Northwestern University

Name: SOLUTIONS

Student ID:

Math 290-1 Final Exam

Autumn Quarter 2012

Wednesday, December 12, 2012

Put a check mark next to your section:

Allen		Cyr (12pm)	
Canez		Peters	
Cyr (10am)			

Question	Possible points	Score
1	15	
2	15	
3	10	
4	10	
5	11	
6	9	
7	10	
8	9	
9	11	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 14 pages, and 9 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) If A is a 3×3 matrix whose columns are linearly independent, then the columns of A^2 are also linearly independent.

true: lin ind. columns $\Rightarrow$
A is invertible $\Rightarrow$
A^2 is invertible ($(A^2)^{-1} = (A^{-1})^2$) $\Rightarrow$
columns of A^2 are lin. ind.

(b) Let A be an $n \times n$ matrix and let λ be an eigenvalue for A with algebraic multiplicity k. Then $\text{rank}(A - \lambda I_n)$ satisfies the inequality

$$n - k \leq \text{rank}(A - \lambda I_n) \leq n - 1.$$

True: geo mult $\leq$ alg. mult $\Rightarrow$
$\dim \ker(A - \lambda I_n) \leq k$ $\Rightarrow$
$\text{rank}(A - \lambda I_n) \geq n-k$ by rank-nullity thm.

Since λ is an eigenvalue,
$\dim \ker(A - \lambda I_n) \geq 1$ $\Rightarrow$
$\text{rank}(A - \lambda I_n) \leq n-1$ (rank-nullity thm. again.)

(c) The matrix $\begin{pmatrix} 3 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 2 \end{pmatrix}$ is only diagonalizable when a and b are both 0.

False: if $a = 0$ and $b = 1$, the matrix is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

this has eigenvalue 3 w/ alg mult = geo mult = 2, and eigenvalue 2 w/ alg mult = geo mult = 1

hence it is diagonalizable

(d) If λ is an eigenvalue of A, then λ is an eigenvalue of A^2.

False: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has eigenvalue 2,

but $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

only has eigenvalue 4.

(e) If A and B are similar 5×5 matrices, then $A + A^2$ is similar to $B + B^2$.

True. If $A = S^{-1}BS$ then

$$A + A^2 = S^{-1}BS + (S^{-1}BS)(S^{-1}BS)$$
$$= S^{-1}BS + S^{-1}B^2S$$
$$= S^{-1}(B + B^2)S$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) For two 2×2 matrices A and B, which are both of rank 2, the matrix

$$A + B$$

is of rank 2.

$\boxed{\text{Sometimes}}$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, they both have rank 1, and so does $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, they both have rank 1, but

$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not.

(b) If A is an invertible $n \times n$ matrix and $\vec{v}$ is an eigenvector for A, then $A\vec{v}$ is also an eigenvector for A.

$\boxed{\text{Always}}$

Since A is invertible, $A\vec{v} \neq \vec{0}$. Then, if λ is the eigenvalue for $\vec{v}$,

$$A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v})$$

So $A\vec{v}$ is an eigenvector for A with eigenvalue λ.

(c) If A is a 2×2 matrix and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation $T(\vec{x}) = A\vec{x}$, the matrix of T relative to a non-standard basis $\mathcal{B}$ of $\mathbb{R}^2$ is not equal to A itself.

$\boxed{\text{Sometimes}}$

Let $\mathcal{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note $\mathcal{B}$ is a non-standard basis.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then the $\mathcal{B}$-matrix for T is

$$B = \left[\left[A\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{\mathcal{B}} \left[A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} \right] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A.$$

(the identity transformation is given by I_2 in any basis)

If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then the $\mathcal{B}$-matrix for T is

$$B = \left[\left[A\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{\mathcal{B}} \left[A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} \right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \neq A.$$

(d) Two 2-by-2 matrices which have the same eigenvalues are similar.

$\boxed{\text{Sometimes}}$

If A and B are two diagonalizable 2×2 matrices with the same eigenvalues λ_1, λ_2; then there are two invertible 2×2 matrices S_A and S_B such that

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = S_A^{-1} A S_A$$
$$= S_B^{-1} B S_B$$

(note S_A may be different from S_B)

So $A = S_A \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S_A^{-1} = S_A (S_B^{-1} B S_B) S_A^{-1} = (S_A S_B^{-1}) B (S_A S_B^{-1})^{-1}$ and A and B are similar.

This is no longer true if they are not both diagonalizable. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, both A and B have eigenvalue $\lambda = 1$ with algebraic multiplicity 1. But, if $B = S^{-1}AS$, then $B = S^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}S = S^{-1}S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is not true.

(e) Suppose $\vec{u} = \begin{pmatrix}1\\1\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\1\end{pmatrix}$. For an invertible 2×2 matrix A, the area of the parallelogram determined by $A\vec{u}$ and $A\vec{v}$ is $|\det(A)|$.

$\boxed{\text{Never}}$

Let Ω be the parallelogram determined by $\vec{u}$ and $\vec{v}$.
Then $A(\Omega)$ is the parallelogram determined by $A\vec{u}$ and $A\vec{v}$.
We know

$$|\det A| = \frac{\text{Area of } A(\Omega)}{\text{Area of } \Omega}$$

Now, Area of $\Omega = \left|\det\begin{bmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{bmatrix}\right| = \left|\det\begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}\right| = 2$

So Area of $A(\Omega) = 2\,|\det A|$; and,

$2|\det A| \neq |\det A|$ since A is invertible.

3. Find a 3×3 matrix A such that **both** of the following conditions hold

- the matrix equation $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is consistent $\Leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \operatorname{Im} A$
- the matrix equation $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is inconsistent. $\Leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin \operatorname{Im} A$

Justify your answer.

$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ works.

4. Determine whether or not

$$A = \begin{pmatrix} -1 & 0 & -3 \\ 2 & 2 & -3 \\ 1 & 0 & 3 \end{pmatrix}$$

is diagonalizable. If it is, find an invertible 3×3 matrix S and a diagonal 3×3 matrix D such that

$$A = SDS^{-1}.$$

If not, explain why not.

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 0 & -3 \\ 2 & 2-\lambda & -3 \\ 1 & 0 & 3-\lambda \end{pmatrix} = (2-\lambda)\big((-1-\lambda)(3-\lambda)+3\big)$$

$$= (2-\lambda)\,\lambda(\lambda-2)$$

So $\lambda = 0$ mult. 1 (alg. mult.)
$\lambda = 2$ mult. 2

We need to check if geometric mult. of $\lambda = 2$ is 2

$$\ker(A-2I) = \ker \begin{pmatrix} -3 & 0 & -3 \\ 2 & 0 & -3 \\ 1 & 0 & 1 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$ because $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$

are linearly independent. So $\lambda = 2$ has geometric mult $1 < 2$.

Therefore, A is not diagonalizable.

5. Suppose that a 3×3 matrix A has eigenvalues $-1, 2, 2$ with corresponding eigenvectors

$$\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Find A^{-1}. Your answer should be an explicit matrix.

$A = SDS^{-1}$ so $A^{-1} = (SDS^{-1})^{-1} = SD^{-1}S^{-1}$

$$S = \begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 3 & -3 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 3 & -3 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & 3 & 5 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$S^{-1} = \begin{pmatrix} 1/3 & 1/2 & 5/6 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A^{-1} = SD^{-1}S^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & & \\ & -1 & \\ & & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & & \\ & -1 & \\ & & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/2 & 5/6 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/2 & 9/4 & 9/4 \\ 0 & -1 & -3/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

6. Determine whether or not each of the following sets is a **subspace** of $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x + 3y - z = 0 \text{ and } x + y + z = 0 \right\} = \ker \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ so it is a subspace of $\mathbb{R}^3$

Yes $2 \cdot 0 + 3 \cdot 0 - 0 = 0$, $0 + 0 + 0 = 0$, so $\vec{0}$ is in.

(b) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2y - 3x + 7z = 1 \right\} \not\ni \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

No.

$V =$ (c) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : y \leq z \right\}$ not closed under scalar multiplication

No. $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \in V$, but $(-1)\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \notin V$.

7. Suppose A is a 3×3 matrix with eigenvalues 2, 1 and 0 and eigenvectors $\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\1\end{pmatrix}$, respectively.

(a) Find a basis for im(A).

$A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ so $\begin{pmatrix}2\\1\\1\end{pmatrix} \in \ker(A)$

(so $\dim(\ker(A)) \geq 1$).

$\left.\begin{array}{l} A\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\2\\4\end{pmatrix} \\ A\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}\end{array}\right\} \Rightarrow \begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix} \in \text{im}(A)$

(since $\left\{\begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ is lin. indep.,
$\dim(\text{im}(A)) \geq 2$).

$\dim(\ker(A)) + \dim(\text{im}(A)) = 3 = \#$ cols. of A

So $\dim(\ker(A)) = 1 \leftarrow$ so $\ker(A) = \text{Span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$

$\dim(\text{im}(A)) = 2 \leftarrow$ so $\text{im}(A) = \text{Span}\left\{\begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$

$\boxed{\mathcal{B} = \left\{\begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}}$

(b) Give an example of a vector $\vec{b} \in \mathbb{R}^3$ for which $A\vec{x} = \vec{b}$ has no solutions.

$\begin{pmatrix}a\\b\\c\end{pmatrix} \in \text{Span}\left\{\begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ iff there is <u>not</u> a pivot in the rightmost column of

$$\left(\begin{array}{cc|c} 2 & 1 & a \\ 2 & 2 & b \\ 4 & 1 & c \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & a \\ 0 & 1 & b-a \\ 0 & -1 & c-2a \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & 1 & a \\ 0 & 1 & b-a \\ 0 & 0 & -3a+b+c \end{array}\right)$$

So $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \text{im}(A)$ iff $-3a+b+c=0$.

so $\begin{pmatrix}1\\0\\0\end{pmatrix} \notin \text{im}(A)$ and $A\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ has no solutions.

8. Given that

$$\operatorname{rank}\left(\begin{pmatrix} 1 & -1 & -1 & 0 \\ -5 & -1 & -3 & -1 \\ 5 & 3 & 5 & 1 \\ 6 & 4 & 4 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\right) = 3$$

$$\operatorname{rank}\left(\begin{pmatrix} 1 & -1 & -1 & 0 \\ -5 & -1 & -3 & -1 \\ 5 & 3 & 5 & 1 \\ 6 & 4 & 4 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right) = 2,$$

$$\operatorname{rank}\left(\begin{pmatrix} 1 & -1 & -1 & 0 \\ -5 & -1 & -3 & -1 \\ 5 & 3 & 5 & 1 \\ 6 & 4 & 4 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = 3,$$

find the eigenvalues (and their geometric multiplicities) of

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -5 & -1 & -3 & -1 \\ 5 & 3 & 5 & 1 \\ 6 & 4 & 4 & 3 \end{pmatrix}.$$

λ is an eigenvalue if and only if $\operatorname{rank}(A - \lambda I) < n = 4$

So $3, 2, 1$ are eigenvalues.

geometric multiplicity $= \dim \ker(A - \lambda I) = n - \operatorname{rank}(A - \lambda I)$

So geometric multiplicity of $1 = 4 - 3 = 1$

of $2 = 4 - 2 = 2$

of $3 = 4 - 3 = 1$

9. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects a point over the y-axis and then reflects that image over the line $y = x$.

(a) Find the standard matrix of T.

Lets find $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$

So $\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}-1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\-1\end{pmatrix}$

and $\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix}$

So $A = \left[T\begin{pmatrix}1\\0\end{pmatrix} \; T\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

(b) Show that if A is the standard matrix of T, then the matrix A^{10} represents a 180° rotation of $\mathbb{R}^2$ (about the origin).

Given $A = \begin{bmatrix} & 1\\ -1 & \end{bmatrix}$ notice that $A^2 = \begin{bmatrix}-1 & \\ & -1\end{bmatrix}$

So $A^{10} = (A^2)^5 = \begin{pmatrix}-1 & \\ & -1\end{pmatrix}^5 = \begin{pmatrix}-1 & \\ & -1\end{pmatrix}$

It is enough to check that $\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$ represents a 180° rotation of $\mathbb{R}^2$.

For example, it sends $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}-1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}0\\-1\end{pmatrix}$.