities, either $B \not\sim_l D$ or $A \not\sim_m C$, or both, by Lemma 5.4 and Lemma 3.12. □

6. The congruence relation $\cong$

For points A and B, the *segment* AB is defined as the subset of the plane

$$AB = \{A, B\} \cup (A, B).$$

Note that $AB = BA$. We have the undefined binary relation $\cong$ on segments. If $AB \cong CD$, we say that the segment AB is *congruent* to the segment CD. Hilbert's axioms for congruence of segments are

- **C1.** Take a segment s and a ray $\overrightarrow{OZ}$ defined by distinct points O and Z. Then there is a unique point $P \in \overrightarrow{OZ} - \{O\}$ with $OP \cong s$.
- **C2.** $\cong$ is an equivalence relation on the set of segments.
- **C3.** If $B \in (A, C)$, $B' \in (A', C')$, $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.

We call C1 the segment construction axiom and C3 the segment addition axiom. In this paper, we never use axiom C3, but in our HOL Light code, we need C3 to prove results of Greenberg's which we cite without proof in this paper.

We write $AB < CD$ if there exists a point $E \in (C, D)$ such that $AB \cong CE$. Greenberg [6, Prop. 3.13] proves the segment order relation $<$ satisfies a trichotomy law, a transitive property and is well defined up to congruence of segments.

We have the undefined binary relation $\cong$ on angles, and Hilbert's axioms are

- **C4.** Take an angle α and a ray $\overrightarrow{OA}$. Let $l = \overline{OA}$, and take a point $Y \notin l$. Then there is a unique ray $\overrightarrow{OB}$ so that $B \sim_l Y$ and $\measuredangle AOB \cong \alpha$.
- **C5.** $\cong$ is an equivalence relation on the set of angles.
- **C6.** Take triangles $\triangle ABC$ and $\triangle A'B'C'$. If $BA \cong B'A'$, $BC \cong B'C'$ and $\measuredangle ABC \cong \measuredangle A'B'C'$, then $\measuredangle BCA \cong \measuredangle B'C'A'$.

We will often use an obvious angle symmetry property. Given a triangle $\triangle ABC$, we have $\measuredangle ABC = \measuredangle CBA$, because

$$\measuredangle ABC = \{\overrightarrow{BA}, \overrightarrow{BC}\} = \{\overrightarrow{BC}, \overrightarrow{BA}\} = \measuredangle CBA.$$

We prove a result about axiom C4 analogous to Lemma 2.1.

Lemma 6.1. *Take distinct points O and A and let $l = \overline{OA}$. Take points $B, P \notin l$ with $P \sim_l B$. If $\measuredangle AOP \cong \measuredangle AOB$, then $\overrightarrow{OB} = \overrightarrow{OP}$.*

Proof. $B \sim_l B$ by Lemma 3.2. $\measuredangle AOB \cong \measuredangle AOB$ by axiom C5. Then $\overrightarrow{OB} = \overrightarrow{OP}$ by the uniqueness of axiom C4 applied to $\measuredangle AOB$, ray $\overrightarrow{OA}$, and the point $B \notin l$. □

We write $\triangle ABC \cong \triangle A'B'C'$, and say the two triangles are congruent, if the corresponding segments and angles are congruent. Now we prove that the SAS criterion implies congruent triangles, following Hilbert's proof.

Theorem 6.2 (SAS)**.** *Take triangles $\triangle ABC$ and $\triangle A'B'C'$. If $BA \cong B'A'$, $BC \cong B'C'$ and $\measuredangle ABC \cong \measuredangle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. Let $c = \overline{AB}$, $a = \overline{BC}$ and $b = \overline{AC}$. Since A, B and C are non-collinear points, $c \cap b = \{A\}$ and $a \cap b = \{C\}$, by Lemma 2.1. Then $\measuredangle BCA \cong \measuredangle B'C'A'$, by axiom C6. We can apply axiom C6 to the triangles $\triangle CBA$ and $\triangle C'B'A'$, since $\measuredangle CBA \cong \measuredangle C'B'A'$, because $\measuredangle CBA = \measuredangle ABC$, $\measuredangle ABC \cong \measuredangle A'B'C'$, and $\measuredangle A'B'C' = \measuredangle C'B'A'$. It is not necessary to use axiom C5 here: equal mathematical objects can be substituted for each other. Thus $\measuredangle BAC \cong \measuredangle B'A'C'$ by axiom C6.

By the segment construction axiom C1, there is a point $Y \in \overrightarrow{AC} - \{A\}$ with $AY \cong A'C'$. Then $\overrightarrow{AY} = \overrightarrow{AC}$, by Lemma 4.3, so $\measuredangle YAB = \measuredangle CAB$. Hence $\measuredangle YAB \cong \measuredangle C'A'B'$, since $\measuredangle BAC \cong \measuredangle B'A'C'$. We used $\measuredangle CAB = \measuredangle BAC$ and $\measuredangle B'A'C' = \measuredangle C'A'B'$. Apply axiom C6 to the triangles $\triangle YAB$ and $\triangle C'A'B'$. Thus $\measuredangle ABY \cong \measuredangle A'B'C'$. Hence $\measuredangle ABY \cong \measuredangle ABC$, by hypothesis and axiom C5. But $A \notin (Y, C)$, since $Y \in \overrightarrow{AC}$. Therefore $Y \sim_c C$, by Lemma 3.1, since $Y, C \in b - \{A\}$. Thus $\overrightarrow{AY} = \overrightarrow{AC}$, by Lemma 6.1. Then $Y \in a \cap b$. Thus $C = Y$, so $AC \cong A'C'$. Hence $\triangle ABC \cong \triangle A'B'C'$. □

Greenberg takes Theorem 6.2 as an axiom instead of C6. His proof [6, Prop. 3.17] of the ASA theorem is similar to Hilbert's proof of Theorem 6.2. His proof [6, Prop. 3.19] of the angle addition result seems difficult, and does not follow Hilbert's treatment. Trying to follow Hilbert, we instead prove an angle subtraction result.

Lemma 6.3. *Take triangles $\triangle AOB$ and $\triangle A'O'B'$, and points $G \in \operatorname{int} \measuredangle AOB$ and $G' \in \operatorname{int} \measuredangle A'O'B'$. Suppose we have the angle congruences $\measuredangle AOB \cong \measuredangle A'O'B'$ and $\measuredangle AOG \cong \measuredangle A'O'G'$. Then $\measuredangle BOG \cong \measuredangle B'O'G'$.*

Proof. For any point $B_0 \in \overrightarrow{OB} - \{O\}$, Lemma 4.3 implies $\measuredangle AOB_0 = \measuredangle AOB$ and $\measuredangle B_0OG = \measuredangle BOG$. We may replace B by B_0, because this does not change the statement of our lemma. Similarly we may replace A, G, A', B', and G'. By the segment construction axiom C1, and the Crossbar Theorem 3.10, we may assume that $OA \cong O'A'$, $OB \cong O'B'$, $G \in (A, B)$ and $G' \in (A', B')$. By the SAS Theorem 6.2, $\triangle AOB \cong \triangle A'O'B'$, so $AB \cong A'B'$, $\measuredangle OAB \cong \measuredangle O'A'B'$ and $\measuredangle OBA \cong \measuredangle O'B'A'$. By ASA [6, Prop. 3.17], $\triangle AOG \cong \triangle A'O'G'$, so $AG \cong A'G'$. Then $BG \cong B'G'$ by the segment subtraction result [6, Prop. 3.11]. By SAS, $\triangle BOG \cong \triangle B'O'G'$. Thus $\measuredangle BOG \cong \measuredangle B'O'G'$. □

We prove the angle analogue of a segment result of Greenberg [6, Prop. 3.12].

Lemma 6.4. *Take triangles $\triangle AOB$ and $\triangle A'O'B'$ with $\measuredangle AOB \cong \measuredangle A'O'B'$ and a point $G \in \operatorname{int} \measuredangle AOB$. There exists $G' \in \operatorname{int} \measuredangle A'O'B'$ such that $\measuredangle AOG \cong \measuredangle A'O'G'$.*

Proof. We only sketch the proof, as it is similar to the above proof. We may assume $OA \cong O'A'$, $OB \cong O'B'$, $AB \cong A'B'$, $\measuredangle OAB \cong \measuredangle O'A'B'$ and $G \in (A, B)$. By Greenberg's result [6, Prop. 3.12], there is a point $G' \in (A', B')$ with $AG \cong A'G'$. Then $G' \in \operatorname{int} \measuredangle A'O'B'$ by Lemma 3.6. By SAS, $\triangle OAG \cong \triangle O'A'G'$. □

We deduce the angle addition result from Lemma 6.3 and Lemma 6.4.

Lemma 6.5. *Take triangles $\triangle AOB$ and $\triangle A'O'B'$, and points $G \in \operatorname{int} \measuredangle AOB$ and $G' \in \operatorname{int} \measuredangle A'O'B'$. Suppose we have the angle congruences $\measuredangle AOG \cong \measuredangle A'O'G'$ and $\measuredangle GOB \cong \measuredangle G'O'B'$. Then $\measuredangle AOB \cong \measuredangle A'O'B'$.*

Proof. Let $a = \overline{OA}$ and $g = \overline{OG}$. Then $G \sim_a B$ by Definition 3.3. By axioms C4, there exists a point X so that $X \notin a$, $X \sim_a G$ and $\measuredangle AOX \cong \measuredangle A'O'B'$. By axiom C5 and Lemma 6.4, there exists $Y \in \operatorname{int} \measuredangle AOX$ so that $\measuredangle A'O'G' \cong \measuredangle AOY$. By axiom C5, $\measuredangle AOY \cong \measuredangle AOG$. Furthermore $Y \sim_a X$, so $Y \sim_a G$, by Lemma 3.2. By Lemma 6.1, $\overrightarrow{OG} = \overrightarrow{OY}$, so $G \in \overrightarrow{OY} - \{O\}$. Thus $G \in \operatorname{int} \measuredangle AOX$ by Lemma 3.7. By Lemma 6.3, $\measuredangle GOX \cong \measuredangle G'O'B'$. By hypothesis and axiom C5, $\measuredangle GOX \cong \measuredangle GOB$.

By the Crossbar Theorem 3.10, the intervals (A, X) and (A, B) both intersect the ray $\overrightarrow{OG}$. So $A \nsim_g X$ and $A \nsim_g B$. Thus $X \sim_g B$ by Lemma 4.13. Hence $\overrightarrow{OX} = \overrightarrow{OB}$, by Lemma 6.1. Therefore $\measuredangle AOB \cong \measuredangle A'O'B'$. □

Given an angle $\measuredangle AOB$ and a point A' with $O \in (A, A')$, the angle $\measuredangle BOA'$ is called the *supplement* of $\measuredangle AOB$, and we say $\measuredangle AOB$ and $\measuredangle BOA'$ are *supplementary* angles. We say an angle is a *right* angle if it is congruent to its supplement. Then

Lemma 6.6. *Supplements of congruent angles are congruent. An angle congruent to a right angle is a right angle. All right angles are congruent. Take a line $h = \overline{OH}$ and two right angles $\measuredangle AOH$ and $\measuredangle HOB$, with $A \nsim_h B$. Then $O \in (A, B)$.*

Proof. Greenberg proves the first statement [6, Prop. 3.14]. This implies the second statement, by axiom C5. The third statement, due to Hilbert, is [6, Prop. 3.23].

Let $v = \overline{OA}$. Then $h \cap v = \{O\}$, by Lemma 2.1, since $A \notin h$. By axiom B2', there exists a point $X \in v$ with $O \in (A, X)$. Then $X \notin h$, since $X \neq O$, and $A \not\sim_h X$. Thus $X \sim_h B$, by Corollary 4.13, since $B \notin h$.

$\measuredangle HOX \cong \measuredangle AOH$, since $\measuredangle AOH$ is a right angle with supplement $\measuredangle HOX$. By axiom C5, $\measuredangle HOX \cong \measuredangle HOB$, since all right angles are congruent. Thus $\overrightarrow{OX} = \overrightarrow{OB}$, by Lemma 6.1, since $X \sim_h B$. Thus $B \in v$. Hence $O \in (A, B)$, by Lemma 3.1 and a short proof by contradiction, since $A, B \in v - \{O\}$. □

We reprove Greenberg's trichotomy and congruence angle ordering result [6, Prop. 3.21], as his exercise proof seems to require Lemma 6.4, which he did not state. We first prove the parts of [6, Prop. 3.21] we actually use.

Lemma 6.7. *Take angles α and β with $\alpha < \beta$. Then $\alpha \ncong \beta$.*

Proof. Assume (for a contradiction) that $\alpha \cong \beta$. Write $\beta = \measuredangle AOB$. Since $\alpha < \beta$, there is a point $G \in \operatorname{int} \measuredangle AOB$ so that $\alpha \cong \measuredangle AOG$. Let $a = \overline{OA}$ and $b = \overline{OB}$. Then $B \notin a$, since A, O and B are non-collinear. Furthermore $G \notin a$, $G \notin b$, and $G \sim_a B$. By axiom C5, $\measuredangle AOG \cong \measuredangle AOB$, since $\alpha \cong \beta$. Thus $\overrightarrow{OB} = \overrightarrow{OG}$, by Lemma 6.1, so $G \in b$. This is a contradiction. Therefore $\alpha \ncong \beta$. □

Next we have an obvious corollary of Lemma 6.4.

Lemma 6.8. *Take angles α, β and γ with $\alpha < \beta$ and $\beta \cong \gamma$. Then $\alpha < \gamma$.*

Proof. Let $\beta = \measuredangle AOB$ and $\gamma = \measuredangle A'O'B'$. Since $\alpha < \beta$, there exists a point $G \in \operatorname{int} \measuredangle AOB$ so that $\alpha \cong \measuredangle AOG$. By Lemma 6.4, there exists $G' \in \operatorname{int} \measuredangle A'O'B'$ such that $\measuredangle AOG \cong \measuredangle A'O'G'$. By axiom C5, $\alpha \cong \measuredangle A'O'G'$. Thus $\alpha < \gamma$. □

Now we prove transitivity for the angle order relation $<$.

Lemma 6.9. *Take angles α, β and γ with $\alpha < \beta$ and $\beta < \gamma$. Then $\alpha < \gamma$.*

Proof. Let $\gamma = \measuredangle AOB$. Since $\beta < \gamma$, there exists $G \in \operatorname{int} \measuredangle AOB$ with $\beta \cong \measuredangle AOG$. Then $\alpha < \measuredangle AOG$, by Lemma 6.8, since $\alpha < \beta$. Thus there exists $F \in \operatorname{int} \measuredangle AOG$ with $\alpha \cong \measuredangle AOF$. Then $F \in \operatorname{int} \measuredangle AOB$, by Lemma 4.15. Hence $\alpha < \gamma$. □

We finish proving [6, Prop. 3.21], although we only use the simpler results above.

Lemma 6.10. *Given angles α and β, exactly one of the following statements is true: $\alpha \cong \beta$, $\alpha < \beta$, and $\beta < \alpha$.*

Proof. If $\alpha < \beta$ or $\beta < \alpha$, then $\alpha \ncong \beta$, by Lemma 6.7 and axiom C5. Assume (for a contradiction) that $\alpha < \beta$ and $\beta < \alpha$. Then $\alpha < \alpha$ and $\alpha \cong \alpha$, by Lemma 6.9 and axiom C5. By Lemma 6.7, this is a contradiction. Thus $\alpha \not< \beta$ or $\beta \not< \alpha$. Hence at most one of the statements $\alpha \cong \beta$, $\alpha < \beta$, and $\beta < \alpha$ is true.

Let $\alpha = \measuredangle POA$, and let $a = \overline{OA}$. $P \notin a$ because P, O and A are non-collinear. By axiom C4, there exists a point Q such that $Q \notin a$, $Q \sim_a P$ and $\measuredangle AOQ \cong \beta$. We have two cases. First assume that P, Q and O are collinear. $O \notin (Q, P)$, since $Q \sim_a P$. Thus $Q \in \overrightarrow{OP} - \{O\}$, so $\overrightarrow{OP} = \overrightarrow{OQ}$, by Lemma 4.3. Hence $\measuredangle QOA = \measuredangle POA$. Thus $\alpha \cong \beta$. Assume P, Q and O are non-collinear. Then $P \in \operatorname{int} \measuredangle QOA$ or $Q \in \operatorname{int} \measuredangle POA$ by Lemma 3.8. Then $\measuredangle POA < \measuredangle QOA$ or $\measuredangle QOA < \measuredangle POA$, by axiom C5. By axiom C5 and Lemma 6.8, $\alpha < \beta$ or $\beta < \alpha$. The two cases show that $\alpha \cong \beta$, $\alpha < \beta$, or $\beta < \alpha$. □

A simple but useful result combining axioms C1, C4 and B2' is

Lemma 6.11. *Take an angle α, distinct points on a line $O, A \in l$, a point off the line $Z \notin l$, and distinct points P and Q. Then there exists a point N such that $O \neq N$, $N \notin l$, $Z \not\sim_l N$, $ON \cong PQ$, and $\measuredangle AON \cong \alpha$.*

Proof. Since $Z \neq O$, there exists a point Y with $O \in (Z, Y)$, by axiom B2'. By axiom B1', $O \neq Y$ and Z, O and Y are collinear. Thus $Y \notin l$ by axiom I1, since $O \in l$ and $Z \notin l$. By axiom C4, there exists a point B such that $O \neq B$, $B \notin l$, $B \sim_l Y$, and $\measuredangle AOB \cong \alpha$. By axiom C1, there exists a point $N \in \overrightarrow{OB} - \{O\}$ such that $ON \cong PQ$. By Lemma 4.3, $\overrightarrow{ON} = \overrightarrow{OB}$. Thus $\measuredangle AON = \measuredangle AOB$, and so $\measuredangle AON \cong \alpha$. Now $Z \not\sim_l Y$, since $O \in (Z, Y)$. Hence $Z \not\sim_l N$ by Lemma 3.1 and a short proof by contradiction, since $N \sim_l Y$. □

7. Euclid, Hilbert and the parallel axiom

Hartshorne [11] does an excellent job explaining how Hilbert's work rigorizes book I of Euclid's Elements [5]. Here we fill some gaps in Hartshorne's proofs.

Hartshorne gives the mild Hilbert rigorization [11, p. 97] needed for Euclid's Prop. I.5 (an isosceles triangle has congruent base angles). A proof requiring no Hilbert rigorization [16, §6.2, Thm. 1] is due to Pappus [12, p. 254] in 300 C.E.

Lemma 7.1. *Take a triangle $\triangle ABC$ with $BA \cong BC$. Then $\measuredangle CAB \cong \measuredangle ACB$.*

Proof. $BC \cong BA$ by axiom C2, and $\measuredangle ABC \cong \measuredangle CBA$ by axiom C5. Thus $\triangle ABC \cong \triangle CBA$, by SAS. □

Hartshorne explains how to rigorize Euclid's Prop. I.6, the converse of Lemma 7.1, but it should be noted that Prop. I.6 follows immediately from ASA [6, Prop. 3.17].

Greenberg [6, Prop. 3.22] and Moise [16, §6.2, Thm. 3] give the same Hilbert rigorization of Euclid's Prop. I.8, the SSS Theorem about congruent triangles.

Theorem 7.2. *Take triangles $\triangle ABC$ and $\triangle A'B'C'$ with $AB \cong A'B'$, $AC \cong A'C'$, and $BC \cong B'C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Hartshorne's proof [11, Prop. 10.1] of the SSS Theorem 7.2 is perhaps too sketchy.

Hartshorne's Hilbert rigorization of Euclid's Prop. I.9 [11, p. 100], the existence of angle bisectors, has a gap which we fill in the second paragraph of our proof

Lemma 7.3. *Given an angle $\measuredangle BAC$, there exists a point $F \in \operatorname{int}\measuredangle BAC$ so that $\measuredangle BAF \cong \measuredangle FAC$.*

Proof. There exists a point D with $B \in (A, D)$, by axiom B2'. Thus $D \in \overrightarrow{AB} - \{A\}$, by Lemma 3.14. By axiom C1, there exists a point $E \in \overrightarrow{AC} - \{A\}$ with $AE \cong AD$. Then $\overrightarrow{AD} = \overrightarrow{AB}$ and $\overrightarrow{AE} = \overrightarrow{AC}$, by Lemma 4.3. D, A and E are non-collinear, by axiom I1, since B, A and C are non-collinear. Applying Lemma 7.1 to $\triangle EAD$ yields $\measuredangle DEA \cong \measuredangle EDA$. Furthermore $E \neq D$. Let $h = \overline{DE}$. Thus $A \notin h$, by non-collinearity. By Lemma 6.11, there exists a point F so that $\measuredangle EDF \cong \measuredangle EDA$, $F \notin h$, $A \not\sim_h F$, and $DF \cong DA$. Thus $A \neq F$, by Lemma 3.2, and E, D and F are non-collinear. By the SAS Theorem 6.2, $\triangle EDF \cong \triangle EDA$, since $DE \cong DE$ and $FA \cong FA$ by axiom C2, so $FE \cong AE$ and $\measuredangle FED \cong \measuredangle AED$. By axiom C5, $\measuredangle EDA \cong \measuredangle DEA$, $\measuredangle EDA \cong \measuredangle EDF$, and $\measuredangle DEA \cong \measuredangle DEF$. There exists $G \in h$ with $G \in (A, F)$ since $A \not\sim_h F$. Thus $F \in \overrightarrow{AG} - \{A\}$. Let $v = \overline{AF}$. Then $v \cap h = \{G\}$, by Lemma 2.1, since $A \notin h$.

We must show $D, E \notin v$. Assume (for a contradiction) that $D \in v$. Then $D = G$, since $D \in h$ and $v \cap h = \{G\}$. Thus $D \in (A, F)$. Hence $\measuredangle EDA$ is a right angle, as it is congruent to its supplement $\measuredangle EDF$. By Lemma 6.6, $\measuredangle AED$ and $\measuredangle DEF$ are right angles, and $E \in (A, F)$. Since $v \cap h = \{D\}$, we have $E = D$, since $(A, F) \subset v$ and $E \in h$. This is a contradiction, since $E \neq D$. Hence $D \notin v$. A similar proof shows $E \notin v$. Thus $D, E \notin v$. Hence we have triangles $\triangle FAD$ and $\triangle FAE$.

By axiom C2, $FE \cong AD$, $AD \cong FD$, and $FE \cong FD$. Thus $\triangle FAE \cong \triangle FAD$, by the SSS Theorem 7.2, so $\measuredangle FAE \cong \measuredangle FAD$. By axiom C5, $\measuredangle DAF \cong \measuredangle FAE$. Then $\measuredangle BAF \cong \measuredangle FAC$, since $\overrightarrow{AB} = \overrightarrow{AD}$ and $\overrightarrow{AC} = \overrightarrow{AE}$. We need to prove that $E \not\sim_v D$.

Assume (for a contradiction) that $E \sim_v D$. Then $\overrightarrow{AD} = \overrightarrow{AE}$, by Lemma 6.1, so D, A and E are collinear. This is a contradiction, so $E \not\sim_v D$. Thus there exists a point $H \in v \cap (E, D)$. Then $H = G$, since $(E, D) \subset h$, and $v \cap h = \{G\}$. Thus $G \in (E, D)$, so $G \in \operatorname{int} \measuredangle EAD$, by Lemma 3.6. Hence $F \in \operatorname{int} \measuredangle EAD$, by Lemma 3.7. Then $F \in \operatorname{int} \measuredangle BAC$, by Lemma 3.4 and Lemma 3.5, since $B \in \overrightarrow{AD} - \{A\}$ and $C \in \overrightarrow{AE} - \{A\}$. Thus $F \in \operatorname{int} \measuredangle BAC$ and $\measuredangle BAF \cong \measuredangle FAC$. □

Take distinct points B and C. Hartshorne constructs [11, Prop. 10.2] an isosceles triangle $\triangle BAC$, and then proves the existence of midpoints by using the Crossbar Theorem 3.10, Lemma 7.3 and SAS to find the midpoint of (A, B). This, together with axiom B2', proves Hilbert's axiom II.2, which is stronger than our axiom B2:

II.2. Given two distinct points A and C, there exists points B and D such that $B \in (A, C)$ and $C \in (A, D)$.

Greenberg, who takes II.2 as an axiom himself, gives a much simpler but less appealing proof [6, p. 86, Ex. 6] of II.2 using only axiom B2'.

Hartshorne gives a nice Hilbert rigorization of Euclid's Prop. I.16 [11, Prop. 10.3]

Theorem 7.4. *Take a triangle $\triangle ABC$ with a point D such that $C \in (B, D)$. Then $\measuredangle CAB < \measuredangle ACD$.*

This result (see also [17, Thm. 6.3.2]) is called the Exterior Angle Theorem, because $\measuredangle ACD$, the supplement of $\measuredangle BCA$, is called an *exterior* angle of $\triangle ABC$. We deduce Euclid's Prop. I.17, a useful version of the Exterior Angle Theorem, which follows from [6, Prop. 3.15], which says that vertical angles are congruent:

Lemma 7.5. *Take a triangle $\triangle ABC$ with a point D such that $C \in (B, D)$. Then $\measuredangle ABC < \measuredangle ACD$.*

Proof. By axiom B3', there exists a point E with $C \in (A, E)$. By Theorem 7.4, $\measuredangle ABC < \measuredangle BCE$. The *vertical angles* $\measuredangle BCE$ and $\measuredangle ACD$ are congruent by Lemma 6.6 (i.e. [6, Prop. 3.14]), because they are supplements of $\measuredangle ACB$ and $\measuredangle BCA$, which are congruent by axiom C5. Hence $\measuredangle ABC < \measuredangle ACD$, by Lemma 6.8. □

Hartshorne's Hilbert rigorization of Euclid's Prop. I.7 [11, p. 35 & Ex. 9.4] is insufficient, so we reprove Prop. I.7, adding details to Hartshorne's proof. Afterward we will give a much simpler proof of the result.

Proposition 7.6. *Take distinct points A and B in the plane, and let $a = \overline{AB}$. Take distinct points C and D in the plane with $C, D \notin a$ and $C \sim_a D$, and suppose that $AC \cong AD$. Then $BC \ncong BD$.*

Proof. The points A, B and C are non-collinear, as are D, A and B, by axiom I1, since $A \neq B$, and $C, D \notin a$, which also shows $A \neq C$, $A \neq D$, $B \neq C$ and $B \neq D$. Assume (for a contradiction) that A, D and C are collinear. Then $C \in \overrightarrow{AD} - \{A\}$, since $A \notin (C, D)$, since $C \sim_a D$. Thus $C = D$, by axiom C1, since $AC \cong AD$ and $D \in \overrightarrow{AD} - \{A\}$. This is a contradiction. Hence A, D and C are non-collinear. If B, D and C are collinear, then a similar argument shows that $BC \ncong BD$, since $B \neq D$, and we are done. So assume that B, D and C are non-collinear. Note that $AD \cong AC$ by by axiom C2, and $D \sim_a C$ by Lemma 3.2.

Thus $\square ABCD$ is a tetralateral with $C \sim_a D$. By Lemma 5.5, either $\square ABCD$ is a convex quadrilateral, $C \in \operatorname{int} \triangle DAB$, $\square ABDC$ is a convex quadrilateral, or $D \in \operatorname{int} \triangle ABC$. We have four cases, and prove the first two cases in detail.

Case 1: Suppose $\square ABCD$ is a convex quadrilateral. Then $B \in \operatorname{int} \measuredangle CDA$ and $A \in \operatorname{int} \measuredangle BCD$. $\triangle DAC$ is isosceles, so $\measuredangle CDA \cong \measuredangle DCA$, by Lemma 7.1. $A \in \operatorname{int} \measuredangle DCB$ by Lemma 3.4. $\measuredangle DCA \cong \measuredangle DCA$ and $\measuredangle CDB \cong \measuredangle CDB$ by axiom C5. By Definition 4.16, $\measuredangle DCA < \measuredangle DCB$ and $\measuredangle CDB < \measuredangle CDA$. Thus $\measuredangle CDB < \measuredangle DCB$, by Lemmas 6.8 and 6.9.

Thus $\measuredangle DCB \ncong \measuredangle CDB$, by Lemma 6.7 and axiom C5. Applying Lemma 7.1 to $\triangle CBD$ shows that $BC \ncong BD$, and we are done.

Case 2: Suppose $C \in \operatorname{int}\triangle DAB$. Then $C \in \operatorname{int}\measuredangle DAB$ and $C \in \operatorname{int}\measuredangle ADB$ by Definition 4.16 and Lemma 3.4. $\measuredangle CDA \cong \measuredangle DCA$, by Lemma 7.1. There exists a point E so that $D \in (A, E)$, by axiom B2'. $B \in \operatorname{int}\measuredangle CDE$, by Lemma 3.9. There exists a point $F \in (B, D)$ with $F \in \overrightarrow{AC} - \{A\}$ by the Crossbar Theorem 3.10 and axiom B1'. $F \in \overrightarrow{DB} - \{D\}$ by Lemma 3.14. Then $C \in \operatorname{int}\measuredangle ADF$ by Lemma 3.5. Thus $C \in (A, F)$, by Lemma 3.11. $\measuredangle ADC$ and $\measuredangle CDE$ are supplementary angles, as are $\measuredangle ACD$ and $\measuredangle DCF$. Supplements of congruent angles are congruent [6, Prop. 3.14], so $\measuredangle CDE \cong \measuredangle DCF$, since $\measuredangle ADC \cong \measuredangle ACD$. Thus $\measuredangle CDB < \measuredangle CDE \cong \measuredangle DCF < \measuredangle DCB$, since $B \in \operatorname{int}\measuredangle CDE$ and $F \in \operatorname{int}\measuredangle BCD$. Hence $\measuredangle CDB < \measuredangle DCB$, by Lemmas 6.8 and 6.9. The argument of case 1 shows that $BC \ncong BD$.

Cases 3 and 4: Suppose $\square ABDC$ is a convex quadrilateral or $D \in \operatorname{int}\triangle ABC$. Then $BD \ncong BC$, as we see by switching C and D in the proofs of cases 1 and 2. By axiom C2, $BC \ncong BD$. □

Hartshorne explains that Euclid's proof of Prop. I.7 has a gap to be filled by showing that $A \in \operatorname{int}\measuredangle DCB$ and $B \in \operatorname{int}\measuredangle CDA$ in our case 1, but he ignores the other three cases. Fitzpatrick's literal translation of the Elements `http://farside.ph.utexas.edu/euclid.html` gives only Hartshorne's one case. The version of the Elements we use [5] gives our cases 1 and 2. In what is considered to be the definitive version of the Elements, Heath explains that [12, p. 259–260] a literal translation of Euclid's statement of Prop. I.7 is "apparently obscure" and "vague"[1], that this second case of [5] is due to the 5th Century mathematician Proclus, and that Euclid had "the general practice of giving only one case, and that the most difficult, and leaving the others to be worked out by the reader." For Prop. I.7 this seems a bad practice, because of the Hilbert rigorization needed.

Now we give a simple proof of Euclid's Prop. I.7 which we think is well known.

Proof of Proposition 7.6. Assume (for a contradiction) that $BC \cong BD$. We have triangles $\triangle ACB$ and $\triangle ADB$ because $C, D \notin a$. Then $\triangle ACB \cong \triangle ADB$, by the SSS Theorem 7.2, since $AB \cong AB$, by axiom C2. Hence $\measuredangle BAC \cong \measuredangle BAD$. By axiom C4, $\overrightarrow{AC} = \overrightarrow{AD}$, since $C \sim_a D$. Then $C = D$, by axiom C1, since $AC \cong AD$. Since C and D are distinct, we have a contradiction. Hence $BC \ncong BD$. □

Hartshorne gives Euclid's Prop. I.12 as an exercise [11, Ex. 10.4] with no hints. We add details to Greenberg's proof [6, Prop. 3.16], which only considers Case 2.2.

Lemma 7.7. *Take a line l point $P \notin l$. Then there exists distinct points $E, Q \in l$ such that $\measuredangle PQE$ is a right angle.*

Proof. There exists distinct points $A, B \in l$ by axiom I2. Then $A \neq P$. By Lemma 6.11, there exists a point P' such that $A \neq P'$, $P' \notin l$, $P \not\sim_l P'$, $AP' \cong AP$ and $\measuredangle BAP' \cong \measuredangle BAP$. Thus there exists a point $Q \in l$ such that $Q \in (P, P')$. By axiom C5, $\measuredangle BAP \cong \measuredangle BAP'$. We have two cases: $A = Q$ and $A \neq Q$.

Case 1: Suppose $A = Q$. Then $A \in (P, P')$ and $\measuredangle PAB \cong \measuredangle BAP'$. Thus $\measuredangle PAB$ is a right angle, and we are done.

Case 2: Suppose $A \neq Q$. Then $AQ \cong AQ$ and $AP \cong AP'$ by axiom C2. We have two cases.

Case 2.1: Suppose $A \in (Q, B)$. By Lemma 6.6, $\measuredangle PAQ \cong \measuredangle P'AQ$. Thus $\triangle PAQ \cong \triangle P'AQ$ by the SAS Theorem 6.2. This implies $\measuredangle PQA \cong \measuredangle AQP'$. Hence $\measuredangle PQA$ is a right angle, since $Q \in (P, P')$, and we are done.

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[1]Fitzpatrick's translation is "On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines."

Case 2.2: Suppose $A \notin (Q, B)$. Then $Q \in \overrightarrow{AB} - \{A\}$. Thus $\overrightarrow{AQ} = \overrightarrow{AB}$ by Lemma 4.3. Hence $\measuredangle QAP \cong \measuredangle QAP'$, and so $\measuredangle PAQ \cong \measuredangle QAP'$. Therefore $\measuredangle PAQ$ is a right angle, and we are done. □

Euclid avoids the parallel axiom (axiom P below), by his use the Exterior Angle Theorem and the following two result (Prop. I.18–9) [6, Prop. 4.5, Ex. 4.21].

Lemma 7.8. *Take a triangle $\triangle ABC$ with $AC < AB$. Then $\measuredangle ABC < \measuredangle BCA$.*

Proof. There exists a point $D \in (A, B)$ such that $AC \cong AD$. Then $D \in (B, A)$, so $\overrightarrow{BD} = \overrightarrow{BA}$, by Lemma 4.3. $D \in \operatorname{int}\measuredangle ACB$, by Lemma 3.6. $\measuredangle CDA \cong \measuredangle ACD$ by Lemma 7.1. $\measuredangle CDA < \measuredangle ACB$ by Definition 4.16. $\measuredangle BDC$ and $\measuredangle CDA$ are supplementary angles. $\measuredangle CBD < \measuredangle CDA$, by Lemma 7.5. $\measuredangle CBD < \measuredangle ACB$, by Lemma 6.9. Now we are done, since $\measuredangle CBD = \measuredangle DBC = \measuredangle ABC$, since $\overrightarrow{BD} = \overrightarrow{BA}$. □

Greenberg points out that the next result, Euclid's Prop. I.19, follows from Lemmas 7.8, 7.1, 6.10, and the segment trichotomy property [6, Prop. 3.13] .

Lemma 7.9. *Take a triangle $\triangle ABC$ with $\measuredangle ABC < \measuredangle BCA$. Then $AC < AB$.*

We discuss Hartshorne's treatment of circles [11, §11]. Given a point O and a segment AB, let Γ be the *circle* with center O and radius AB defined as

$$\Gamma = \{C \mid OC \cong AB\}.$$

We say $X \in \operatorname{int}\Gamma$ if $X = O$ or $OX < OA$, and X is in the exterior of Γ if $OX > OA$.

As Hartshorne explains, Euclid's Prop. I.7 says that two circles can not have two intersection points in the same half-plane of the line between their centers. We work Hartshorne's circle convexity exercise [11, Ex. 11.1(a)], which Greenberg proves in an exercise [6, Ex. 4.29]. Our proof is similar but simpler and covers more cases.

Lemma 7.10. *Take points $O \neq R$ and $B \in (A, C)$. Let Γ be the circle with center O and radius OR. Assume $A, C \in \operatorname{int}\Gamma$. Then $B \in \Gamma$.*

Proof. $A = O$ or $OA < OR$, since $A \in \operatorname{int}\Gamma$, and $C = O$ or $OC < OR$, since $C \in \operatorname{int}\Gamma$. $\overrightarrow{AB} = \overrightarrow{AC}$ and $\overrightarrow{CB} = \overrightarrow{CA}$, by Lemma 4.3, since $B \in (C, A)$. We have three cases, and then subcases, depending primarily on whether A, O and C are collinear.

Case 1: Suppose $O = A$. Then $B \in (O, C)$, so $OB < OC$ and $O \neq C$. Thus $OB < OR$, by the transitive property of $<$ [6, Prop. 3.13]. Thus $B \in \Gamma$.

Case 2: Suppose $O = C$. The proof is almost identical to the proof of Case 1.

Case 3: Suppose $O \neq A$ and $O \neq C$. Then $OA < OR$ and $OC < OR$. By the argument of Case 1, it suffices to show that $B = O$, $OB < OA$ or $OB < OC$. We have two cases. (Greenberg only considers Case 3.1.)

Case 3.1: Suppose A, O and C are non-collinear. By the trichotomy property of $<$ [6, Prop. 3.13], $OA < OC$, $OA \cong OC$ or $OC < OA$. We have three cases.

Case 3.1.1: Suppose $OA < OC$. Then $\measuredangle OCA < \measuredangle CAO$, by Lemma 7.8. Thus $\measuredangle BCO < \measuredangle OAB$. By Lemma 7.5, $\measuredangle OAB < \measuredangle OBC$. Then $\measuredangle BCO < \measuredangle OBC$ by Lemma 6.9. Hence $OB < OC$, by Lemma 7.9, and we are done.

Case 3.1.2: Suppose $OA \cong OC$. Then $OC \cong OA$ by axiom C2, so $\measuredangle OCA \cong \measuredangle CAO$, by Lemma 7.1. We are done by the proof of Case 3.1.1, with one change. Having proved $\measuredangle BCO \cong \measuredangle OAB$ and $\measuredangle OAB < \measuredangle OBC$, we deduce $\measuredangle BCO < \measuredangle OBC$ by axiom C5 and the definition of angle ordering.

Case 3.1.3: Suppose $OC < OA$. By the proof of Case 3.1.1, $OB < OA$, and we are done.

Case 3.2: Suppose A, O and C are collinear. We have two cases.

Case 3.2.1: Suppose $B \in (O, A)$ or $B \in (O, C)$. Then $OB < OA$ or $OB < OC$. Then $B \in \Gamma$ as in Case 1, by the transitive property of $<$.

Case 3.2.2: Suppose $B \notin (O, A)$ and $B \notin (O, C)$. Let $l = \overline{AC}$. Then $B, O \in l$, so $O \in \overrightarrow{BA} \cap \overrightarrow{BC}$. Thus $O \in \{B\}$, by Lemma 4.5, so $O = B$. Hence $B \in \Gamma$. □

We say lines l and m in a plane are *parallel*, and write $l \parallel m$, if $l \cap m = \emptyset$. The Exterior Angle Theorem 7.5 allows us to construct parallel lines.

Take lines determined by distinct points $l = \overline{EA}$ and $m = \overline{BC}$. Assume $A \neq B$. The line $t = \overline{AB}$ is called a *transversal* which *cuts* l and m. Assume also that $E, C \notin t$ and $C \not\sim_t E$. Then $\measuredangle EAB$ and $\measuredangle CBA$ are called *alternate interior angles* [17, p. 107]. Venema [17, Thm. 6.8.1] gives a nice proof of the following result, Euclid's Prop. I.27, which is called the Alternate Interior Angles Theorem.

Theorem 7.11. *Let E, A, B, C, l, m, t be as above, with $\measuredangle EAB \cong \measuredangle CBA$. Then $l \parallel m$.*

Proof. E, A and B are non-collinear, by axiom I1, since $E \notin t$ and $A \neq B$. Similarly C, B and A are non-collinear, since $C \notin t$. Thus $B \notin l$ and $A \notin m$, by axiom I1, since $E \neq A$ and $B \neq C$. Assume (for a contradiction) that $l \nparallel m$, so there exists a point $G \in l \cap m$. Then $G \neq A$ and $G \neq B$. The points A, G and B are non-collinear, by Lemma 2.3. Similarly B, G and A are non-collinear. Thus $G \notin t$. By Lemma 3.2, $E \not\sim_t C$, since $C \not\sim_t E$. Note that $C, G \in m - \{B\}$ and $E, G \in l - \{A\}$.

Assume (for a contradiction) that $G \sim_t E$. Then $A \notin (G, E)$, and $G \in \overrightarrow{AE} - \{A\}$. Thus $\overrightarrow{AG} = \overrightarrow{AE}$, by Lemma 4.3. $C \not\sim_t G$ by Lemma 3.2 and a short proof by contradiction. Thus $C \notin \overrightarrow{BG} - \{B\}$, by Lemma 3.13 and a short proof by contradiction. Thus $B \in (C, G)$, since $C \neq B$. By Theorem 7.4, $\measuredangle GAB < \measuredangle CBA$. Hence $\measuredangle EAB < \measuredangle CBA$. This is a contradiction, by Lemma 6.7. Hence $G \not\sim_t E$.

By axiom C5, $\measuredangle CBA \cong \measuredangle EAB$. We obtain a contradiction by the argument above, switching the pairs l and m, A and B, and E and C. This switch works because of the symmetry in our lemma. Hence $l \parallel m$. □

Hartshorne [11, p. 102] erroneously describes Euclid's proof of Theorem 7.11 as "ok." Euclid can not even define alternate interior angles. We prove part of Euclid's Prop. I.21, as Hartshorne [11, Thm. 10.4] fails to explain that Euclid's proof requires Hilbert rigorization.

Lemma 7.12. *Take a triangle $\triangle ABC$ with a point $D \in \operatorname{int} \triangle ABC$. Then $\measuredangle ABC < \measuredangle CDA$.*

Proof. $D \in \operatorname{int} \measuredangle BAC$ and $D \in \operatorname{int} \measuredangle CBA$ by Definition 4.16 and Lemma 3.4. There exists a point $E \in (B, C)$ with $E \in \overrightarrow{AD} - \{A\}$ by the Crossbar Theorem 3.10. Then $\overrightarrow{BE} = \overrightarrow{BC}$ by Lemma 4.3. Thus $D \in \operatorname{int} \measuredangle ABE$, by Lemmas 3.5 and 3.4. $D \in (A, E)$ by Lemma 3.11, since A, D and E are collinear. $\overrightarrow{ED} = \overrightarrow{EA}$ by Lemma 4.3, since $D \in (E, A)$. $\measuredangle ABE < \measuredangle AEC$ by Lemma 7.5, since $E \in (B, C)$. Then $\measuredangle ABC < \measuredangle CED$. Thus $\measuredangle CED < \measuredangle CDA$ by Lemma 7.5, since $D \in (E, A)$. We are done by Lemma 6.9. □

As Hartshorne points out [11, Ex. 10.10], Lemma 7.11 implies, without the parallel axiom, that a quadrilateral with opposite sides congruent is a parallelogram:

Proposition 7.13. *A quadrilateral $\square ABCD$ with $AB \cong CD$ and $BC \cong AD$ is a parallelogram.*

Proof. We use the conventions of §5. Since $\square ABCD$ is a tetralateral, we have lines $a = \overline{AB}$, $b = \overline{BC}$, $c = \overline{CD}$, $d = \overline{DA}$, $l = \overline{AC}$, and $m = \overline{BD}$, with $B \notin l$, $D \notin l$, $A \notin m$, and $C \notin m$. By Lemma 5.6, either $B \not\sim_l D$ or $A \not\sim_m C$. We have two cases.

Case 1: Suppose $B \not\sim_l D$. By axiom C2, $D \not\sim_l B$. By the SSS Theorem 7.2, $\triangle ABC \cong \triangle CDA$, so $\measuredangle BCA \cong \measuredangle DAC$ and $\measuredangle CAB \cong \measuredangle ACD$. The transversal l cuts b and d. Then $b \parallel d$, by Theorem 7.11, since $D \not\sim_l B$ and $\measuredangle BCA \cong \measuredangle DAC$. Similarly $a \parallel c$, since the transversal l also cuts a and c, and $\measuredangle BAC \cong \measuredangle DCA$. Hence $\square ABCD$ is a parallelogram.

Case 2: Suppose $A \not\sim_m C$. By SSS, $\triangle BCD \cong \triangle DAB$, so $\measuredangle CDB \cong \measuredangle ABD$ and $\measuredangle CBD \cong \measuredangle ADB$. By the argument of Case 1 applied to the transversal m, $c \parallel a$ and $b \parallel d$, and $\square ABCD$ is a parallelogram. □

A typical reader of Hartshorne's could not be expected to work this exercise [11, Ex. 10.10], as Hartshorne does not mention the betweenness issues we dealt with, and merely says "Join the midpoints of AB and CD, then use (I.27)." We also prove

Lemma 7.14. *A quadrilateral $\square ABCD$ with $\measuredangle ABC \cong \measuredangle CDA$ and $\measuredangle DAB \cong \measuredangle BCD$ is a parallelogram.*

Proof. We again use the conventions of §5. Since $\square ABCD$ is a tetralateral, the points A, B, C and D are distinct, and the sets $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$ and $\{B, C, D\}$ are non-collinear. Thus $A, B \notin c$, $A, D \notin b$ and $B, C \notin d$. By axiom C5, $\measuredangle CDA \cong \measuredangle ABC$ and $\measuredangle BCD \cong \measuredangle DAB$. Assume (for a contradiction) that $A \in \operatorname{int} \triangle BCD$. Then $\measuredangle DAB < \measuredangle BCD$ by Lemma 7.12. This is a contradiction by Lemma 6.7. Hence $A \notin \operatorname{int} \triangle BCD$. Similarly, $B \notin \operatorname{int} \triangle CDA$, $C \notin \operatorname{int} \triangle DAB$ and $D \notin \operatorname{int} \triangle ABC$. Therefore $\square ABCD$ is a convex quadrilateral, by Lemma 5.6. Thus $A \in \operatorname{int} \measuredangle BCD$, $B \in \operatorname{int} \measuredangle CDA$, $C \in \operatorname{int} \measuredangle DAB$ and $D \in \operatorname{int} \measuredangle ABC$. Hence $B \sim_c A$, $B \sim_d C$ and $A \sim_b D$.

Assume (for a contradiction) that $a \nparallel c$. There exists a point $G \in a \cap c$. Then $G \neq A$, because $A \notin c$. Similarly, $G \neq B$, $G \neq C$ and $G \neq D$. Then B, G and C are non-collinear, by axiom I1, because $C \notin a$. Similarly A, D and G are non-collinear. Thus $G \notin b$ and $G \notin d$. Now $G \notin (B, A)$, since $B \sim_c A$. By axiom B3', $A \in (G, B)$ or $B \in (A, G)$.

Assume (for a contradiction) that $A \in (G, B)$. Then $A \in (B, G)$, so $\overrightarrow{BA} = \overrightarrow{BG}$, by Lemma 4.3. Thus $B \not\sim_d G$, so $C \not\sim_d G$, by Lemma 3.2. Hence $C \notin \overrightarrow{DG}$, by Lemma 3.13, and a short proof by contradiction, since $G \notin d$ and $C \neq D$. Hence $D \in (C, G)$, since $C, D, G \in c$. There exists a point M such that $C \in (D, M)$, by axiom B2'. Thus $C \in (M, G)$ by Lemma 4.6, since $C \in (M, D)$ and $D \in (G, C)$. Hence $C \in (G, M)$. Now $\measuredangle BCD$ has supplement $\measuredangle MCB$, and $\measuredangle DAB$ has supplement $\measuredangle GAD$. Thus $\measuredangle MCB \cong \measuredangle GAD$, by Lemma 6.6. Hence $\measuredangle GBC < \measuredangle MCB$ and $\measuredangle GAD < \measuredangle CDA$, by Theorem 7.4. Then $\measuredangle GBC < \measuredangle GAD$, by Lemma 6.8, so $\measuredangle GBC < \measuredangle CDA$, by Lemma 6.9. Hence $\measuredangle ABC < \measuredangle CDA$. This is a contradiction by Lemma 6.7. Hence $A \notin (G, B)$.

Similarly $B \notin (A, G)$. This contradicts axiom B3'. Hence $a \parallel c$. Since $\square BCDA$ is also a quadrilateral, this argument shows that $b \parallel d$. Hence $\square ABCD$ is a parallelogram. □

Greenberg proves the Alternate Interior Angles Theorem [6, Thm. 4.1] by using axiom C4, and then deduces from it the Exterior Angle Theorem [6, Thm. 4.2], the existence of midpoints and angle bisectors [6, Prop. 4.3–4]. We recommend these interesting proofs.

For the rest of the paper we adopt Venema's version of Euclid's parallel axiom.

P. Given a line l in the plane and a point $P \notin l$, there exists a unique line m in the plane such that $P \in m$ and $m \parallel l$.

We can now prove the converse of Theorem 7.11, Euclid's Prop. I.29, which Hartshorne sketches a proof of [11, p. 113] without mentioning any Hilbert betweenness issues.

Lemma 7.15. *Take points and lines $A, E \in l$, $B, C \in m$ and $A, B \in t$, with $A \neq E$, $B \neq C$, $A \neq B$, and $E, C \notin t$. Assume $C \not\sim_t E$ and $l \parallel m$. Then $\measuredangle EAB \cong \measuredangle CBA$.*

Proof. C, B and A are non-collinear, by axiom I1, because $C \notin t$. Hence $A \notin m$. By Lemma 6.11, there exists a point D such that $A \neq D$, $D \notin t$, $C \not\sim_t D$, $AD \cong AE$ and $\measuredangle BAD \cong \measuredangle CBA$. Let $k = \overline{AD}$. Then $k \parallel m$, by Theorem 7.11. Thus $k = l$, by axiom P. Now $D \sim_t E$, by Corollary 4.13, so $A \notin (D, E)$. Then $D \in \overrightarrow{AE} - \{A\}$, since $D, E, A \in l$. Thus $\overrightarrow{AD} = \overrightarrow{AE}$, by Lemma 4.3. Hence $\measuredangle EAB \cong \measuredangle CBA$. □

We prove the hard part of the triangle sum theorem (Lemma 10.1), which Hartshorne merely says is "without difficulty" [11, p. 113]. Venema and Greenberg [6, Prop. 4.11] give it as an exercise, Greenberg giving the hint of the standard picture. But the proof is tricky enough to merit writing up, and ours does not use the Crossbar theorem 3.10.

Lemma 7.16. *Take a triangle $\triangle ABC$. Then there exists points E and F with $B \in (E, F)$, $C \in \operatorname{int} \measuredangle ABF$, $\measuredangle EBA \cong \measuredangle CAB$, and $\measuredangle CBF \cong \measuredangle BCA$.*

Proof. Let $l = \overline{AC}$, $x = \overline{AB}$ and $y = \overline{BC}$. Then $C \notin x$, since A, B and C are non-collinear. By Lemma 6.11, there exists a point E such that $B \neq E$, $E \notin x$, $C \nsim_x E$, $BE \cong AB$, and $\measuredangle ABE \cong \measuredangle CAB$. Let $m = \overline{EB}$. Hence $m \parallel l$, by Theorem 7.11, because $\measuredangle EBA$ and $\measuredangle CAB$ are congruent alternate interior angles when m and l are cut by the transversal x. Thus $m \cap l = \emptyset$, so $A, C \notin m$. There exists $F \in m$ with $B \in (E, F)$, by axiom B2'. A, B and F are non-collinear, by axiom I1, because $A \notin m$ and $B \neq F$. Thus $F \notin x$. Now $E \nsim_x F$ and $E \nsim_y F$, since $B \in (E, F)$. Thus $C \sim_x F$, by Corollary 4.13. $m \cap (C, A) = \emptyset$, because $m \cap l = \emptyset$ and $(C, A) \subset l$, so $C \sim_m A$. Thus $C \in \operatorname{int} \measuredangle ABF$, so $C \in \operatorname{int} \measuredangle FBA$, by Lemma 3.4. Hence $A \in \operatorname{int} \measuredangle CBE$, by Lemma 3.9, since $B \in (F, E)$. Thus $A \notin y$ and $A \sim_y E$. Now $E \notin y$, by axiom I1, because $E \neq B$ and $C \notin m$. Similarly $F \notin y$. By Lemma 3.2, $E \sim_y A$. Thus $A \nsim_y F$, by Lemma 3.2 and a short proof by contradiction. Hence $\measuredangle FBC \cong \measuredangle ACB$, by Lemma 7.15, because the transversal y cuts the parallel lines m and l. Thus $\measuredangle CBF \cong \measuredangle BCA$. □

8. 3-dimensional geometry

Hartshorne constructs the 5 Platonic solids and the 13 Archimedean solids [11, Ch. 8], but does not "take the time to set up axiomatic foundations for this solid geometry," and recommends Euclid's book XI [5]. We explain Euclid's work with Hilbert's 3-dimensional axioms [17, App. B1]. We give the SMSG [1, Thm. 8.2] proof of Euclid's Proposition XI.4, which improves on Euclid's proof by not using the converse of the Pythagorean theorem. Following the SMSG insight, we similarly improve Euclid's proofs of his Propositions XI.6 and XI.8.

We explain Hilbert's 3-dimensional axioms [17, App. B1]. We are given a set of *points* called *Space*. We are given subsets of Space called *lines* and *planes*. Points in the set Space are called *coplanar* if there is some plane in Space containing them. All but two of our axioms for the set Space together with its subsets are

- **J1.** Given distinct points A and B in Space, there is a unique line l such that $A \in l$ and $B \in l$.
- **J2.** Given a line l in Space, there exists two distinct points A and B in the plane such that $A \in l$ and $B \in l$.
- **I4.** If the points A, B and C are not contained in some line, there exists a unique plane α containing A, B and C.
- **I5.** Take a plane α and distinct points $A, B \in \alpha$, and let l be the unique line containing A and B. Then $l \subset \alpha$.
- **I6.** If two distinct planes intersect, their intersection is a line.
- **I7.** There exists four distinct non-coplanar points that do not lie on a line.

Axiom I5 requires axiom J1. A plane α in Space is a set of points, as it is a subset of Space, and α contains lines, the lines in Space that are subsets of α. By axioms J1, J2 and I5, α satisfies axioms I1 and I2. We can now use the notation $\overline{AB}$ unambiguously to mean the line guaranteed by axiom J1 or axiom I1. We assume every plane α has a betweenness relation $*$. We assume the axiom (M for model)

- **M1.** Every plane α in Space satisfies axioms I3, B1–4 and P.

The results of §3 and §4 now hold in the plane α, and define segments, rays, angles and half-planes in α. We assume our other model axiom

- **M2.** Every plane α in Space satisfies axioms C1–6.

Thus every plane α in Space satisfies axioms I1–3, B1–4, C1–6 and P.

Our axioms differ from Hilbert's, because we did not want to now change our earlier axioms, and claim, "an easy modification of our proofs show our earlier results hold in any plane in Space." The wording of most of our earlier axioms would remain the same, but the meanings would change, as now the points would be in Space, rather than in the plane. Our axiom B4 would be now changed to

- Take a plane α, a line $l \subset \alpha$, and non-collinear points $A, B, C \in \alpha - l$. If there exists $D \in l$ so that $A * D * C$, there exists an $X \in l$ such that $A * X * B$ or $B * X * C$.

To avoid this trouble, we do not now modify our earlier axioms. Another option we did not take is correctly stating Hilbert's axioms from the outset. Our approach seems suited for learning plane geometry first and then 3-dimensional geometry.

Suppose distinct lines l and m intersect. By axioms I4 and I5, they determine a plane. We say l and m are *perpendicular*, and write $l \perp m$, if they meet at a right angle in this plane. Take a plane α, a line l and a point O with $\alpha \cap l = \{O\}$. We say l is *perpendicular* to α, and write $l \perp \alpha$, if l is perpendicular to every line m such that $O \in m \subset \alpha$. We prove Euclid's [5, Prop. XI.4], following [1, Thm. 8.2].

Lemma 8.1 (Euclid)**.** *Take a plane α, lines p, q, n and a point O such that $p \subset \alpha$, $q \subset \alpha$, $p \cap q = \{O\}$ and $n \cap \alpha = \{O\}$. If $n \perp p$ and $n \perp q$, then $n \perp \alpha$.*

Proof. Take a line $x \subset \alpha$ through O, with $x \neq p$ and $x \neq q$. Take $X \in x - \{O\}$ so that $x = \overline{OX}$. Then $X \notin p$ and $X \notin q$. Choose points $P, P' \in p$ so that $O \in (P, P')$. Since $P \not\sim_q P'$, then $X \not\sim_q P$ or $X \not\sim_q P'$, by Lemma 3.2. We may assume that $X \not\sim_q P$, switching P and P' if necessary. Thus there is a point $Q \in (P, X) \cap q$.

Take points $N, N' \in n$ so that $O \in (N, N')$ and $NO \cong N'O$. By axiom C2, $OP \cong OP$ and $OQ \cong OQ$. We have right angles $\measuredangle NOP$ and $\measuredangle NOQ$, since $n \perp p$ and $n \perp q$. Thus $\measuredangle NOP \cong \measuredangle N'OP$ and $\measuredangle NOQ \cong \measuredangle N'OQ$. Therefore $\triangle NOP \cong N'OP$ and $\triangle NOQ \cong N'OQ$ by the SAS Theorem 6.2. Hence $NP \cong N'P$ and $NQ \cong N'Q$. Thus $\triangle NPQ \cong \triangle N'PQ$, by SSS [6, Prop. 3.22], so $\measuredangle NPQ \cong \measuredangle N'PQ$. Hence $\measuredangle NPX \cong \measuredangle N'PX$, since $\overrightarrow{PQ} = \overrightarrow{PX}$, since $P \notin (Q, X)$. Hence $\triangle NPX \cong \triangle N'PX$ by SAS, so $NX \cong NX'$. Thus $\triangle NOX \cong \triangle N'OX$ by SSS, so $\measuredangle NOX \cong \measuredangle N'OX$. So $\measuredangle NOX$ is a right angle, and $n \perp x$. Therefore $n \perp \alpha$. □

We prove [5, Prop. XI.8] without the converse of the Pythagorean theorem.

Lemma 8.2 (Euclid)**.** *Take planes α and β and lines $l, m \subset \beta$ such that $l \parallel m$ and $l \perp \alpha$. Then $m \perp \alpha$.*

Proof. Since $l \perp \alpha$, α and β are distinct planes, so they intersect in a line $z = \alpha \cap \beta$, by axiom I5. Take $A \in l \cap \alpha$. Then $z \cap l = \{A\}$, by axiom I1, since $l \perp z$. By axiom P, $m \cap z \neq \emptyset$, or else lines z and l are distinct lines through A parallel to m. Then $m \perp z$, by the converse to the Alternate Interior Angles Theorem [17, Thm. 6.8.1], since $l \parallel m$. Take $B \in m \cap z$. Then $z = \overline{AB}$. Choose $C \in m - \{B\}$, and then $C' \in m$ with $B \in (C, C')$ and $BC \cong BC'$. Then $\triangle ABC$ and $\triangle ABC'$ are congruent right triangles by the SAS Theorem 6.2, so $CA \cong C'A$. Choose $X \in \alpha$ such that $\measuredangle XAB$ is a right angle, and let $n = \overline{AX}$. Then $n \perp z$, and $n \perp l$, since $l \perp \alpha$. Thus $n \perp \beta$, by Lemma 8.1, since $z \cap l = \{A\}$. Thus $\measuredangle CAX$ and $\measuredangle C'AX$ are right angles. Thus $\triangle CAX \cong \triangle C'AX$ by SAS, and so $CX \cong C'X$. Hence $\triangle CBX \cong \triangle C'BX$ by SSS, so the supplementary angles $\measuredangle CBX$ and $\measuredangle C'BX$ are congruent. Hence $\measuredangle CBX$ is a right angle. Let $p = \overline{BX}$, so $m \perp p$. Then $p \cap z = \{B\}$, by Lemma 2.1, since $X \notin z$. Then $m \perp \alpha$, by Lemma 8.1. □

Using Lemma 8.2, Euclid proves [5, Prop. XI.11] that given a plane α and a point $A \notin \alpha$, there exists a line $l \ni A$ such that $l \perp \alpha$. Using his Proposition XI.11, Euclid proves [5, Prop. XI.12] that given a point $B \in \alpha$, there exists a line $m \ni B$ such that

$m \perp \alpha$. Euclid's simple proof by contradiction proves [5, Prop. XI.13] that given $B \in \alpha$, there exists at most one line $m \ni B$ such that $m \perp \alpha$. We now remove the converse of the Pythagorean theorem from the proof of a third result of Euclid [5, Prop. XI.6], using Lemma 8.2 and ideas of its proof.

Lemma 8.3 (Euclid)**.** *Take a plane α and distinct lines l and m with $l \perp \alpha$ and $m \perp \alpha$. Then $l \parallel m$.*

Proof. There exists a points $A, B \in \alpha$ such that $l \cap \alpha = \{A\}$ and $m \cap \alpha = \{B\}$. Let $z = \overline{AB}$. By axiom I4 there is a unique plane β containing the lines l and z. There exists a line $n \subset \beta$ such that $B \in n$ and $n \perp z$. Then $l \parallel n$ by the Alternate Interior Angles Theorem [17, Thm. 6.8.1]. Then $n \perp \alpha$ by Lemma 8.2. Then $m = n$, by Euclid's Proposition XI.13, since $B \in n$. Therefore $l \parallel m$. □

9. Birkhoff's alternative to Hilbert's work

Venema uses Birkhoff's [4] ruler and protractor postulates [17, Axioms 5.4.1 & 5.6.2], which measure segments and angles with the real line $\mathbb{R}$. Venema's book is primarily a text on Hilbert's axioms, where Birkhoff's two real line axioms relieve us of the labor [11, §4] of using Hilbert's axioms to construct the real line. We explain how Birkhoff's ingenious ideas give a rigorous treatment of geometry different from Hilbert's, following MacLane [15] improvements on Birkhoff's work.

We use MacLane's [15] set $\mathbb{R}/360$ of equivalence classes $[x]$ of points $x \in \mathbb{R}$, under the equivalence relation $x \sim x + k360$ for any integer k.

Our first four Birkhoff-MacLane type postulates are

BM1. There is a unique line containing any two distinct points A and B.

BM2. For any line l, there is a 1-1 correspondence from $\mathbb{R}$ to l. If $x, y \in \mathbb{R}$ map to points $P, Q \in l$, then the distance from P to Q is $d(PQ) := |x - y|$. Given segments PQ and AB, we say that $PQ \cong AB$ iff $d(PQ) = d(AB) \in \mathbb{R}$.

BM3. There is a 1-1 correspondence from $\mathbb{R}/360$ to the set of rays $\overrightarrow{OP}$ starting at O. If $\alpha, \beta \in \mathbb{R}/360$ map to $\overrightarrow{OA}, \overrightarrow{OB}$, then $\measuredangle AOB := \alpha - \beta \in \mathbb{R}/360$. Two angles are said to be congruent if they are equal in $\mathbb{R}/360$.

BM4. $\measuredangle AOB = [180]$ iff $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays on a line.

Birkhoff defines betweenness using the ruler postulate: B is between A and C iff A, B and C are distinct collinear points and $d(AC) = d(AB) + d(BC)$. Given two points A and B, we have the segment AB and the open interval (A, B) as before.

Birkhoff's angles satisfy $\measuredangle AOB = -\measuredangle BOA$, and are called *directed* angles.

Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if there is a positive real number k and number $\epsilon = \pm 1$ such that $\measuredangle ABC = \epsilon \measuredangle A'B'C'$, $\measuredangle BCA = \epsilon \measuredangle B'C'A'$, $\measuredangle CAB = \epsilon \measuredangle C'A'B'$, $kd(AB) = d(A'B')$, $kd(BC) = d(B'C')$, $kd(CA) - d(C'A')$. The triangles are congruent if this holds with $k = 1$. MacLane's SAS postulate is

BM5. The triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if $\measuredangle ABC = \epsilon \measuredangle A'B'C'$, $d(AB) = kd(A'B')$ and $d(BC) = kd(B'C')$, for $\epsilon = \pm 1$ and $k \in \mathbb{R}$ positive.

For any $\alpha \in \mathbb{R}/360$ we have $\alpha = [x]$ for some $x \in [-180, 180]$, and x will be unique unless $\alpha = [180]$, in which case x must be either 180 or -180. So for any $\alpha \in \mathbb{R}/360$, we have $\alpha = [x]$ where exactly one of the four statements is true: $x = 0$, $x = 180$, $x \in (0, 180)$, and $x \in (-180, 0)$. By axiom BM3, $\measuredangle AOB = [0]$ iff $\overrightarrow{OA} = \overrightarrow{OB}$. Thus by axiom BM4, if A, O and B are non-collinear points, then $\measuredangle AOB = [x]$ where exactly one of the statements $x \in (0, 180)$ and $x \in (-180, 0)$ is true. MacLane calls $\measuredangle AOB$ a *proper* angle if $x \in (0, 180)$, and an *improper* angle if $x \in (-180, 0)$, and orders the proper angles by the order of $(0, 180)$. We will abuse notation by replacing $[x]$ by its preferred representative $x \in (-180, 180]$.

Birkhoff's protractor postulate is stronger than our axiom BM3, and MacLane defines a separate axiom which amounts to the crossbar theorem and its converse.

BM6. Let $\measuredangle AOB$ be proper. If $0 < \measuredangle AOC < \measuredangle AOB$, then $\overrightarrow{OC} \cap (A, B) \neq \emptyset$. Conversely, if D is between A and B, then $0 < \measuredangle AOD < \measuredangle AOB$.

Birkhoff's directed angles allow a short proof of the angle addition theorem [4, p. 332]. Given rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, axiom BM3 implies $\measuredangle AOB + \measuredangle BOC = \measuredangle AOC$, regardless of betweenness of rays. Notice that the proof Lemma 6.5 is much more difficult. Birkhoff [4, Thm. 5] also has no difficulty summing the angles of a triangle. He uses axiom BM5 and a proof of existence of a centroid (divide $\triangle ABC$ into four congruent triangles similar to $\triangle ABC$ with scale factor $\frac{1}{2}$) to show

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 180. \tag{1}$$

But now we pay for using directed angles, having to work quite hard to prove Birkhoff's result [4, 3, Thm. 5], whose proof MacLane improves on [15, Thm. 6]

Lemma 9.1. *Given a triangle $\triangle ABC$, either all three angles are proper, or else all three angles are improper.*

Proof. Assume that $\triangle ABC$ has both a proper and an improper angle. We will obtain a contradiction. We may assume that $\measuredangle ABC$ is proper and that $\measuredangle CAB$ is improper. By (1) we know that $\measuredangle ABC + \measuredangle CAB \neq 0$, for otherwise $\measuredangle BCA = 180$, but this violates axiom BM4, as A, B and C are non-collinear. We may assume that $\measuredangle ABC + \measuredangle CAB \in (0, 180)$, so that $0 < -\measuredangle CAB < \measuredangle ABC$. By axiom BM6, there exists a point $X \in (A, C)$ so that $\measuredangle ABX = -\measuredangle CAB$. Applying (1) to the triangle $\triangle ABX$, we have $\measuredangle ABX + \measuredangle BXA + \measuredangle XAB = 180$. But $\measuredangle XAB = \measuredangle CAB$, so $\measuredangle BXA = 180$. As A, B and X are also non-collinear, this violates axiom BM4. Thus $\triangle ABC$ can not have both a proper and an improper angle. □

Birkhoff shows any model of his axioms is essentially the Cartesian (x, y) plane $\mathbb{R}^2$. Hartshorne [11, §21] also proves that any model of his collection of Hilbert's axioms is $\mathbb{R}^2$, and uses Hilbert theory to construct a version of $\mathbb{R}$ geometrically.

Birkhoff's protractor postulate has an additional requirement that a function to be continuous. Unfortunately continuous functions taking values in $\mathbb{R}/360$ are only treated in a Point-set Topology course, long after Calculus. Birkhoff only uses his continuity requirement in his proof of Lemma 9.1, which contained a mistake, and his erratum [3] is also very difficult to read. MacLane used his axiom BM6 to show that a function is continuous, and used this to give a clean version of Birkhoff's proof. Our proof of Lemma 9.1 involves no mention of continuous functions.

10. Venema's axioms and US Geometry textbooks

We now adopt Venema's axioms [17, App. C]: axiom BM2, his Ruler Postulate; his SAS axiom (our SAS Theorem 6.2); his Protractor Postulate, this version of axiom BM3.

V. For three distinct points A, O and B with $\overrightarrow{OA}$ and $\overrightarrow{OB}$ not opposite rays, there is a real number $\mu(\measuredangle AOB)$, called the measure of $\measuredangle AOB$, such that the following conditions are satisfied.
- (a) $0^\circ \leq \mu(\measuredangle AOB) < 180^\circ$.
- (b) $\mu(\measuredangle AOB) = 0$ iff $\overrightarrow{OA} = \overrightarrow{OB}$.
- (c) For each real number $r \in [0, 180)$, and for each half-plane bounded by $\overline{OA}$ there exists a unique ray $\overrightarrow{OE}$ such that $E \in H$ and $\mu(\measuredangle AOB) = r^\circ$.
- (d) If $D \in \operatorname{int} \measuredangle AOB$ then $\mu(\measuredangle AOB) = \mu(\measuredangle AOD) + \mu(\measuredangle DOB)$.

Venema calls (d) the Angle Addition Postulate. Venema does not define degree, as in r°, and degrees are only used in this paper to synchronize with [2]. It would be better to always write r instead of r°. If one wants to use degrees, one needs to remember that one degree is the real number $\pi/180$, so that $r^\circ := r\pi/180$.

It's an exercise of Venema that his axioms imply our axioms I1–3, B1–4, C1–6, where two angles are called congruent if their measures are equal, and betweenness is following Birkhoff's definition. So all our earlier results still hold.

Venema partly leaves this well-known result as an exercise [17, Thm. 7.2.3].

Lemma 10.1. *The sum of the measures of a triangle* $\triangle ABC$ *is* 180°.

Proof. Let $l = \overline{AC}$, $x = \overline{AB}$ and $y = \overline{BC}$, and let m be the unique (by axiom P) line such that $B \in m$ and $m \parallel l$. By Lemma 7.16, there exists points $E, F \in m - \{B\}$ with $B \in (E, F)$, $E \not\sim_x C$, $F \not\sim_y A$ and $C \in \operatorname{int} \measuredangle ABF$. Let $\measuredangle 1 = \measuredangle EBA$, $\measuredangle 2 = \measuredangle ABC$ and $\measuredangle 3 = \measuredangle CBF$. By Lemma 7.16, $\measuredangle 1 \cong \measuredangle CAB$ and $\measuredangle 3 \cong \measuredangle BCA$. Thus $\mu\measuredangle 1 = \mu\measuredangle CAB$, and $\mu\measuredangle 2 + \mu\measuredangle 3 = \mu\measuredangle FBA$, by the Angle Addition Postulate. Since $B \in (E, F)$, $\measuredangle FBA$ and $\measuredangle 1$ are supplementary angles, and so form a *linear pair*. By Venema's Linear Pair Theorem [17, Thm. 5.7.18], $\mu\measuredangle 1 + \mu\measuredangle FBA = 180^\circ$. Hence $\mu\measuredangle 1 + \mu\measuredangle 2 + \mu\measuredangle 3 = 180^\circ$. Thus $\mu\measuredangle CAB + \mu\measuredangle ABC + \mu\measuredangle BCA = 180^\circ$. □

The UCSMP high school Geometry text [2] makes an effort at mathematical rigor, but fails to prove the triangle sum theorem and two parallelogram results. We contend there is little point in an axiomatic Geometry course unless the proofs are rigorous.

All widely used US Geometry texts seem to follow Birkhoff by assuming that the real line measures distances and angles. For UCSMP these are postulates Number Line Assumption, Distance Postulate and Angle Measure Postulate. UCSMP does not however have a postulate like Venema's [17, p. 408] Plane Separation Postulate, our Proposition 4.14.

UCSMP [2, p. 116] gives an imprecise definition of the interior of a triangle

- Every non-zero angle separates the plane into two nonempty sets other than the angle itself. If the angle is not a straight angle, exactly one of those sets is convex. This convex set is called the *interior of the angle*. The nonconvex set is called the *exterior of the angle*. For a straight angle, both sets are convex and either set may be considered the interior.

This definition has no precise meaning. The last sentence about a straight angle amounts to Venema's Plane Separation Postulate, and there is no such UCSMP postulate. UCSMP defines convex [2, p. 62], but there is no justification for the last sentence of the above definition, and no proof that a non-straight angle separates the plane into two sets, one convex and the other nonconvex.

UCSMP [2, p. 126] gives a correct definition of adjacent angles, modulo their imprecise definition of the interior of a triangle.

- Two nonstraight angles are adjacent if a common side is in the interior of the angle formed by the noncommon sides.

UCSMP [2, p. 127] correctly states their Angle Addition Assumption postulate, modulo the new adjacent angle imprecision.

- If $\measuredangle AVC$ and $\measuredangle CVB$ are adjacent, then $\mu\measuredangle AVC + \mu\measuredangle CVB = \mu\measuredangle AVB$.

UCSMP [2, p. 271] gives an imprecise definition of alternate interior angles.

- We say that $\measuredangle 4$ and $\measuredangle 6$ are alternate interior angles because they are interior angles on opposite sides of the transversal and they have different vertices.

Opposite side here has no precise meaning since UCSMP has no plane separation postulate. Thus their Corresponding Angles Postulate [2, p. 146] is also imprecise. Otherwise UCSMP has an adequate proof [2, p. 271] that if two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

Let's turn to the UCSMP [2, p. 285] proof of Lemma 10.1. It's same as the proof we gave, except for the missing Plane Separation Postulate, Crossbar Theorem, and rigorous definitions of angle interiors and alternate interior angles. Therefore UCSMP does not have a rigorous proof of the triangle sum theorem Lemma 10.1.

UCSMP also falsely claims that the parallelogram results Propositions 7.13 and 7.14 are trivial, and seems to be making the unstated assumption that the quadrilateral is convex. UCSMP's proof of Proposition 7.14 uses their quadrilateral-sum theorem [2, p. 289], the sum of angles in a convex quadrilateral is 360°, which requires the parallel axiom.

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