Northwestern University

Name: Solutions

Student ID:

Math 290-1 Final Exam

Fall Quarter 2013

Wednesday, December 11, 2013

Put a check mark next to your section:

Allen		Cañez	
Broderick 10:00		Davis	
Broderick 12:00			

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 13 pages, and 8 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

Question	Possible points	Score
1	20	
2	24	
3	9	
4	10	
5	8	
6	10	
7	8	
8	11	
TOTAL	100	

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) Every 2×2 matrix for which $\vec{e}_1$ and $\vec{e}_2$ are eigenvectors is a diagonal matrix.

TRUE

If $A\vec{e_1} = \lambda_1 \vec{e_1}$ and $A\vec{e_2} = \lambda_2 \vec{e_2}$,

then $A = \begin{bmatrix} | & | \\ \lambda_1 \vec{e_1} & \lambda_2 \vec{e_2} \\ | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$,

which is diagonal.

(b) If A is a 3×3 matrix such that $\ker(A) = \operatorname{span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\0\\4\end{bmatrix}\right)$, then $\operatorname{rank}(A) = 1$.

FALSE

Since $\begin{bmatrix}2\\0\\4\end{bmatrix} = 2\begin{bmatrix}1\\0\\2\end{bmatrix}$, $\ker A = \operatorname{span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)$,

so $\dim(\ker A) = 1$, so by the rank-nullity theorem

$\operatorname{rank}(A) = 3 - 1 = 2$

(c) Let A and B be 2×2 matrices, and let Ω be a parallelogram in $\mathbb{R}^2$. If $A^2 = B^2$, then $\text{Area}(A(\Omega)) = \text{Area}(B(\Omega))$.

TRUE

If $A^2 = B^2$, then $\det(A^2) = \det(B^2)$,

so $(\det A)^2 = (\det B)^2$, so

$|\det A| = |\det B|$.

Hence,

$\text{Area}(A(\Omega)) = |\det A| \, \text{Area}(\Omega)$

$= |\det B| \, \text{Area}(\Omega) = \text{Area}(B(\Omega))$

(d) Let $\mathfrak{B}$ be the basis $\mathfrak{B} = (2\vec{e}_1, 3\vec{e}_2, 5\vec{e}_3)$ for $\mathbb{R}^3$. Then for a 3×3 invertible matrix A, if B is its $\mathfrak{B}$-matrix,

$$\det(B) = 30 \det(A).$$

FALSE

Since A and B are similar

$\det B = \det A$,

and since A is invertible, $\det A \neq 0$,

so $30 \det A \neq \det A$.

(e) Let A be a 3×3 matrix, and let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be a basis for $\mathbb{R}^3$. Let $\mathfrak{B}'$ be the basis $\mathfrak{B}' = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$, where $\vec{w}_1 = \vec{v}_3$, $\vec{w}_2 = \vec{v}_1$, and $\vec{w}_3 = \vec{v}_2$. If

$$B = \begin{bmatrix} 0 & 57 & \sqrt{2} \\ 57 & 0 & \pi \\ 2 & 1 & 0 \end{bmatrix}$$

is the $\mathfrak{B}$-matrix for A, then

$$B' = \begin{bmatrix} \sqrt{2} & 0 & 57 \\ \pi & 57 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

is the $\mathfrak{B}'$-matrix for A.

FALSE

the $\mathfrak{B}'$-matrix is $\begin{bmatrix} | & | & | \\ [A\vec{v_3}]_{\mathfrak{B}'} & [A\vec{v_1}]_{\mathfrak{B}'} & [A\vec{v_2}]_{\mathfrak{B}'} \\ | & | & | \end{bmatrix}$

$\cancel{A\vec{v_3}}$ The $\mathfrak{B}$-matrix is $\begin{bmatrix} | & | & | \\ [A\vec{v_1}]_{\mathfrak{B}} & [A\vec{v_2}]_{\mathfrak{B}} & [A\vec{v_3}]_{\mathfrak{B}} \\ | & | & | \end{bmatrix}$

So $A\vec{v_3} = \sqrt{2}\vec{v_1} + \pi\vec{v_2} + 0\vec{v_3} = 0\vec{w_1} + \sqrt{2}\vec{w_2} + \pi\vec{w_3}$

$A\vec{v_1} = 0\vec{v_1} + 57\vec{v_2} + 2\vec{v_3} = 2\vec{w_1} + 0\vec{w_2} + 57\vec{w_3}$

$A\vec{v_2} = 57\vec{v_1} + 0\vec{v_2} + 1\cdot\vec{v_3} = 1\cdot\vec{w_1} + 57\vec{w_2} + 0\vec{w_3}$.

Hence, the $\mathfrak{B}'$-matrix is $\begin{bmatrix} 0 & 2 & 1 \\ \sqrt{2} & 0 & 57 \\ \pi & 57 & 0 \end{bmatrix}$.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) For an invertible matrix A and an eigenvalue λ of A, λ is also an eigenvalue of A^{-1}.

Sometimes

If $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, and 1 is an eigenvalue of both.

If $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}$

$A\vec{e}_1=2\vec{e}_1$ so 2 is an eigenvalue of A,

but $\det(A^{-1}-\lambda I_2)=(\frac{1}{2}-\lambda)^2$, so $\frac{1}{2}$ is the only eigenvalue of A^{-1}!

(b) For a 3×3 matrix that has $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-2\\0\\3\end{bmatrix}$, and $\begin{bmatrix}-1\\1\\4\end{bmatrix}$ as eigenvectors, their corresponding eigenvalues are 2, 2, and 3 respectively.

Never

If $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}-2\\0\\3\end{bmatrix}$ are in E_2, then

$\begin{bmatrix}-1\\1\\4\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}-2\\0\\3\end{bmatrix}$ is also in E_2, not E_3.

(c) For a 7×7 matrix A with characteristic polynomial $f_A(\lambda) = (x-3)^2(x-1)^3(x+5)^2$, the matrix $A - I_7$ has rank 3.

Never

1 is an eigenvalue with almu$(1) = 3$, so gemu$(1) \leq$ almu$(1) = 3$. Thus $\dim(\ker(A - I_7)) \leq 3$, so by the rank-nullity theorem, $\text{rank}(A - I_7) \geq 4$.

(d) For an $n \times m$ matrix A and an $m \times n$ matrix B such that $AB = I_n$, $BA = I_m$.

Sometimes

If $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it's true.

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$,

so it's false.

(e) For a 4×2 matrix A, the set of solutions of the linear system $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ is $\{[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}], [\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}]\}$.

Never

The solution set of a linear system must have 0, 1, or infinitely many elements.

(f) Given a non-invertible 4×4 matrix A for which $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \\ 9 \\ 0 \end{bmatrix}$ are all eigenvectors with eigenvalue 1, A is diagonalizable.

Always

Since $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \\ 9 \\ 0 \end{bmatrix}$ are linearly independent, $\text{gemu}(1) \geq 3$.

And since A is not invertible, $\text{gemu}(0) \geq 1$.

So $\text{gemu}(3) + \text{gemu}(1) \geq 4$, so A is diagonalizable.

3. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ -3 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} -1 \\ -6 \\ 5 \\ -6 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ -6 \end{bmatrix}.$$

Is $\vec{x}$ in span$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$? Justify your answer.

We need to determine whether the system

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & -6 \\ -1 & 2 & 5 \\ 0 & -3 & -6 \end{bmatrix} \vec{y} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ -6 \end{bmatrix}$$

is consistent.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & -2 & -6 & -2 \\ -1 & 2 & 5 & 4 \\ 0 & -3 & -6 & -6 \end{array}\right] \sim \begin{array}{c} \\ -2(\mathrm{I}) \\ +(\mathrm{I}) \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -2 & -4 & -6 \\ 0 & 2 & 4 & 6 \\ 0 & -3 & -6 & -6 \end{array}\right] \sim \begin{array}{c} \\ \times -\frac{1}{2} \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & -3 & -6 & -6 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \\ -2(\mathrm{II}) \\ +3(\mathrm{II}) \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The system is not consistent, so $\vec{x}$ is not in span$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

4. Diagonalize $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 1 \\ -3 & 5 & -2 \end{bmatrix}$, or show that it is not diagonalizable.

The characteristic polynomial is

$$\det\begin{bmatrix} 3-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ -3 & 5 & -2-\lambda \end{bmatrix} = (3-\lambda)\det\begin{bmatrix} 2-\lambda & 1 \\ 5 & -2-\lambda \end{bmatrix} - 0 + 0$$

$$= (3-\lambda)\left[(2-\lambda)(-2-\lambda) - 5\right] = (3-\lambda)\left[\lambda^2 - 9\right] = (3-\lambda)(\lambda+3)(\lambda-3)$$

The eigenvalues are $\lambda = 3$ (w/ algebraic multiplicity 2)
and $\lambda = -3$ (w/ algebraic multiplicity 1)

$$E_3 = \ker\begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 1 \\ -3 & 5 & -5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 5 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -3 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

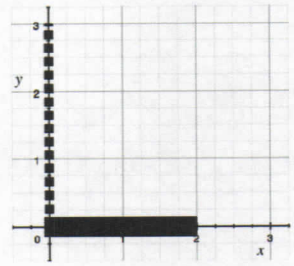
$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{7}{2} & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

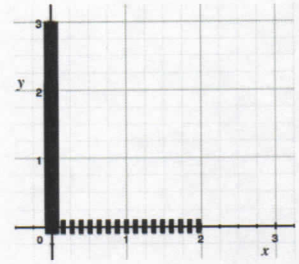
so $E_3 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$, so

$$\text{gemu}(3) = 1 < 2 = \text{almu}(3).$$

Hence, the matrix is not diagonalizable.

5. Logos, Inc. has introduced a very creative new corporate logo. Unfortunately, the image is being displayed flipped and distorted on their webpage. It should look like the picture on the right below but is instead appearing as the one on the left. Find a 2×2 matrix that will send the figure on the left to the one on the right.





$A(2\vec{e_1}) = 3\vec{e_2}$

and $A(3\vec{e_2}) = 2\vec{e_1}$, so $A = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{3}{2} & 0 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 2 & a & b \\ 0 & -1 & -a \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues of A and their algebraic multiplicities.

$\det(A - \lambda I_3) = (2-\lambda)(-1-\lambda)(2-\lambda)$

Eigenvalues are 2 (w/ algebraic multiplicity 2)

and -1 (w/ algebraic multiplicity 1)

(b) For each of the eigenvalues you found, find the values of a and b for which its geometric multiplicity is 1.

gemu(-1) is <u>always</u> 1, since -1 is an eigenvalue and almu$(-1)=1$, so $1 \le \text{gemu}(-1) \le 1$.

$\ker(A - 2I_3) = \ker \begin{bmatrix} 0 & a & b \\ 0 & -3 & -a \\ 0 & 0 & 0 \end{bmatrix}$

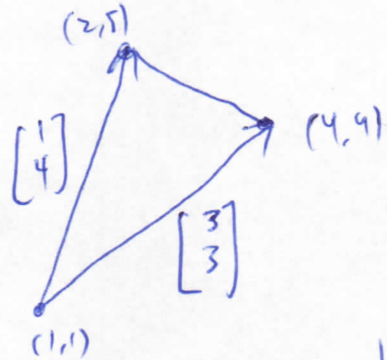
$$\begin{bmatrix} 0 & a & b \\ 0 & -3 & -a \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -3 & -a \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & \frac{a}{3} \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & a/3 \\ 0 & 0 & b-\frac{a^2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

So gemu$(2)=1$ if and only if $\text{rank}(A - 2I_3) = 2$, which occurs if and only if $b - \frac{a^2}{3} \neq 0$.

(c) For which values of a and b is A diagonalizable?

A is diagonalizable if and only if gemu$(2)=2$, i.e. (from part (b)) when $b = \frac{a^2}{3}$

7. Find the area enclosed by the triangle in the plane with vertices $(1,1)$, $(2,5)$, and $(4,4)$.



$$\text{Area} = \frac{1}{2}\left|\det\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}\right| = \frac{1}{2}\left|3-12\right| = \frac{9}{2}.$$

8. Find $\overset{A^{964}}{\overset{\shortparallel}{\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}^{964}}}$. You must express your answer as a single matrix with entries written explicitly.

$$\det \begin{bmatrix} -3-\lambda & 1 \\ -6 & 2-\lambda \end{bmatrix} = (-3-\lambda)(2-\lambda) + 6 = \lambda^2 + \lambda = \lambda(\lambda+1)$$

$$E_0: \left[\begin{array}{cc|c} -3 & 1 & 0 \\ -6 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad E_0 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

$$E_{-1}: \left[\begin{array}{cc|c} -2 & 1 & 0 \\ -6 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$S = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad S^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$A^{964} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}^{964}\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}.$$