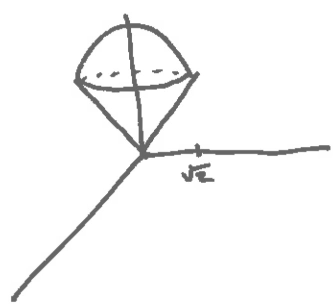
1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 e^{\rho}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{2}} e^{\sqrt{r^2+z^2}}\,r\,dz\,dr\,d\theta.$

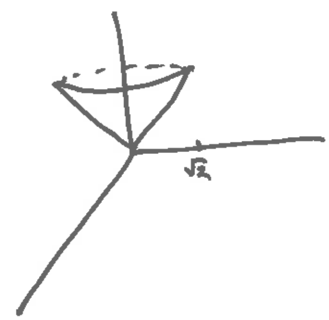
FALSE

$$= \iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV \qquad\qquad = \iiint_D e^{\sqrt{x^2+y^2+z^2}}\,dV$$

where E is



where D is



(b) Suppose $\mathbf{F} = (xy, e^x, \sqrt{1-y})$ and C is the curve which connects $(0,0,1)$ and $(1,1,0)$ along the intersection of $y = x^2$ and $z = 1-x$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_0^{\frac{\pi}{2}} (\sin^3(t), e^{\sin(t)}, \cos(t))\cdot(\cos(t), 2\sin(t)\cos(t), -\cos(t))\,dt$$

TRUE Using parametric equations

$$\vec{x}(t) = (\sin t, \sin^2 t, 1-\sin t), \quad 0 \le t \le \pi/2$$

$$\vec{F}(\vec{x}(t)) = \left(\sin^3 t, e^{\sin t}, \underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\right)$$

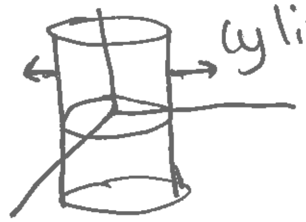
$$\vec{x}'(t) = (\cos t, 2\sin t\cos t, -\cos t)$$

And $t=0$ give start point $(0,0,1)$

$t=\pi/2$ given end point $(1,1,0)$

(c) The flux of $\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + \frac{x\ln(y^2+1)}{z^2+2}\mathbf{k}$ through the surface $x^2 + y^2 = 1$, $-3 \le z \le 3$ is zero.

TRUE



cylinder has normal vector with no $\vec{k}$-component regardless of orientation,

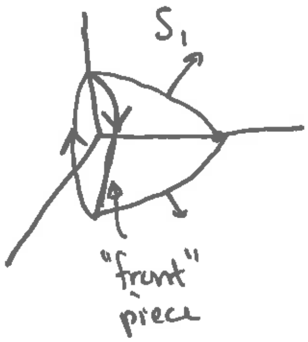
So $\vec{F} \cdot \text{normal vector} = 0$

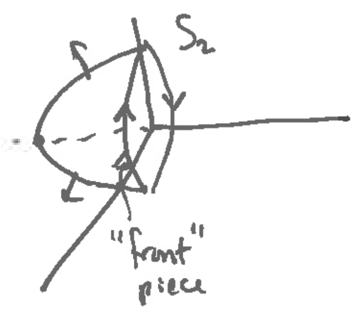
and $\iint_S \vec{F} \cdot d\vec{S}$ thus equals 0.

(d) Let S_1 be the oriented surface $y = 4 - x^2 - z^2$ with $y \ge 0$ and outward pointing unit normal. Let S_2 be the oriented surface $y = -4 + x^2 + z^2$ with $y \le 0$ and outward pointing unit normal. Then for any differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\iint_{S_1} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = -\iint_{S_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

TRUE





Both have same boundary which is the circle of radius 2 on xz-plane

but S_1 gives this circle the opposite orientation that S_2 does so Stokes's Theorem says

$$\iint_{S_1} \text{curl}\,\vec{F} \cdot d\vec{S} = \int_{\partial S_1} \vec{F} \cdot d\vec{s} = -\int_{\substack{\partial S_1 \\ \text{opposite}}} \vec{F} \cdot d\vec{s} = -\iint_{S_2} \text{curl}\,\vec{F} \cdot d\vec{S}.$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) If R is a closed rectangle in $\mathbb{R}^2$ and $f, g : R \to \mathbb{R}$ are non-negative integrable functions, then

$$\iint_R (\cos^7(x^2y)f(x,y) + 5^{1+x^2+y^2}g(x,y))\,dA < \iint_R -f(x,y)\,dA + 5\iint_R g(x,y)\,dA.$$

NEVER:

$$\iint_R \cos^7(x^2y)f(x,y) + 5^{1+x^2+y^2} g(x,y)\,dA \geq \iint_R (-1)f(x,y)\,dA \iint 5\, g(x,y)\,dA$$

(b) For a circle C of radius 1 in the xy-plane, the line integral of $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$ over C is nonzero.

SOMETIMES: $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = \dfrac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \dfrac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = 0$

and F is defined everywhere but at the origin; so if C does not encompass the origin, $\int_C \vec{F}\cdot d\vec{s} = 0$, since $\vec{F}$ is conservative (or by Green's thm.).

On the other hand, consider the unit circle centered at $(0,0)$: $(x,y) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Then $\int_C \vec{F}\cdot d\vec{s} = \int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \sin^2 t\,dt + \cos^2 t\,dt = 2\pi \neq 0.$

(c) For an oriented surface S which encloses a bounded region D, the surface integral of $\mathbf{F} = (2xz - y^9z^2)\mathbf{i} - (x + z\cos e^x)\mathbf{j} + (z - z^2 + yx)\mathbf{k}$ over S equals the volume of D.

$\vec{\nabla}\cdot\vec{F} = 2z + 0 + 1 - 2z = 1$, so by Gauss's/divergence theorem,

$\text{area}(D) = \iiint_D \vec{\nabla}\cdot\vec{F} = \iint_{\partial D} \vec{F}\cdot d\vec{S} = \pm \iint_S \vec{F}\,d\vec{S}$

with + or − depending on orientation.

SOMETIMES.

(d) If $x = f(y, z)$ is a C^1 function which gives a smooth, bounded surface for (y, z) in some bounded subset D of the yz plane, then the surface area of the graph of $f(y, z)$ over D is given by

$$\iint_D \sqrt{f_y^2 + f_z^2 + 1}\,\mathrm{dy}\,\mathrm{dz}.$$

ALWAYS: parametrize the graph by

$(f(s,t), s, t)$ for $(s,t) \in D$.

Then

$$\vec{N}(s,t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_s(s,t) & 1 & 0 \\ f_t(s,t) & 0 & 1 \end{vmatrix} = \vec{i} - f_s(s,t)\vec{j} - f_t(s,t)\vec{k}$$

so Surface area $= \iint_D \sqrt{1^2 + f_y^2 + f_x^2}\,dx\,dy$

3. Maximize the function $f(x, y, z) = x + y$ subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 + z^2 = 1 \quad \text{and} \quad g_2(x, y, z) = x + y + z = 1.$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$(1, 1, 0) = \lambda_1 (2x, 2y, 2z) + \lambda_2 (1, 1, 1)$$

gives
$$1 = 2\lambda_1 x + \lambda_2$$
$$1 = 2\lambda_1 y + \lambda_2$$
$$0 = 2\lambda_1 z + \lambda_2$$

First two gives
$$0 = 2\lambda_1 (x - y)$$

If $\lambda_1 = 0$, 3rd gives $\lambda_2 = 0$ which can't satisfy first two equations.

Thus $\lambda_1 \neq 0$ So $\boxed{x = y}$.

Constraints become $2x^2 + z^2 = 1$

$2x + z = 1 \rightsquigarrow z = 1 - 2x$

So $2x^2 + (1 - 2x)^2 = 1 \Rightarrow 6x^2 - 4x + 1 = 1$

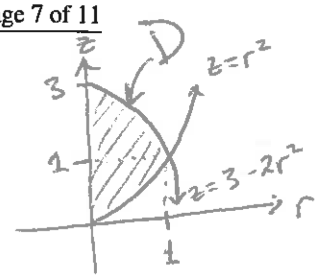
$\Rightarrow x = 0$ or $\frac{2}{3}$.

Then $y = 0$ or $2/3$ and $z = 1$ or $-1/3$.

Critical points: $(0, 0, 1)$ and $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$.

$f(0, 0, 1) = 0$ $f(2/3, 2/3, -1/3) = 4/3$

So $\boxed{\text{max } 4/3 \text{ at } (\frac{2}{3}, \frac{2}{3}, -1/3)}$



4. The center of mass of an object D of uniform density is given by

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{\iiint_D x\,dV}{\iiint_D dV}, \frac{\iiint_D y\,dV}{\iiint_D dV}, \frac{\iiint_D z\,dV}{\iiint_D dV}\right).$$

Assuming uniform density, find the center of mass of the region which lies above $z = x^2 + y^2$ and below $z = 3 - 2x^2 - 2y^2$. (Hint: use symmetry to simplify some of these computations.)

$x^2+y^2=r^2$; $\left.\begin{array}{l} z = x^2+y^2 \iff z=r^2 \\ z = 3-2(x^2+y^2) \iff z = 3-2r^2 \end{array}\right\}$ intersect at $r=z=1$

total mass:

$$\underset{D}{\iiint} \underset{\substack{\uparrow \\ \text{mass density} \\ \text{(constant)}}}{K}\, dV = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=r^2}^{z=3-2r^2} K\,r\,dz\,d\theta\,dr$$

$$= 2\pi K\int_{r=0}^{1} 3r-3r^3\,dr = 2\pi K\left(\frac{3}{2}-\frac{3}{4}\right) = 2\pi K\left(\frac{3}{4}\right) = \boxed{\frac{3}{2}\pi K} = M$$

$$\underset{D}{\iiint} x\,dV = K\int_{r=0}^{1}\int_{z=r^2}^{3-2r^2} r^2\underbrace{\left(\int_0^{2\pi}\cos\theta\,d\theta\right)}_{=0} dz\,dr$$ since $x = r\cos\theta$

$= 0.$ Similarly, $\underset{D}{\iiint} y\,dV = 0$

$$\underset{D}{\iiint} z\,dV = K\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=r^2}^{3-2r^2} r z\,dz\,d\theta\,dr = 2\pi K\int_0^1 \left.\frac{rz^2}{2}\right|_{z=r^2}^{z=3-2r^2} dr$$

$$= K\pi\int_0^1 9r-12r^3+3r^5\,dr = K\pi\left(\frac{9}{2}-3+\frac{1}{2}\right) = 2K\pi.$$

$$\bar{z} = \frac{1}{M}\underset{D}{\iiint} z\,dv = \frac{2K\pi}{\frac{3}{2}K\pi} = \boxed{\frac{4}{3}} \implies \boxed{(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{4}{3}\right)}$$

5. Compute the line integral of the vector field

$$\mathbf{F} = \frac{2x}{e^y + x^2}\mathbf{i} + \left(\frac{e^y}{e^y + x^2} - z\cos(y)\right)\mathbf{j} + \sin(y)\mathbf{k}$$

along the curve (oriented counter-clockwise when viewed from the positive z-axis) which is given by the intersection of the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

and the plane

$$x + \sqrt{\pi}y - 100z = 0.$$

This question was much more difficult than intended. We were <u>very</u> lenient in grading it, and any reasonable attempt using conservative vector fields or Stokes's Theorem received most (if not all) the points.

Let $f(x,y,z) = \ln(e^y + x^2) - z\sin(y)$. Note f is defined everywhere since $e^y + x^2 > 0$.

Then

$$\nabla f = \left(\frac{2x}{e^y + x^2}, \frac{e^y}{e^y+x^2} - z\cos(y), -\sin(y)\right) = F - G,$$

where $G(x,y,z) = (0, 0, 2\sin(y))$. Since our curve C is closed,

$$\int_C \nabla f \cdot d\vec{s} = 0 \quad \text{and} \quad \int_C F \cdot d\vec{s} = \int_C G \cdot d\vec{s}$$

We can parametrize C by

$$\vec{x}(t) = (-2\sqrt{\pi}\sin(t) + 300\cos(t), 2\sin(t), 3\cos(t)) \qquad t \in [0, 2\pi]$$

Then $\int_C F \cdot d\vec{s} = \int_C G \cdot d\vec{s}$

$$= \int_0^{2\pi} G(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt$$

$$= \int_0^{2\pi} (0, 0, 2\sin(2\sin(t))) \cdot (\underset{\substack{\downarrow \\ \text{not important}}}{\_\_}, \underset{\downarrow}{\_\_}, -3\sin(t))\,dt$$

$$= \int_0^{2\pi} -6\sin(t)\sin(2\sin(t))\,dt$$

= Very hard to compute.

6. Let C be the circle in the plane $x + y + z = 0$ centered at $(0, 0, 0)$ and of radius 1 (this is the intersection of the plane $x + y + z = 0$ with the sphere $x^2 + y^2 + z^2 = 1$). Orient C counterclockwise about the vector $(1, 1, 1)$. Let

$$\mathbf{F}(x, y, z) = (4x^3y^3 + 2xz^2 + z)\mathbf{i} + (3x^4y^2 + 3y^2z^4 + x)\mathbf{j} + (2x^2z + 4y^3z^3 + y)\mathbf{k}.$$

Find

$$\int_C \mathbf{F} \cdot \cancel{d\mathbf{S}} \; d\vec{s}$$

Can we use Stokes' thm?

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x^3y^3 + 2xz^2 + z & 3x^4y^2 + 3y^2z^4 + x & 2x^2z + 4y^3z^3 + y \end{vmatrix} =$$

$$= \vec{i}\left(12y^2z^3 + 1 - (12y^2z^3)\right) - \vec{j}\left(4xz - (4xz + 1)\right) + \vec{k}\left(12x^3y^2 + 1 - (12x^3y^2)\right)$$

$$= \vec{i} + \vec{j} + \vec{k}$$

Let S be the disk in the plane $x+y+z=0$ bounded by C. Then by Stokes' thm,

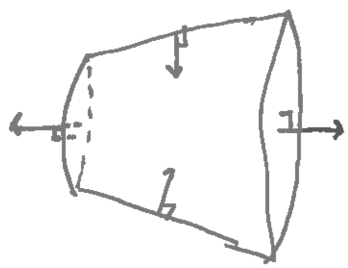
$$\int_C \vec{F} \cdot d\vec{s} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$$

and for the plane $x+y+z=0$, $\vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

So $\int_C \vec{F} \cdot d\vec{s} = \iint_S (1,1,1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) dS = \frac{3}{\sqrt{3}} \iint_S dS$

$= \frac{3}{\sqrt{3}} \cdot \text{area}(\text{circle of radius } 1) = \sqrt{3}\,\pi.$ $\square$

8. Compute the surface integral of $\mathbf{F} = (x - z^{y\sin z})\mathbf{i} + (2-y)\mathbf{j} + (zx - \sin(e^{xy}))\mathbf{k}$ over the piece of the cone $y^2 = x^2 + z^2$ lying between $y = 1$ and $y = 4$ oriented with normal vectors pointing towards the y-axis.



Let S be this piece of cone.

Let S_1 be: $x^2+z^2 \leq 1$ and $y=1$ oriented with unit normal $(0,-1,0)$.

Let S_2 be: $x^2+z^2 \leq 16$ and $y=4$ oriented with unit normal $(0,1,0)$.

Let D be the solid region bounded by $S \cup S_1 \cup S_2$.

S_1 and S_2 have outward pointing unit normal but S has inward pointing unit normal. So, the divergence theorem implies

$$\iiint_D \operatorname{div}(F)\, dV = \iint_{S_1} F \cdot dS + \iint_{S_2} F \cdot dS - \iint_S F \cdot dS$$

$$\operatorname{div}(F) = \frac{\partial}{\partial x}(x - z^{y\sin z}) + \frac{\partial}{\partial y}(2-y) + \frac{\partial}{\partial z}(zx - \sin(e^{xy})) = 1 - 1 + x = x.$$

D is symmetric about the plane $x=0$, so

$$\iiint_D \operatorname{div}(F)\, dV = \iiint_D x\, dV = 0.$$

So, $\iint_S F\cdot dS = \iint_{S_1} F \cdot dS + \iint_{S_2} F\cdot dS.$

$$= \iint_{S_1} (F \cdot (0,-1,0))\, dS + \iint_{S_2} (F \cdot (0,1,0))\, dS$$

$$= \iint_{S_1} (y-2)\, dS + \iint_{S_2} (2-y)\, dS$$

$$= \iint_{S_1} (-1)\, dS + \iint_{S_2} -2\, dS$$

$$= -(\text{Surface area of } S_1) - 2(\text{Surface area of } S_2) = -33\pi.$$