A PROOF OF THE STANDARD REDUCTION THEOREM IN THE LAMBDA CALCULUS

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ABSTRACT. Felleisen's Standard Reduction Theorem for the λ_v Calculus yields an algorithm that models the Scheme interpreter. The β-nf λ Calculus analogue is Barendregt's Normalization Theorem. We give a simple proof of this by porting and simplifying Felleisen's proof.

1. INTRODUCTION

Barendregt [Ba, Thm. 13.2.2] proves a *Normalization Theorem* for the λ Calculus: if a Λ expression has a β-nf. Plotkin [Fe, Thm. 3.1.4] proved the analogue for his λ-value Calculus (which models Scheme): his standard reduction algorithm always produces a value, if an expression has one. Barendregt deduces the Normalization Theorem from a Standardization Theorem [Ba, Thm. 11.4.7] [Ha, Thm. 3.37] (Felleisen and Plotkin follow suit [Fe, Thm. 3.1.8]). We give a direct proof of the Normalization Theorem, by porting and simplifying Felleisen's proof.

We additionally simplify Felleisen's proof by replacing his size estimate Lemma [Fe, Lem. 3.1.13] by Lemma 2.4 below. We found Felleisen's Lemma difficult because it was not clear what the domain of his size function was, or what the size function was measuring.

We think we have a significantly shorter than Barendregt's proof. However, Barendregt's proof is of independent interest, as his proof of the Standardization Theorem grows out of a "longer but more perspicacious" proof of the Church-Rosser theorem [Ba, Thm. 11.1.10] than the usual Tait and Martin-Löf proof. Barendregt reduces all the residuals of a given redex, from the inside out, in order to prove a diamond-like property.

We begin with the definition of our λ Calculus β-nf seeking leftmost reduction algorithm. Following Felleisen [Fe, Def. 3.1.3], we will call each step of the reduction process the *standard reduction arrow*.

Thanks to Paul Burchard for the diagram package, which uses XY-pic arrows.

Definition 1.1. The standard reduction arrow $\mapsto$ is defined to be the smallest relation satisfying

(i) $M \mapsto N$ if $M \;\beta\; N$

(ii) $\lambda x.M \mapsto \lambda x.N$ if $M \mapsto N$

(iii) $MN \mapsto M'N$ if $M \mapsto M'$ and M is not a λ abstraction

(iv) $MN \mapsto MN'$ if $N \mapsto N'$ and M is a β-nf but not a λ abstraction

The following lemma, which we leave as an exercise, shows that the standard reduction arrow is an algorithm.

Lemma 1.2. *For any Λ term M which is not a β-nf, there exists a unique N such that $M \mapsto N$.*

We will now write $M \mapsto_\beta N$ rather than $M \;\beta\; N$, since the relation $\mapsto$ contains β. Recall that β, now written $\mapsto_\beta$, is the relation, or arrow

$$(\lambda x.M)U \mapsto_\beta M[x := U]$$

The arrow $\to$ is defined to be the syntactic closure of $\mapsto_\beta$. The arrow $\twoheadrightarrow$ is defined to be the reflexive transitive closure of $\to$. The reflexive transitive closure of the standard reduction arrow $\mapsto$ will be denoted by $\mapsto\!\!\!\!\twoheadrightarrow$.

The λ Calculus Normalization Theorem is now:

Theorem A. *If $M \twoheadrightarrow N$, with N a β-nf, then $M \mapsto\!\!\!\!\twoheadrightarrow N$.*

Thus the standard reduction arrow $\mapsto$ gives an algorithm for evaluation of expressions in the λ Calculus, returning β-nfs.

2. Replacement for Felleisen's size estimate lemma

The arrow $\twoheadrightarrow$ is also the reflexive transitive closure of a relation which following Hankin we will call *grand reduction*:

Definition 2.1. The grand reduction arrow $\to$ is defined to be the smallest relation given by

(1) $x \to x$

(2) $\lambda x.M \to \lambda x.N$ if $M \to N$

(3) $MN \to M'N'$ if $M \to N$ and $M' \to N'$

(4) $(\lambda x.M)U \to N[x := V]$ if $M \to N$ and $U \to V$

Remark 2.2. Grand reduction is the key definition of Tait and Martin-Löf's proof of the Church-Rosser theorem. Barendregt [Ba, Def. 3.2.3], Hankin [Ha, Def. 3.14] and Felleisen [Fe, Def. 2.5.4] define axiom (1) to be $M \to M$. Our apparently more restrictive definition is equivalent to theirs, because we can deduce $M \to M$ from axioms (1–3), by induction on M.

Let $\mapsto\!\!\!\twoheadrightarrow_\beta$ mean the reflexive transitive closure of $\longmapsto_\beta$, or β. Since β is a subset of the Standard Reduction arrow $\longmapsto$, the arrow $\mapsto\!\!\!\twoheadrightarrow_\beta$ is a subset of $\mapsto\!\!\!\twoheadrightarrow$.

We give a mild extension of a lemma [Ba, Lem. 3.2.4] [Fe, Lem. 2.5.6] which Hankin [Ha, p. 37] states as Property (1).

Lemma 2.3. (1) *Assume that $M \to N$ and $U \to V$. Then*

$$M[x := U] \to N[x := V]$$

and this grand reduction has the same type as the grand reduction $M \to N$, unless the type of $M \to N$ is (1).

(2) *If $M \longmapsto_\beta N$, then for any Λ expression U*

$$M[x := U] \longmapsto_\beta N[x := U]$$

Proof. The proof of part 1 follows immediately from the argument of Barendregt [Ba, Lem. 3.2.4] and Felleisen [Fe, Lem. 2.5.6]. A simple version of their argument proves part 2, which we give for completeness.

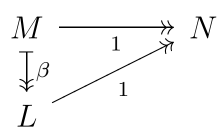
We're given the β-reduction $M = (\lambda y.P)V \longmapsto_\beta P[y := V] = N$. Then

$$\begin{aligned} M[x := U] &= (\lambda y.P[x := U])\, V[x := U] \longmapsto_\beta P[x := U][y := V[x := U]] \\ &= P[y := V][x := U] = N[x := U] \end{aligned}$$

by the Subsitition Lemma [Ha, Lem. 2.11], since by the variables convention, $y \notin FV(U)$. □

Now we give the λ Calculus port for a replacement for Felleisen's size estimate Lemma [Fe, Lem. 3.1.13]. Our replacement works because the purpose of Felleisen's lemma is to deal with grand reductions of type (4). We think that our Lemma 2.4 is a much cleaner way to do so.

Lemma 2.4. *For any grand reduction $M \to N$, there exists L with*



where the grand reduction $L \to N$ is of type (1), (2) or (3).

Proof. We use induction on M. If $M \to N$ is a grand reduction of type (1), (2) or (3), we're done. So assume that $M \to N$ is of type (4), say

$$M = (\lambda x.P)U \to Q[x := V] = N, \qquad \text{with } P \to Q \text{ and } U \to V.$$

By induction, there exists K with $P \mapsto\!\!\!\twoheadrightarrow_\beta K$, and a grand reduction $K \to Q$ of type (1), (2) or (3). By Lemma 2.3, we have

$$M = (\lambda x.P)U \longmapsto_\beta P[x := U] \mapsto\!\!\!\twoheadrightarrow_\beta K[x := U] \to Q[x := V] = N$$

and the last grand reduction has the same type as $K \to Q$, unless $K \to Q$ has type (1). So if $K \to Q$ is of type (2) or (3), then we're done: we can take $L = K[x := U]$.

Now let's assume that $K \to Q$ is of type (1), that is, $K = Q = y$, for some variable y. We have two cases now.

If $y = x$, then $K[x := U] \to Q[x := V]$ is $U \to V$. The inductive hypothesis applies to $U \to V$, so there exists L with

$$U \mapsto\!\!\!\rightarrow_\beta L \to V = N,$$

and $L \to N$ a grand reduction of type (1), (2) or (3), and we're done:

$$M = (\lambda x.P)U \longmapsto_\beta P[x := U] \mapsto\!\!\!\rightarrow_\beta U \mapsto\!\!\!\rightarrow_\beta L \to N$$

Now if instead $y \neq x$, then $K[x := U] \to Q[x := V]$ is $y \to y$. And so we're done, we have

$$M = (\lambda x.P)U \longmapsto_\beta P[x := U] \mapsto\!\!\!\rightarrow_\beta y \to y = N,$$

where the grand reduction is of type (1). □

Using our replacement for Felleisen's size estimate Lemma 3.1.13, we port one of his two lemmas [Fe, Lem. 3.1.11–2] to the λ Calculus.

Lemma 2.5. (a) *If $M \to x$, then $M \mapsto\!\!\!\rightarrow_\beta x$.*

(b) *Given $M \to \lambda x.N$, there exists a grand reduction $L \to N$ such that*

$$M \mapsto\!\!\!\rightarrow_\beta \lambda x.L \to \lambda x.N$$

(c) *Given $M \to V$, with V a β-nf, then $M \mapsto\!\!\!\rightarrow V$.*

Proof. Given $M \to \lambda x.N$, by Lemma 2.4 there exists L such that

$$M \mapsto\!\!\!\rightarrow_\beta L \to \lambda x.N,$$

with the second arrow a grand reduction of type (2), since (1) and (3) are impossible. So by definition, there exists K such that $L = \lambda x.K \to \lambda x.N$, with $K \to N$. This proves (b). The proof of (a) is similar, but even easier.

For (c), assume $M \to V$, with V a β-nf. We argue by induction on V. By Lemma 2.4 there exists L such that

$$M \mapsto\!\!\!\rightarrow_\beta L \to V,$$

with the 2nd arrow a grand reduction of type (1), (2) or (3).

If type (1), we're done, since $L = V$.

If type (2), then by definition, $L = \lambda x.K \to \lambda x.U = V$, with $K \to U$.

U is then a β-nf, and by induction we have that $K \mapsto\!\!\!\rightarrow U$, and we're done.

If type (3), then $L = (HK) \to (AB) = V$, with $H \to A$ and $K \to B$, and A, B β-nf, with A not a λ abstraction. By induction we have that

$H \mapsto\!\!\!\!\twoheadrightarrow A$ and $K \mapsto\!\!\!\!\twoheadrightarrow B$, and since A is not a λ abstraction, no term in the sequence $H \mapsto\!\!\!\!\twoheadrightarrow A$ can be a λ abstraction, including H, so we have that

$$(HK) \mapsto\!\!\!\!\twoheadrightarrow (AK) \mapsto\!\!\!\!\twoheadrightarrow (AB). \quad \square$$

For the λ_v Calculus, Lemma 2.4 suffices to replace Felleisen's size estimate Lemma 3.1.13. However, in in the λ Calculus it's convenient to take Lemma 2.4 one step farther, to simplify the grand reductions of type (3). For this we need some terminology suggested by Hankin's sketch of Barendregt's proof of the Standardization Theorem [Ha, Thm. 3.37].

Any application M can be written uniquely as $M = M_1 M_2 \cdots M_k$, for some $k \geq 2$, with M_1 either a variable or a λ abstraction. If M_1 is a variable, then M is called a *weak head normal form* [Ha, p. 49]. If $M_1$is a λ abstraction, then $M_1 M_2$ is called the *head redex* of M.

We define a *weak head reduction*, written $\longmapsto_w$, by

$$(\lambda x.M)\, U\, M_2\, \cdots\, M_k \longmapsto_w M[x := U]\, M_2\, \cdots\, M_k$$

The transitive reflexive closure of $\longmapsto_w$ will be written $\mapsto\!\!\!\!\twoheadrightarrow_w$ as usual.

We define the *weak internal reduction* arrow $\longrightarrow_i$ by

$$M_1 M_2 \cdots M_k \longrightarrow_i N_1 N_2 \cdots N_k \quad \text{if } M_t \to N_t \text{ for } t = 1, \ldots k$$

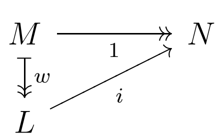
and if $M_1 \to N_1$ a grand reduction of type (1) or (2).

We collect some simple properties of weak head reduction and leave the proof as an exercise.

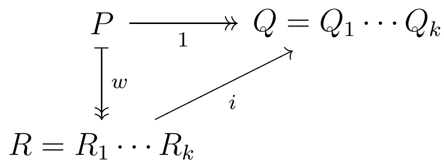
Lemma 2.6. (1) *Given* $M \mapsto\!\!\!\!\twoheadrightarrow_w N$ *and* H, *we have* $(MH) \mapsto\!\!\!\!\twoheadrightarrow_w (NH)$.
(2) *Given* $M \longrightarrow_i N$ *and* $H \longrightarrow_i K$, *we have* $(MH) \longrightarrow_i (NK)$.
(3) $\mapsto\!\!\!\!\twoheadrightarrow_w$ *is a subset of* $\mapsto\!\!\!\!\twoheadrightarrow$.

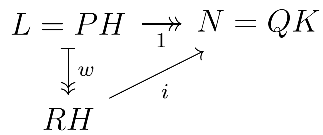
Now we have our extension of Lemma 2.4:

Lemma 2.7. *For any grand reduction* $M \to N$, *there exists* L *such that*



Proof. We use induction on N. If N is a variable, then the result follows from Lemma 2.5(a), since $\mapsto\!\!\!\!\twoheadrightarrow_\beta$ is a subset of $\mapsto\!\!\!\!\twoheadrightarrow_w$. If N is a λ abstraction, then then the result follows from Lemma 2.5(b). If $N = (QK)$ is an application, then Lemma 2.4 implies that there exists $L = (PH)$ with $M \mapsto\!\!\!\!\twoheadrightarrow_\beta L \to N$, the second grand reduction of type (3). Now by induction (on N), we have the left-hand diagram:





where $R_i \to Q_i$ and R_1 either a variable or a λ abstraction. Then by Lemma 2.6, we have the right-hand diagram. □

3. PROOF OF THEOREM A

Using Lemma 2.7, we can port Felleisen's main Lemma [Fe, Lem. 3.1.11] to the λ Calculus.

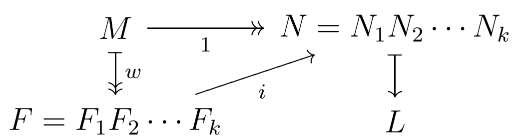
Lemma 3.1. (1) *Given* $M \to N \mapsto L$, *there exists* N^* *such that*

$$M \mapsto\!\!\!\twoheadrightarrow N^* \to L.$$

(2) *Given* $M \to N \mapsto\!\!\!\twoheadrightarrow L$, *there exists* N^* *such that*

$$M \mapsto\!\!\!\twoheadrightarrow N^* \to L.$$

Proof. Part 2 follows easily from part 1 by building "ladders". We now prove part 1 by induction on N. By Lemma 2.7, there exists

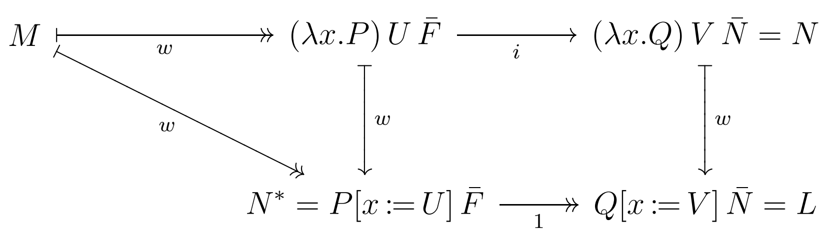


with grand reductions $F_t \to N_t$, and with F_1 either a λ abstraction or a variable. We consider the two cases separately.

If F_1 is a λ abstraction, then $F_1 = \lambda x.P \to \lambda x.Q = N_1$, for some grand reduction $P \to Q$. We consider the two cases $k = 1$ and $k > 1$ separately.

If $k = 1$, then $L = \lambda x.R$, with a standard reduction $Q \mapsto R$. Now by induction (on N) we have Q^* with $P \mapsto\!\!\!\twoheadrightarrow Q^* \to R$, and we're done.

If $k > 1$, then call $U = F_2$, $V = N_2$, $\bar{F} = F_3 \cdots F_k$, and $\bar{N} = N_3 \cdots N_k$. Then by Lemma 2.3, we have a grand reduction $P[x := U] \to Q[x := V]$, and therefore by reducing the head redexes



This finishes the case where F_1 a λ abstraction.

If instead $F_1 = x = N_1$, then suppose that $N_a \mapsto D$, and that N_t is a β-nf for $1 < t < a$, so

$$N \mapsto L = x \cdots N_{a-1} D N_{a+1} \cdots N_k.$$

We apply induction to $F_a \to N_a \mapsto D$. So there exists Z such that $F_a \mapsto\!\!\!\twoheadrightarrow Z \to D$. Furthermore by Lemma 2.5(c), $F_t \mapsto\!\!\!\twoheadrightarrow N_t$ for $1 < t < a$, so

$$\begin{aligned} F = xF_2 \cdots F_k &\mapsto\!\!\!\twoheadrightarrow xN_2 \cdots F_k \mapsto\!\!\!\twoheadrightarrow \cdots \mapsto\!\!\!\twoheadrightarrow xN_2 \cdots N_{a-1} F_a \cdots F_k \\ &\mapsto\!\!\!\twoheadrightarrow xN_2 \cdots N_{a-1} Z F_{a+1} \cdots F_k \to xN_2 \cdots N_{a-1} D N_{a+1} \cdots N_k, \end{aligned}$$

where the last grand reduction is obtained from the grand reductions $Z \to D$ and $F_t \to N_t$ for $a < t \leq k$. □

Now finally we have

Proof of Theorem A. Given $M \twoheadrightarrow N$, for a β-nf N, there's a sequence of grand reductions

$$M = M_0 \to M_1 \to M_2 \to \cdots \to M_l \to N$$

We use induction on l. For $l = 0$, Lemma 2.5(c) proves our result. For $l > 0$, by induction $M \to M_1 \mapsto\!\!\!\twoheadrightarrow U$. By Lemma 3.1(b), there exists N^* such that $M \mapsto\!\!\!\twoheadrightarrow N^* \to U$. By Lemma 2.5(c), $N^* \mapsto\!\!\!\twoheadrightarrow U$, and we're done. □

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