Northwestern University Name: SOLUTIONS

Math 290-1: Final Exam

Fall Quarter 2014

Wednesday, December 10, 2014

Put a check mark next to your section:

Davis (10am)		Canez	
Alongi		Peterson	
Graham		Davis (12pm)	

Question	Possible points	Score
1	18	
2	24	
3	10	
4	8	
5	10	
6	10	
7	8	
8	12	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 15 pages, and 8 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **six** parts.)

   (a) Let V be the subspace of $\mathbb{R}^4$ consisting of all vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ simultaneously satisfying the following equations:

   $$x - y + 3z = 0, \quad 2x + y + 3z = 0, \quad 7x - 3y + 5z + 2w = 0, \quad 3x + 2y + 4z = 0.$$

   Then $V = \{\vec{0}\}$.

   Answer: False

This system can be written as

$\begin{pmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ 7 & -3 & 5 & 2 \\ 3 & 2 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \vec{0}$ or $A\vec{x} = \vec{0}$ for $A =$ to this matrix.

Note $V = \ker(A)$ and

$\det A = \det\begin{pmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ 7 & -3 & 5 & 2 \\ 3 & 2 & 4 & 0 \end{pmatrix} = -2\det\begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{pmatrix} = -2\det\begin{pmatrix} 1 & -1 & 3 \\ 0 & 3 & -3 \\ 0 & 5 & -5 \end{pmatrix} = (-2)(1)\det\begin{pmatrix} 3 & -3 \\ 5 & -5 \end{pmatrix} = 0,$

which means that $V = \ker(A) \neq \{\vec{0}\}$ by AAT.

   (b) For all 2×2 matrices A and B, $\text{rref}(AB) = \text{rref}(A)\,\text{rref}(B)$.

   Answer: False

One counterexample is

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad$ $\text{rref}(A) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $\quad$ $\text{rref}(B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad$ $\text{rref}(AB) = \begin{pmatrix} 1 & 1/3 \\ 0 & 0 \end{pmatrix}$

$\text{rref}(A)\,\text{rref}(B) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1/3 \\ 0 & 0 \end{pmatrix} = \text{rref}(AB).$

(c) If a subspace V of $\mathbb{R}^3$ does not contain any of the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V is the zero subspace.

Answer: False

One counterexample is $V = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.
Note that $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are not elements of V and V is not the zero subspace.

(d) The set of real eigenvectors of an $n \times n$ matrix must span $\mathbb{R}^n$.

Answer: FALSE

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Only eigenvalue is 0 which has eigenspace equal to $\text{Span}\begin{pmatrix}1\\0\end{pmatrix}$ and these do not span $\mathbb{R}^2$.

(e) The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 2 & -6 \\ 1 & 1 & -21 \\ 2 & -4 & 6 \end{bmatrix}$.

Answer: TRUE

$$\begin{bmatrix} 2 & 2 & -6 \\ 1 & 1 & -21 \\ 2 & -4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ -12 \end{bmatrix} = 12\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector

with eigenvalue 12

(f) Suppose that A_1 and A_2 are both diagonalizable and that they are both similar to the same diagonal matrix D. Then A_1 and A_2 must have the same eigenvectors.

Answer: FALSE

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

these are both similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

but A_2 has $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ~~as~~ as an

eigenvector which is not an

eigenvector of A_1

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **six** parts.)

   (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto a plane in $\mathbb{R}^3$ through the origin. Then $\dim \ker T = 1$.

   Answer: Always

   im(T) is a plane, so rank(T) = dim(im(T)) = 2.
   By Rank-Nullity Theorem, dim(ker(T)) = 3 - dim(im(T)) = 1.

   (b) Let A be an $n \times n$ matrix such that the linear transformation $T(\vec{x}) = A\vec{x}$ has expansion factor 5. Then the columns of A are linearly independent.

   Answer: Always

   Expansion factor of $A = 5 \Rightarrow |\det(A)| = 5$
   $\Rightarrow \det(A) \neq 0$
   By AAT $\Rightarrow$ A is invertible
   $\Rightarrow$ cols. of A are linearly independent.

(c) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, and if $T : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$T(\vec{x}) = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_1 \\ | & | & | & | \end{bmatrix} \vec{x},$$

then T is invertible.

Answer: Never

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is in $\ker(T)$, so $\ker(T) \neq \{\vec{0}\}$, so T is not invertible.

Or

Cols. of $\begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_1 \\ | & | & | & | \end{bmatrix}$ are linearly dependent, so T is not invertible.

(d) If the standard basis vectors $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$ of $\mathbb{R}^3$ are eigenvectors of a 3×3 matrix A, then A is a diagonal matrix.

Answer: Always

$$A = \begin{bmatrix} | & | & | \\ A\vec{e}_1 & A\vec{e}_2 & A\vec{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1\vec{e}_1 & \lambda_2\vec{e}_2 & \lambda_3\vec{e}_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(e) If A is a diagonalizable matrix, then A is invertible.

Answer: Sometimes

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← diagonalizable and invertible.

Counterexample: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ← diagonalizable and noninvertible.

(f) A diagonalizable 5×5 matrix A with rank 2 has an eigenvalue with algebraic multiplicity 3.

Answer: Always

The matrix is similar to the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\lambda_1, \lambda_2 \neq 0)$$

since $\text{nullity}(A) = 5 - 2 = 3$.

Because similar matrices have the same eigenvalues (with the same algebraic multiplicities), $\text{almu}(0) = 3$ for A.

3. (This problem has **two** parts.)

(a) Find the inverse of the following matrix.

$$A = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 5 & -4 \\ 1 & 0 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ -2 & 5 & -4 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \\ -2(\mathrm{I}) \\ +(\mathrm{I}) \end{array} \rightarrow \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]\begin{array}{l} \times(-1) \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]\begin{array}{l} \\ \\ -2(\mathrm{II}) \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -2 & 1 \end{array}\right]\begin{array}{l} +3(\mathrm{III}) \\ +2(\mathrm{III}) \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 14 & -6 & 3 \\ 0 & 1 & 0 & 8 & -3 & 2 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right]\begin{array}{l} +2(\mathrm{II}) \\ \\ \times(-1) \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 30 & -12 & 7 \\ 0 & 1 & 0 & 8 & -3 & 2 \\ 0 & 0 & 1 & -5 & 2 & -1 \end{array}\right].$$

$$\therefore A^{-1} = \begin{bmatrix} 30 & -12 & 7 \\ 8 & -3 & 2 \\ -5 & 2 & -1 \end{bmatrix}.$$

(b) Solve the following system of equations.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ -2x + 5y - 4z &= 1 \\ x \qquad\quad + 6z &= 1 \end{aligned}$$

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 & -12 & 7 \\ 8 & -3 & 2 \\ -5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 25 \\ 7 \\ -4 \end{bmatrix}$$

4. Find a 2×2 matrix A such that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ geometrically scales by a factor of 2 in the direction of the vector $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and scales by a factor of -3 in the direction of the vector $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$.

$$S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$A = SBS^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4/3 & 5/3 \\ 10/3 & 1/3 \end{bmatrix}.$$

5. Determine the value(s) of k for which the following matrix is invertible.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & k^2 & k & 1 \\ 1 & k & 2k+1 & -1 \\ 1 & k^2 & 0 & 1 \end{bmatrix}$$

$\det A$

$= \det \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & K^2-1 & K & 1 \\ 0 & K-1 & 2K+1 & -1 \\ 0 & K^2-1 & 0 & 0 \end{bmatrix}$

$= -2K(K^2-1) = 0 \iff K = 0, 1, -1$

A invertible $\iff K \neq 0, 1, -1$

6. Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & -1 \\ 1 & 2 & 0 & -1 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & -1-\lambda & 0 & 2 \\ -1 & 1 & 2-\lambda & -1 \\ 1 & 2 & 0 & -1-\lambda \end{bmatrix}$$

$$=(1-\lambda)(2-\lambda)\det\begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$$

$$=(1-\lambda)(2-\lambda)\left((\lambda+1)^2-4\right)$$

$$=(\lambda-1)^2(\lambda-2)(\lambda+3) \qquad \therefore \lambda = 1, 2 \text{ or } -3.$$

$$E_1 = \ker(A-I) = \ker\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 2 \\ -1 & 1 & 1 & -1 \\ 1 & 2 & 0 & -2 \end{bmatrix}$$

$$\therefore \operatorname{rref}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 2 \\ -1 & 1 & 1 & -1 \\ 1 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore E_1 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\therefore$ gemu $(1) <$ almu (1)

$\therefore$ A is not diagonalizable.

7. Suppose that a 3×3 matrix A has eigenvalues $-1, 0$ and 2, with corresponding eigenvectors

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix} \text{ and } \begin{bmatrix}1\\0\\0\end{bmatrix}$$

respectively. Compute $A\begin{bmatrix}4\\0\\1\end{bmatrix}$.

Suppose $\begin{bmatrix}4\\0\\1\end{bmatrix} = a_1\begin{bmatrix}1\\1\\1\end{bmatrix} + a_2\begin{bmatrix}2\\-1\\0\end{bmatrix} + a_3\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 2 & 1 & 4\\1 & -1 & 0 & 0\\1 & 0 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{array}\right]$$

$\therefore\ a_1 = a_2 = a_3$

$$\therefore\ A\begin{bmatrix}4\\0\\1\end{bmatrix} = A\left(\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}2\\-1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$= (-1)\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\cdot\begin{bmatrix}2\\-1\\0\end{bmatrix} + 2\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\-1\\-1\end{bmatrix}.$$

8. (This problem has **two** parts.) Throughout this problem let A be the following matrix, which has eigenvalues 2 and 3:

$$A = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Find a basis for each eigenspace of A.

$$\ker(A-2I) = \ker\begin{pmatrix} -2 & -2 & -2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{basis}: \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\ker(A-3I) = \ker\begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{basis} \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a diagonal matrix which is similar to A^{100}. Justify your answer.

$$A = P\begin{bmatrix} 2 & & \\ & 2 & \\ & & 3 \end{bmatrix}P^{-1}$$

$$A^{100} = P\begin{bmatrix} 2^{100} & & \\ & 2^{100} & \\ & & 3^{100} \end{bmatrix}P^{-1}$$

So $\begin{bmatrix} 2^{100} & & \\ & 2^{100} & \\ & & 3^{100} \end{bmatrix}$ is similar to A^{100}.