Northwestern University

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Math 290-3 Final Exam Solutions

Spring Quarter 2014

Tuesday, June 10, 2014

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) Let $\mathbf{F}(x,y,z) = (y+\cos x, y^2\cos(z^2), e^{x^2+1}+y^2)$ and let S be the [boundary of the] unit cube $[0,1]\times[0,1]\times[0,1]$, oriented with outward-facing normals. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} \le 3.$$

TRUE: By the divergence theorem, this integral is equal to

$$\iiint_{[0,1]\times[0,1]\times[0,1]} \operatorname{div}\mathbf{F}\, dV.$$

But $\operatorname{div}\mathbf{F} = -\sin x + 2y\cos(z^2) + 0$, so for all (x,y,z) in the unit cube we have $\operatorname{div}\mathbf{F} \le 1 + 2(1)(1) = 3$, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} \le 3\cdot \operatorname{vol}(\text{cube}) = 3.$$

(b) The value of the double integral

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (y^2+1)\,dx\,dy$$

is equal to the value of the triple integral

$$\int_0^\pi \int_0^2 \int_{-r}^{r} r\sin(z^9)\,dz\,dr\,d\theta.$$

FALSE: On the one hand, the first integral is over a half disk of radius 2, so

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (y^2+1)\,dx\,dy \ge \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 1\,dx\,dy = \frac{1}{2}\pi(2)^2 = 2\pi > 0.$$

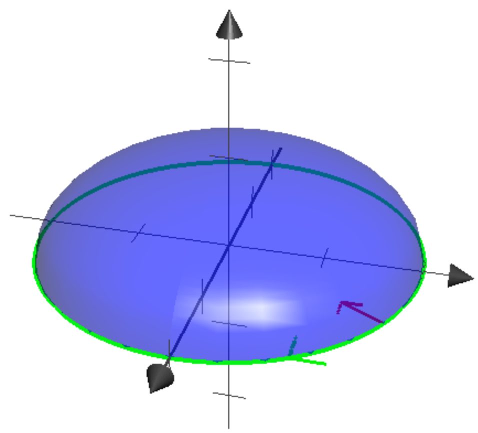
On the other hand $r\sin(z^9)$ is an odd function of z and for each $r \ge 0$, $[-r,r]$ is symmetric around the origin, so

$$\int_0^\pi \int_0^2 \int_{-r}^{r} r\sin(z^9)\,dz\,dr\,d\theta = 0.$$

(c) If $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^3$ and S is the top half of the ellipsoid $z^2 + \frac{x^2}{4} + \frac{y^2}{4} = 1$, oriented with normals pointing toward the origin, then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_0^{2\pi} \mathbf{F}(2\cos t, 2\sin t, 0) \cdot (-2\sin t, 2\cos t, 0)\, dt.$$

FALSE:



In the picture on the right, the blue ellipsoid is S, the green circle is the boundary of S, and the red vector is a normal to S. If this normal vector traverses the boundary in the direction indicated, the surface will be to its left, so Stokes's theorem guarantees that $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the circle of radius 2 centered at the origin, oriented clockwise. The given integral is that same integral but with C oriented counterclockwise, so it is the negative of what we want. If, for example, $\mathbf{F}(x, y, z) = (-y, x, 0)$, then $\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = 4$, so the integrals are nonzero and therefore different.

(d) There exists a C^2 function $f(x, y)$ such that $\nabla f(x, y) \cdot (-y, x) > 0$ for all (x, y) on the unit circle.

FALSE: If f is C^2 then ∇f is a conservative vector field so its integral around the unit circle C (oriented, say, counterclockwise) must be zero (since the circle is closed). But if $\nabla f(x, y) \cdot (-y, x) > 0$, then

$$\int_C \nabla f \cdot d\mathbf{s} = \int_0^{2\pi} \nabla f(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt > 0.$$

So we have a contradiction.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) For a nonzero C^1 vector field $\mathbf{F}$ on $\mathbb{R}^3$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$$

for every closed, smooth oriented surface S.

SOMETIMES: By the divergence theorem, if S is such a surface and $\mathbf{F}$ is such a vector field, then

$$\iint_S \mathbf{F}\, d\mathbf{S} = \iiint_D \operatorname{div} \mathbf{F}\, dV,$$

where D is a solid whose boundary is S and S is oriented with normals pointing out of D. If $\mathbf{F}(x, y, z) = (y, 0, 0)$, then $\operatorname{div} \mathbf{F} = 0$, so the statement is true. If $\mathbf{F}(x, y, z) = (x, 0, 0)$, then $\operatorname{div} \mathbf{F} = 1$, so if S is the unit sphere, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 1 \cdot \left(\frac{4}{3}\pi\right) \neq 0.$$

(b) Given two simple, smooth parameterizations $\mathbf{x}(t)$ $(a \le t \le b)$ and $\mathbf{y}(t)$ $(c \le t \le d)$ of the same curve C with $\|\mathbf{x}'(t)\| > 3$ for all $a \le t \le b$ and $\|\mathbf{y}'(t)\| < 2$ for all $c \le t \le d$, we have

$$\int_a^b \|\mathbf{x}'(t)\|\, dt > \int_c^d \|\mathbf{y}'(t)\|\, dt.$$

NEVER: For *any* simple, smooth parametrization $\mathbf{z}(t)$ $(t_1 \le t \le t_2)$ of C, the integral $\int_{t_1}^{t_2} \|\mathbf{z}'(t)\|\, dt$ gives the arc length of C, so the above two integrals must be equal.

(c) For a solid region E in $\mathbb{R}^3$ (with nonzero volume), we have

$$\iiint_E (x^2+y^2+z^2)\,dV = 3\iiint_E y^2\,dV.$$

SOMETIMES: This is true if E is the unit ball by the symmetry of the ball. If, however, E is the ball of radius 1 centered at $(0,0,11)$, then

$$\iiint_E (x^2+y^2+z^2)\,dV \geq \iiint_E z^2\,dV \geq \iiint_E 100\,dV = 100\left(\frac{4}{3}\pi\right)$$

but

$$3\iiint_E y^2\,dV \leq 3\iiint_E 1\,dV = 3\left(\frac{4}{3}\pi\right) < \iiint_E (x^2+y^2+z^2)\,dV.$$

(d) For $0 < a < \pi/2$, the flux of $\mathbf{F}(x,y,z) = (x,y,z)$ across S_a is zero, where S_a is the part of the surface $\varphi = a$ (in spherical coordinates) that lies below $z = 5$, oriented with outward-facing normals.

ALWAYS: The surface $\varphi = a$ is a (half-)cone so for some $b \in \mathbb{R}$, it can be parameterized by

$$\mathbf{X}(s,t) = (s\cos t, s\sin t, bs),$$

where $bs \leq 5$, so the parameters range over $0 \leq t \leq 2\pi$ and $0 \leq s \leq 5/b$. We have

$$\begin{aligned}
\mathbf{T}_s &= (\cos t, & \sin t, && b) \\
\mathbf{T}_t &= (-s\sin t, & s\cos t, && 0) \\
\mathbf{N} &= (-bs\cos t, & -bs\sin t, && s)
\end{aligned}$$

Hence, the flux is

$$\begin{aligned}
\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{5/b}\int_0^{2\pi} (s\cos t, s\sin t, bs)\cdot(-bs\cos t, -bs\sin t, s)\,dt\,ds \\
&= \int_0^{5/b}\int_0^{2\pi} (-bs^2\cos^2 t - bs^2\sin^2 t + bs^2)\,dt\,ds \\
&= \int_0^{5/b}\int_0^{2\pi} 0\,dt\,ds = 0.
\end{aligned}$$

3. Let S be the part of the cylinder $x^2 + y^2 = 4$ that lies between $z = 0$ and $z = 3$, oriented with outward-facing normals. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = (x^3 + e^{y^2} z, y^3 - e^{x^2} z, zx^2 + zy^2).$$

ANSWER: Note that $\operatorname{div} \mathbf{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4(x^2 + y^2)$, which is much simpler than $\mathbf{F}$, so we'd like to use the divergence theorem, but our surface is not closed. So we "close it off" with the disk S_1 of radius 2 at height 0, oriented with downward-facing normals, and the disk S_2 of radius 2 at height 3, oriented with upward-facing normals. Then

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iiint_D 4(x^2 + y^2)\, dV,$$

where D is the solid cylinder enclosed by our surfaces.

Changing to cylindrical coordinates, the right-hand side is

$$\int_0^3 \int_0^{2\pi} \int_0^2 4r^2 r \, dr \, d\theta \, dz = 3 \cdot 2\pi \cdot 2^4 = 96\pi.$$

To parameterize S_1, let $\mathbf{X}(s, t) = (s \cos t, s \sin t, 0)$, for $0 \le s \le 2$ and $0 \le t \le 2\pi$. Then

$$\begin{array}{rcl} \mathbf{T}_s &=& (\cos t, \quad \sin t, \quad 0) \\ \mathbf{T}_t &=& (-s \sin t, \quad s \cos t, \quad 0) \\ \mathbf{N} &=& (0, \quad 0, \quad s) \end{array}$$

This normal points up, which is the wrong orientation,so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{s} = -\int_0^2 \int_0^{2\pi} (\_, \_, 0) \cdot (0, 0, s)\, dt \, ds = 0.$$

For the other disk, let $\mathbf{X}(s, t) = (s \cos t, s \sin t, 3)$, for $0 \le s \le 2$ and $0 \le t \le 2\pi$. Then we get the same tangent vectors and the same normal. This time 'up' is the correct orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \int_0^{2\pi} (\_, \_, 3s^2) \cdot (0, 0, s)\, dt = \int_0^2 \int_0^{2\pi} 3s^3 \, dt \, ds = 2\pi \cdot \frac{3}{4} 2^4 = 24\pi.$$

Hence,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 96\pi - 0 - 24\pi = 72\pi.$$

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where

$$\mathbf{F}(x, y, z) = (y^2z^3 - yz\sin(xyz), 2xyz^3 - xz\sin(xyz), 3xy^2z^2 - xy\sin(xyz) + x)$$

and C is the curve consisting of the line segment from $(1, 1, 1)$ to $(0, 1, 0)$ followed by the line segment from $(0, 1, 0)$ to $(0, 0, 0)$.

ANSWER:

Our vector field looks like it is "almost" a gradient field. The x term in the third component seems to be the "odd man out." So we try to find a scalar potential for

$$(y^2z^3 - yz\sin(xyz), 2xyz^3 - xz\sin(xyz), 3xy^2z^2 - xy\sin(xyz))$$

Antidifferentiating with respect to x, we get

$$f = \int (y^2z^3 - yz\sin(xyz)\, dx = xy^2z^3 + \cos(xyz) + g(y, z).$$

So

$$2xyz^3 - xz\sin(xyz) = f_y = 2xyz^3 - xz\sin(xyz) + g_y,$$

so $g = h(z)$. Finally,

$$3xy^2z^2 - xy\sin(xyz) = f_z = 3xy^2z^2 - xy\sin(xyz) + h'(z),$$

so we may take $f(x, y, z) = xy^2z^3 + \cos(xyz)$. So parameterizing the two line segments with $\mathbf{x}(t) = (1 - t, 1, 1 - t)$ $(0 \le t \le 1)$ and $\mathbf{y}(t) = (0, 1 - t, 0)$ $(0 \le t \le 1)$, we get

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_C \nabla f \cdot d\mathbf{s} + \int_C (0, 0, x) \cdot d\mathbf{s} \\
&= (f(0, 0, 0) - f(1, 1, 1)) + \int_0^1 (0, 0, 1 - t) \cdot (-1, 0, -1)\, dt + \int_0^1 (0, 0, 0) \cdot (0, -1, 0)\, dt \\
&= (0 + 1) - (1 + \cos(1)) + \int_0 (t - 1)\, dt + 0 = -\cos(1) + \left[\frac{t^2}{2} - t\right]_{t=0}^1 \\
&= -\cos(1) + \frac{1}{2} - 1 = -\frac{1}{2} - \cos(1)
\end{aligned}$$

5. Farmer Steve is replacing the roof on his silo. The new roof will be made of metal and be in the shape of $z = 9 - x^2 - y^2$, for $z \geq 5$ (with distances measured in meters). Find the total amount of metal (in square meters) Steve will need to build the roof.

ANSWER: We can parameterize the paraboloid with $\mathbf{X}(s,t) = (s, t, 9 - s^2 - t^2)$. Since we only want the part with $z \geq 5$, the "shadow" of our surface is $s^2 + t^2 \leq 4$, so letting D be the disk in the st-plane with center $(0,0)$ and radius 2, the surface area is

$$\iint_S 1\, dS = \iint_D \|\mathbf{N}(s,t)\|\, ds\, dt,$$

where $\mathbf{N}$ is the standard normal of our parameterization. We have

$$\begin{aligned} \mathbf{T}_s &= (1, \quad 0, \quad -2s) \\ \mathbf{T}_t &= (0, \quad 1, \quad -2t) \\ \mathbf{N} &= (2s, \quad 2t, \quad 1) \end{aligned}$$

Hence, switching to cylindrical coordinates, we have

$$\begin{aligned} \iint_D \|\mathbf{N}(s,t)\|\, ds\, dt = \iint_D \sqrt{4s^2 + 4t^2 + 1}\, ds\, dt &= \int_0^2 \int_0^{2\pi} \sqrt{4r^2 + 1}\, r\, d\theta\, dr \\ &= \frac{1}{8} \int_0^2 \int_0^{2\pi} 8r\sqrt{4r^2 + 1}\, d\theta\, dr = \frac{\pi}{4} \int_0^2 8r\sqrt{4r^2+1}\, dr \\ &= \frac{\pi}{4} \left[\frac{2}{3}(4r^2 + 1)^{3/2} \right]_{r=0}^2 = \frac{\pi}{6}(17^{3/2} - 1) \end{aligned}$$

6. Find $\oint_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = (\sin(e^{x^2}) + yz, x\cos y, xz^2)$ and C is the rectangular curve consisting of the line segment from $(0, 0, 0)$ to $(0, 2, 0)$, followed by the one from $(0, 2, 0)$ to $(2, 2, 2)$, followed by the one from $(2, 2, 2)$ to $(2, 0, 2)$, followed by the one from $(2, 0, 2)$ to $(0, 0, 0)$.

ANSWER: Integrating $\mathbf{F}$ itself seems difficult because of the $\sin(e^{x^2})$, so let's use Stokes's Theorem to instead integrate curl $\mathbf{F}$. We have

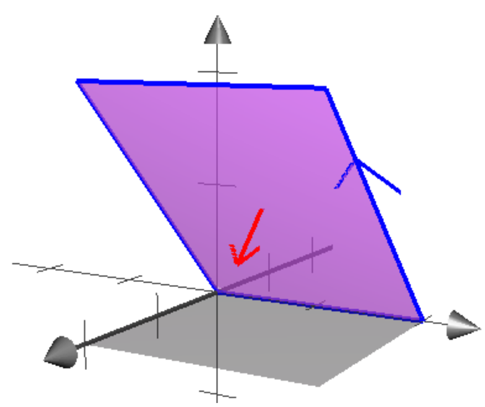
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(e^{x^2}) + yz & x\cos y & xz^2 \end{vmatrix} = (0, y - z^2, \cos y - z).$$

According to Stokes's, our original integral is equal to the integral of this vector field over the rectangular chunk of a plane enclosed by C, which we will call S. To find the equation for the plane, we first find a normal vector. The displacement vectors $(0, 2, 0) - (0, 0, 0) = (0, 2, 0)$ and $(2, 0, 2) - (0, 0, 0) = (2, 0, 2)$ are both parallel to the plane, so we may take

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix} = (4, 0, -4).$$

Our plane is then $4x - 4z = d$ for some constant d, but $(0, 0, 0)$ is on the plane, so $d = 0$ and our equation is $z = x$.

In the image to the right, the purple rectangle is S, the blue boundary is our original oriented curve C, the grey square is "shadow" $[0, 2] \times [0, 2]$ in the xy-plane, which gives the ranges of s and t for our parameterization, and the red vector is a normal vector giving the right orientation to apply Stokes's theorem. So choose $\mathbf{X}(s, t) = (s, t, s)$, which has standard normal $\mathbf{N} = (-1, 0, 1)$. This points the wrong way for Stokes's theorem, so we get



$$\oint_C \mathbf{F} \cdot d\mathbf{s} = -\int_0^2 \int_0^2 (\nabla \times \mathbf{F})(\mathbf{X}(s, t)) \cdot \mathbf{N}(s, t)\, ds\, dt = -\int_0^2 \int_0^2 (0, t - s^2, \cos t - s) \cdot (-1, 0, 1)\, ds\, dt$$

$$= -\int_0^2 \int_0^2 (\cos t - s)\, ds\, dt = -\int_0^2 (2\cos t - 2)\, dt = -[2\sin t - 2t]_{t=0}^2 = -(2\sin(2) - 4) = 4 - 2\sin(2)$$

7. Find the volume of the solid enclosed by the paraboloid $z = y^2 + \frac{x^2}{4}$ and the plane $z = 4$.

ANSWER: The solid enclosed by the two surfaces is most easily described in the following change of variables, which is a slight variant of cylindrical coordinates:

$$x = 2r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Note that on the paraboloid we have $z = r^2\sin^2\theta + \frac{4r^2\cos^2\theta}{4} = r^2$, so $r = \sqrt{z}$. So our region is described by $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{z}$, $0 \le z \le 4$. The volume element for this change of variables is

$$\left|\frac{\partial(x,y,z)}{\partial r,\theta,z}\right| = \begin{vmatrix} 2\cos\theta & \sin\theta & 0 \\ -2r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = |2r\cos^2\theta + 2r\sin^2\theta| = 2r.$$

Hence, the volume is

$$\int_0^{2\pi}\int_0^4\int_0^{\sqrt{z}} 2r\,dr\,dz\,d\theta = 2\pi\int_0^4 \left[r^2\right]_{r=0}^{\sqrt{z}} dz = 2\pi\int_0^4 z\,dz = \left[\pi z^2\right]_{z=0}^4 = 16\pi.$$

8. Let $\mathbf{F}(x, y, z) = (ye^{x^2+y^2-1}, -x + \sin(z^2), x^2y)$ and let S be the top half of the unit sphere, oriented with outward-facing normals. Find $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

   ANSWER: On the boundary of S, we have $z = 0$ and $x^2 + y^2 = 1$, which greatly simplifies $\mathbf{F}$. So let's apply Stokes's Theorem. The counterclockwise orientation of the circle is consistent with the orientation of S (the picture is similar to the one in problem 1a, but with outward-facing normals). So we take $\mathbf{x}(t) = (\cos t, \sin t, 0)$, for $0 \le t \le 2\pi$. Then Stokes's Theorem gives

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_0^{2\pi} ((\sin t)e^0, -\cos t + \sin(0), \cos^2 t \sin t) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = -2\pi.\end{aligned}$$