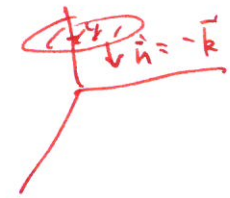
1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **five** parts.)

   (a) Let S be the surface $z = \sqrt{x^2+y^2}$ for $z \leq 4$ with upward orientation and let $\mathbf{F} = (-ye^z, xe^z, 0)$. Then $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$.

   Answer: TRUE

$$\iint_S \vec{F}\cdot d\vec{S} = -\iiint_E \underbrace{\operatorname{div}\vec{F}}_{=0}\, dV - \underbrace{\iint_{S_1} \vec{F}\cdot d\vec{S}}_{\vec{F}\cdot\vec{n} = \vec{F}\cdot(-\vec{k}) = 0}$$

where S_1 is



So $-0-0=0$.

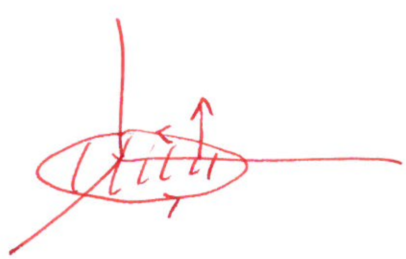
Can also use parametric equation for cone

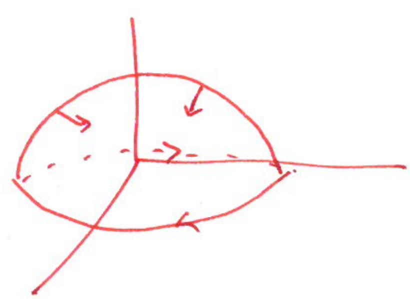
   (b) Let D be the unit disk $x^2+y^2\leq 1$ in the xy-plane with upward orientation, and let S be the top half of the unit sphere $x^2+y^2+z^2=1$ with inward orientation. Then

$$\iint_D \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

   for $\operatorname{curl}\mathbf{F} = (x, -y, 1)$.

   Answer: FALSE





Same boundary but opposite orientation so

$$\iint_D \operatorname{curl}\vec{F}\cdot d\vec{S} = -\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}, \text{ and } \iint_D \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{curl}\vec{F}\cdot\vec{k}\, dS = \iint_D dS = \pi$$

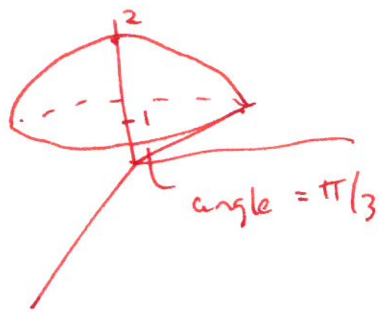
So both nonzero.

(c) The surface area of the "spherical cap" which is the part of the sphere of radius 2 centered at the origin that is above the plane $z = 1$ is

$$\int_0^{2\pi} \int_0^{\pi/2} 2 \sin\phi \, d\phi \, d\theta.$$

Answer: TRUE

$$\hookrightarrow \int_0^{2\pi} -2\cos\phi \Big|_0^{\pi/2} d\theta = 4\pi$$



Use $\vec{X}(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$

$$\|\vec{X}_\phi \times \vec{X}_\theta\| = 4\sin\phi$$

$$\text{area} = \int_0^{2\pi}\int_0^{\pi/3} 4\sin\phi \, d\phi \, d\theta = 4\pi$$

as well.

(d) Let $\mathbf{F} = (yz + ze^{xz}, z^2 + xz, 2yz + xy + xe^{xz})$, and let C be the part of the curve $\mathbf{x}(t) = (t^3 \sin t, 2t, 1 - \cos^2 t)$ with $0 \le t \le \pi$. Then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

Answer: TRUE

$$\vec{F} = \nabla\left(xyz + e^{xz} + yz^2\right)$$

and $\vec{x}(0) = (0,0,0)$ $\quad \vec{x}(\pi) = (0, 2\pi, 0)$

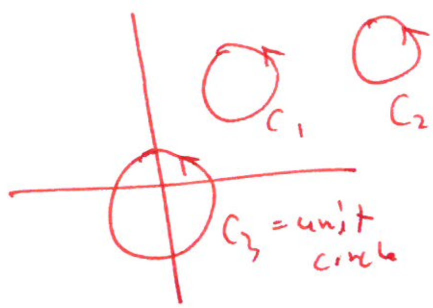
So $\int_C \vec{F} \cdot d\vec{s} = f(0, 2\pi, 0) - f(0,0,0)$

$$= e^0 - e^0 = 0.$$

(e) Let C_1, C_2, and C_3 be three circles in $\mathbb{R}^2$ oriented counterclockwise such that $(0,0)$ does not lie on any of them, and let $\mathbf{F} = \frac{y\mathbf{i}-x\mathbf{j}}{x^2+y^2}$. If $\int_{C_1} \mathbf{F}\cdot d\mathbf{s} = \int_{C_2} \mathbf{F}\cdot d\mathbf{s}$, then $\int_{C_1} \mathbf{F}\cdot d\mathbf{s} = \int_{C_3} \mathbf{F}\cdot d\mathbf{s}$.

Answer: FALSE

If



then since $\text{curl}\,\vec{F} = \vec{0}$

$$\int_{C_1} \vec{F}\cdot d\vec{s} = 0 = \int_{C_2} \vec{F}\cdot d\vec{s}$$

by Green's Thm (which applies since $\vec{F}$ is C^1 on region enclosed by C_1, C_2)

but $\int_{C_3} \vec{F}\cdot d\vec{s} = 2\pi$ using parametric equations $\vec{X}(t) = (\cos t, \sin t)$. $0 \le t \le 2\pi$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **five** parts.)

   (a) For a number k, the vector field $\mathbf{F} = (6x^2y, 4y^2 + kx^3, ze^z)$ has path-independent line integrals in $\mathbb{R}^3$.

   Answer: Sometimes

   Since $\mathbb{R}^3$ is simply-connected, $\vec{F}$ has path independent line integrals exactly when $\operatorname{curl}\vec{F} = \vec{0}$.

   $\operatorname{curl}\vec{F} = \begin{pmatrix} 0-0 \\ 0-0 \\ 3kx^2 - 6x^2 \end{pmatrix}$, so $\operatorname{curl}\vec{F} = \vec{0}$ exactly when $k=2$.

   Example: $k=2$

   counterexample: $k \neq 2$. (Say $k=0$)

   (b) Let C be the unit circle $x^2 + y^2 = 1$, oriented counterclockwise. For a number k,

$$\oint_C (x^2 + k^2 + 1)dx + dy > 0.$$

   Answer: Never

   $\oint_C (x^2+k^2+1)dx + 1dy = \oint_C \nabla f \cdot d\vec{s}$, where

   $f(x,y) = \frac{1}{3}x^3 + (k^2+1)x + y$.

   Since C is closed, the Fundamental Theorem of Line Integrals implies that $\oint_C (x^2+k^2+1)dx + dy = 0$ for all k.

   So, this is never true.

(c) For a number k, define $f(x, y, z) = e^{kx}\sin(ky) + k^2 z$. Then there is a C^1 vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $\nabla f = \operatorname{curl}\mathbf{G}$.

Answer: Always

We know that such a $\vec{G}$ exists exactly when $\operatorname{div}(\nabla f) = 0$.

Now, $\operatorname{div}(\nabla f) = \operatorname{div}\begin{pmatrix} ke^{kx}\sin(ky) \\ ke^{kx}\cos(ky) \\ k^2 \end{pmatrix}$

$= k^2 e^{kx}\sin(ky) - k^2 e^{kx}\sin(ky) + 0 = 0$

at every point (x, y, z), for each k.

So, such a $\vec{G}$ always exists.

(d) Suppose that S is the unit sphere $x^2 + y^2 + z^2 = 1$ with inward orientation. Then for a number k,

$$\iint_S (3xz^4, 2yz^4, (k^2 - 1)z^5 + z^3) \cdot d\mathbf{S} > 0.$$

Answer: Never

Let B be the solid ball $x^2 + y^2 + z^2 \leq 1$.

By Gauss's Theorem, for every k we have

$$\iint_S (\cdots)\cdot d\vec{S} = -\iint_{-S} (\cdots)\cdot d\vec{S}$$

$$= -\iiint_B 3z^4 + 2z^4 + 5(k^2-1)z^4 + 3z^2 \, dV$$

$$= -\iiint_B \underbrace{5k^2z^4 + 3z^2}_{>0 \text{ except on } xy\text{-plane}} \, dV < 0.$$

So, this inequality never holds.

(e) Let S be the (outward oriented) portion of the sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = -1$, and let C be the circle $x^2 + y^2 = 4$ in the plane $z = -1$, oriented counterclockwise when viewed from above. For a C^1 vector field $\mathbf{F}$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \oint_C (\text{curl}\,\mathbf{F}) \cdot d\mathbf{s}.$$

Answer: Sometimes

This is sometimes true. To see why, consider:

Example: $\vec{F} = \vec{0}$ makes each side 0.

Counter example: Take $\vec{F} = \vec{k}$. Then $\text{curl}\,\vec{F} = \vec{0}$,

so $\oint_C (\text{curl}\,\vec{F}) \cdot d\vec{s} = 0$.

On the other hand, $\vec{F} = \text{curl}\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$,

so Stokes' theorem implies that

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl}\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \cdot d\vec{S}$$

$$= \oint_C \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \cdot d\vec{s} \qquad r(t) = (2\cos\theta, 2\sin\theta, -1), \quad -\pi \le \theta \le \pi$$

$$= \int_{-\pi}^{\pi} 2\cos\theta \, 2\cos\theta \, d\theta$$

$$= 8\int_0^{\pi} \cos^2\theta \, d\theta$$

$$= 4\int_0^{\pi} 1 + \cos(2\theta) \, d\theta$$

$= 4\pi \neq 0$, so the statement is false for this $\vec{F}$.

3. Determine the value of the vector line integral

$$\int_C \left(e^x - \frac{4}{3}y^3\right) dx + (3x^3 - \sin(\cos y))\, dy$$

where C is the ellipse $9x^2 + 4y^2 = 36$ oriented clockwise.

Reorient then use Green's Theorem:

The integral is

$-\iint_D (9x^2+4y^2)\,dA$ where D is the interior of the ellipse.

Change of coordinates:

$x = \frac{r\cos\theta}{3}$ $y = \frac{r\sin\theta}{2}$

$r^2 = 36 \Rightarrow r = 6$

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq 6$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{1}{3}\cos\theta & -\frac{r}{3}\sin\theta \\ \frac{1}{2}\sin\theta & \frac{r}{2}\cos\theta \end{bmatrix}$

$= \frac{r\cos^2\theta + r\sin^2\theta}{6} = \frac{r}{6}$

$= -\int_0^{2\pi}\int_0^6 \frac{r^3}{6}\,dr\,d\theta$

$= -2\pi\left(\frac{r^4}{24}\right)\Big]_0^6$

$= -2\pi \frac{6^4}{6\cdot 4}$

$= -108\pi$.

4. Let $\mathbf{F} = (xe^{\sin x}, \sin(\cos y) + y, z + e^{z^4})$. Determine the value of the vector line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

where C is the curve which starts at $(0,0,0)$, follows the spiral with parametric equations $\mathbf{x}(t) = (t\cos t, t\sin t, t^2)$ for $0 \leq t \leq 5\pi$, and then follows the line segment from $(-5\pi, 0, 25\pi^2)$ to $(0,0,0)$.

(1) Show $\vec{F}$ is conservative:

* curl $\vec{F} = \vec{\nabla} \times \vec{F} = \vec{0}$ by direct computation
* The domain, $\mathbb{R}^3$, is simply connected.

$\therefore$ $\vec{F}$ is conservative on $\mathbb{R}^3$.

(2) Since $\vec{F}$ is conservative and C is a closed curve, $\int_C \vec{F} \cdot d\vec{s} = 0$ by defn of conservative.

We could also argue that since $\vec{F}$ is conservative, there exists some potential function $f(x,y,z)$ defined on $\mathbb{R}^3$ such that $\vec{F} = \nabla f$.

By the fundamental thm of line integrals,

$$\int_C \vec{F} \cdot d\vec{s} = \underbrace{f(0,0,0)}_{\text{endpt}} - \underbrace{f(0,0,0)}_{\text{starting pt}} = 0.$$

5. Compute the surface area of the piece of the paraboloid $z = 3 - x^2 - y^2$ where $z \geq 2$.

$\vec{X}(r,\theta) = (r\cos\theta, r\sin\theta, 3-r^2)$ $\quad 0 \leq \theta \leq 2\pi$

$0 \leq r \leq 1$

$\hookrightarrow$ so $2 \leq \underbrace{3-r^2}_{z} \leq 3$

$$\vec{X}_r \times \vec{X}_\theta = (\cos\theta, \sin\theta, -2r) \times (-r\sin\theta, r\cos\theta, 0) = (2r^2\cos\theta, 2r^2\sin\theta, r)$$

$$\|\vec{X}_r \times \vec{X}_\theta\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

Surface area $= \iint \|\vec{X}_r \times \vec{X}_\theta\| \, dr\, d\theta$

$$= \int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(4r^2+1)^{3/2}\Big|_0^1 d\theta = \int_0^{2\pi}\frac{1}{12}(5^{3/2}-1)\, d\theta$$

$$= \boxed{\frac{\pi(5^{3/2}-1)}{6}}$$

6. Let S be the portion of the plane $y+z=2$ which is enclosed by the cylinder $x^2+y^2=4$, oriented upward. Compute the vector line integral

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}=(e^{x^2+y^2}, e^{\sqrt{x^2+y^2+1}}, z+y)$.

$C:\quad X(\theta)=(2\cos\theta, 2\sin\theta, 2-2\sin\theta)\quad 0\le\theta\le 2\pi$

$$\iint_S \text{curl}\,F\cdot dS = \oint_C F\,ds$$

$$= \int_0^{2\pi}(e^4, e^{\sqrt5}, 2)\circ(-2\sin\theta, 2\cos\theta, -2\cos\theta)\,d\theta$$

$$= \int_0^{2\pi}(-2e^4\sin\theta + 2e^{\sqrt5}\cos\theta - 4\cos\theta)\,d\theta$$

$$= 2e^4\cos\theta + 2e^{\sqrt5}\sin\theta - 4\sin\theta\Big]_0^{2\pi}$$

$$= 2e^4 - 2e^4$$

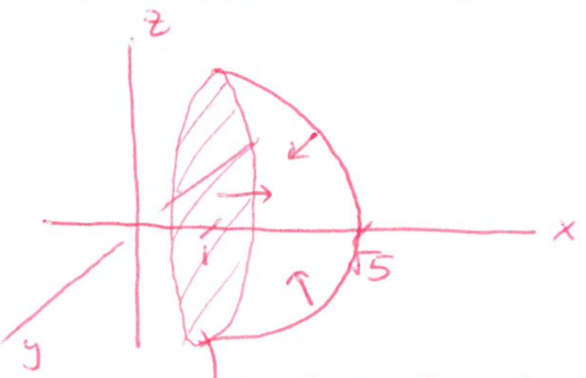
$$= 0$$

7. Let $\mathbf{F} = (y - x\cos(x^4), z + x - \cos(e^y), e^{z^3+z})$. Compute the vector surface integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where S is the piece of the sphere $x^2 + y^2 + z^2 = 5$ where $x \geq 1$, oriented with normal vectors pointing in towards the x-axis.

Method 1 (divergence/Gauss' thm):



$$\iint_S \operatorname{curl} F \cdot d\vec{S} + \iint_{S_1} \operatorname{curl} F \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}(\operatorname{curl} F)}_{=0} dV$$

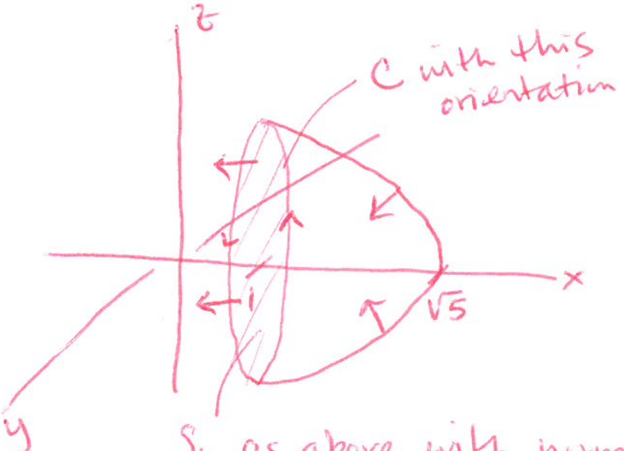
E ← region bounded by S & S_1

S_1: disk at $x=1$, $y^2+z^2 \leq 4$, ie. $X(r,\theta) = \langle 1, r\cos\theta, r\sin\theta \rangle$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$
with normal vector $+\hat{\imath}$.

So, $\displaystyle\iint_S \operatorname{curl} F \cdot d\vec{S} = -\iint_{S_1} \operatorname{curl} F \cdot d\vec{S} = -\iint_{S_1} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} +1 \\ 0 \\ 0 \end{bmatrix} dS_1 = -\iint_{S_1} -dS_1$

$= \text{area of } S_1 = \boxed{4\pi}$

Method 2 (Stokes thm):



$$\iint_S \operatorname{curl} F \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s} = \iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$= 4\pi$

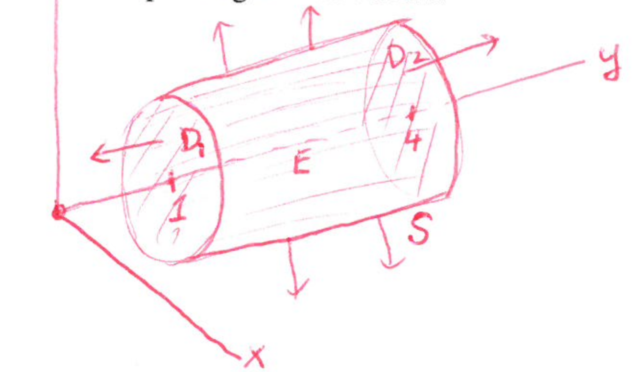
← compute as above with opposite orientation

S_1 as above with normal now $= -\hat{\imath}$

8. Determine the value of the vector surface integral

$$\iint_S \overbrace{(y\cos(e^z),\, yz + y^2,\, e^{x^2+x})}^{F} \cdot d\mathbf{S}$$

where S is the portion of the cylinder $x^2 + z^2 = 1$ with $1 \le y \le 4$, oriented with outward pointing normal vectors.



We want to apply Gauss's Theorem to the solid E inside the cylinder, so we cap off the ends with the unit disk D_1 at $y=1$ with $\vec{n} = [0,-1,0]$ and the unit disk D_2 at $y=4$ with $\vec{n} = [0,1,0]$, with orientations chosen so that the entire surface $S + D_1 + D_2$ of E is oriented out.

By Gauss's Theorem, $$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F}\cdot d\vec{S} + \iint_{D_1} \vec{F}\cdot d\vec{S} + \iint_{D_2} \vec{F}\cdot d\vec{S}$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \underbrace{\iiint_E \operatorname{div}\vec{F}\, dV}_{①} - \underbrace{\iint_{D_1} \vec{F}\cdot d\vec{S}}_{②} - \underbrace{\iint_{D_2} \vec{F}\cdot d\vec{S}}_{③}$$

we compute these:

① $$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E \overset{\text{E is symmetric across } z=0 \text{ and } z \text{ is odd}}{(z+2y)}\, dV = \iiint_E 2y\, dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{y=1}^{4} 2y \cdot r \, dr\, d\theta\, \cancel{dy} = \left(\int_{\theta=0}^{2\pi} d\theta\right)\left(\int_{r=0}^{1} r\,dr\right)\left(\int_{y=1}^{4} 2y\, dy\right)$$

$$= 2\pi \cdot \frac{1}{2} \cdot (16-1) = \underline{\underline{15\pi}}.$$

② $$\iint_{D_1} \vec{F}\cdot\vec{n}\, dS = \iint_{D_1} [\cancel{\ }, \overbrace{z+1}^{y=1 \text{ on } D_1}, \cancel{\ }]\cdot[0,-1,0]\, dS = \iint_{D_1} \overset{D_1 \text{ is symm. across } z=0 \text{ and } z \text{ is odd}}{-(z+1)}\, dS = \iint_{D_1} -1\, dS = -(\text{area of } D_1) = \underline{\underline{-\pi}}.$$

③ $$\iint_{D_2} \vec{F}\cdot\vec{n}\, dS = \iint_{D_2} [\cancel{\ }, \overbrace{4z+16}^{y=4 \text{ on } D_2}, \cancel{\ }]\cdot[0,1,0]\, dS = \iint_{D_2} \overset{D_2 \text{ is symm. across } z=0 \text{ and } 4z \text{ is odd}}{(4z+16)}\, dS = \iint_{D_2} 16\, dS = 16\,(\text{area of } D_2) = \underline{\underline{16\pi}}.$$

So $$\iint_S \vec{F}\cdot d\vec{S} = 15\pi - (-\pi) - 16\pi = \underline{\underline{0}}.$$