Northwestern University **Name:**

Math 290-1: Final Exam

Fall Quarter 2014

Wednesday, December 10, 2014

Put a check mark next to your section:

Davis (10am)		Canez	
Alongi		Peterson	
Graham		Davis (12pm)	

Question	Possible points	Score
1	18	
2	24	
3	10	
4	8	
5	10	
6	10	
7	8	
8	12	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 15 pages, and 8 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **six** parts.)

   (a) Let V be the subspace of $\mathbb{R}^4$ consisting of all vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ simultaneously satisfying the following equations:

   $$x - y + 3z = 0, \quad 2x + y + 3z = 0, \quad 7x - 3y + 5z + 2w = 0, \quad 3x + 2y + 4z = 0.$$

   Then $V = \{\vec{0}\}$.

   Answer:

   (b) For all 2×2 matrices A and B, $\text{rref}(AB) = \text{rref}(A)\,\text{rref}(B)$.

   Answer:

(c) If a subspace V of $\mathbb{R}^3$ does not contain any of the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V is the zero subspace.

Answer:

(d) The set of real eigenvectors of an $n \times n$ matrix must span $\mathbb{R}^n$.

Answer:

(e) The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 2 & -6 \\ 1 & 1 & -21 \\ 2 & -4 & 6 \end{bmatrix}$.

Answer:

(f) Suppose that A_1 and A_2 are both diagonalizable and that they are both similar to the same diagonal matrix D. Then A_1 and A_2 must have the same eigenvectors.

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **six** parts.)

   (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto a plane in $\mathbb{R}^3$ through the origin. Then $\dim \ker T = 1$.

Answer:

   (b) Let A be an $n \times n$ matrix such that the linear transformation $T(\vec{x}) = A\vec{x}$ has expansion factor 5. Then the columns of A are linearly independent.

Answer:

(c) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, and if $T : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$T(\vec{x}) = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_1 \\ | & | & | & | \end{bmatrix} \vec{x},$$

then T is invertible.

Answer:

(d) If the standard basis vectors $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$ of $\mathbb{R}^3$ are eigenvectors of a 3×3 matrix A, then A is a diagonal matrix.

Answer:

(e) If A is a diagonalizable matrix, then A is invertible.

Answer:

(f) A diagonalizable 5×5 matrix with rank 2 has an eigenvalue with algebraic multiplicity 3.

Answer:

3. (This problem has **two** parts.)

(a) Find the inverse of the following matrix.

$$A = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 5 & -4 \\ 1 & 0 & 6 \end{bmatrix}$$

(b) Solve the following system of equations.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ -2x + 5y - 4z &= 1 \\ x \qquad + 6z &= 1 \end{aligned}$$

4. Find a 2×2 matrix A such that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ geometrically scales by a factor of 2 in the direction of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and scales by a factor of -3 in the direction of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

5. Determine the value(s) of k for which the following matrix is invertible.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & k^2 & k & 1 \\ 1 & k & 2k+1 & -1 \\ 1 & k^2 & 0 & 1 \end{bmatrix}$$

6. Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & -1 \\ 1 & 2 & 0 & -1 \end{bmatrix}$$

7. Suppose that a 3×3 matrix A has eigenvalues $-1, 0$ and 2, with corresponding eigenvectors

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix} \text{ and } \begin{bmatrix}1\\0\\0\end{bmatrix}$$

respectively. Compute $A\left[\begin{smallmatrix}4\\0\\1\end{smallmatrix}\right]$.

8. (This problem has **two** parts.) Throughout this problem let A be the following matrix, which has eigenvalues 2 and 3:

$$A = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Find a basis for each eigenspace of A.

(b) Find a diagonal matrix which is similar to A^{100}. Justify your answer.