Northwestern University

Name: ____________________

Student ID:

Math 290-2 Final

Winter Quarter 2013

Monday, March 11, 2013

Put a check mark next to your section:

Allen		Cyr	
Canez		Peters	

Question	Possible points	Score
1	15	
2	15	
3	10	
4	12	
5	9	
6	8	
7	8	
8	11	
9	12	
TOTAL	100	

Instructions:

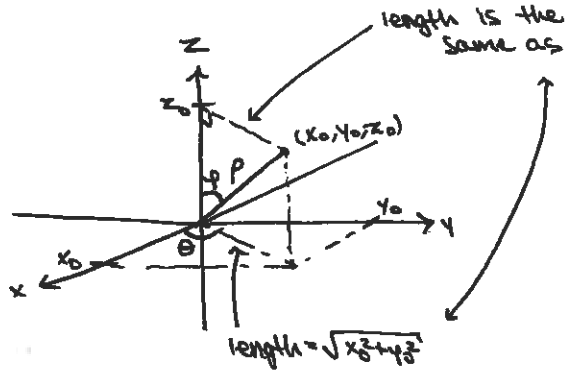
- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 15 pages, and 9 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) The spherical equation $\rho = \frac{2}{\sin\phi}$ describes an infinite cylinder of radius 2.

True.



so $\sin\varphi = \frac{\sqrt{x_0^2+y_0^2}}{\rho} \implies r = \sqrt{x_0^2+y_0^2} = \rho\sin\varphi$.

If $\rho = \frac{2}{\sin\varphi}$ then $\rho\sin\varphi = 2$. The equation is $\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{x^2+y^2} = 2\}$ which is an infinite cyl. of radius 2.

   (b) The directional derivative of $f(x, y) = x\sin xy$ at $(2, \pi)$ in any direction is larger than -4.

False.

The minimum value of $D_{\vec{v}} f(2,\pi)$ is $-\|\nabla f(2,\pi)\|$.

$$\nabla f(x,y) = \left(xy\cos(xy) + \sin(xy),\ x^2\cos(xy)\right)$$

$$\implies \nabla f(2,\pi) = (2\pi, 4).$$

$$-\|(2\pi,4)\| = -\sqrt{4\pi^2+16} < -4.$$

(c) The arc length of $\gamma(t) = (\sin t, 2\cos t, e^{8t})$ between $t = 1$ and $t = 5$ is more than 12.

True.

$$\text{Arc length} = \int_1^5 \|r'(t)\| dt$$
$$= \int_1^5 \sqrt{\cos^2(t) + 4\sin^2(t) + 64e^{16t}}\, dt$$
$$> \int_1^5 \sqrt{64e^{16t}}\, dt$$
$$= \int_1^5 8e^{8t} dt = e^{8t}\Big|_1^5 = e^{40} - e^{8} > 12.$$

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function whose second order partial derivatives are continuous (a.k.a. a C^2 function) and let $\vec{a}$ in $\mathbb{R}^2$ be a critical point for f. If there are nonzero vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ such that

$$Hf(\vec{a})\vec{v}_1 = -7\vec{v}_1 \text{ and } Hf(\vec{a})\vec{v}_2 = 5\vec{v}_2,$$

then f has a saddle point at $\vec{a}$.

True. The statements above show that -7 and 5 are eigenvalues for $Hf(\vec{a})$. Therefore $Hf(\vec{a})$ is the matrix of an indefinite quadratic form and $\vec{a}$ is a saddle point.

(e) The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y, z) = x^4 + y^4 + z^4$$

has a local minimum at $(0, 0, 0)$.

True. $f(0,0,0) = 0$.

$f(x,y,z) > 0$ for any $(x,y,z) \neq (0,0,0)$.

So $(0,0,0)$ is a local minimum for f

(actually it is the global minimum too).

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) If $r_1, r_2 : [a, b] \to \mathbb{R}^3$ are two curves whose images are contained in the xy-plane, then the curve γ defined by $\gamma(t) = r_1(t) \times r_2(t)$ is contained in the z-axis.

ALWAYS. If $r_1(t)$ and $r_2(t)$ are parallel, then $r_1(t) \times r_2(t) = \vec{0}$, which is in the z-axis.

If $r_1(t)$ and $r_2(t)$ are not parallel, then the span the xy-plane. Since $r_1(t) \times r_2(t)$ is ~~parall~~ orthogonal to $r_1(t)$ and $r_2(t)$, it is orthogonal to the xy-plane; hence, is contained in the z-axis.

(b) The matrix of first order partial derivatives of $f(x, y, z) = (xe^{yz}, ye^{xz}, ze^{xy})$ at a point (x, y, z) is orthogonally diagonalizable.

SOMETIMES

$$Df(x,y,z) = \begin{bmatrix} e^{yz} & xze^{yz} & xye^{yz} \\ yze^{xz} & e^{xz} & xye^{xz} \\ yze^{xy} & xze^{xy} & e^{xy} \end{bmatrix}$$

This is orthogonally diagonalizable $\Leftrightarrow$ it is symmetric.

$Df(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is, but $Df(0,1,1) = \begin{bmatrix} e & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is not.

(c) For the path $\vec{x}(t) = (e^t \sin t, e^t \cos t)$, the tangential component of acceleration is larger than the normal component of acceleration.

NEVER $\vec{x}'(t) = (e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t)$, so $s'(t) = \|\vec{x}'(t)\| = \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2}$

$= \sqrt{2}e^t$

Then Tangential component $= s''(t) = \sqrt{2}e^t$

$\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|} = \dfrac{1}{\sqrt{2}}(\sin t + \cos t, \cos t - \sin t)$ and $\vec{T}'(t) = \dfrac{1}{\sqrt{2}}(\cos t - \sin t, -\sin t - \cos t)$

$\kappa(t) = \dfrac{\|\vec{T}'(t)\|}{s'(t)} = \dfrac{\|\frac{1}{\sqrt{2}}(\cos t - \sin t, -\sin t - \cos t)\|}{\sqrt{2}e^t} = \dfrac{1}{\sqrt{2}e^t}$

Normal component $= (s'(t))^2 \kappa(t) = (\sqrt{2}e^t)^2 \left(\dfrac{1}{\sqrt{2}e^t}\right) = \sqrt{2}e^t$.

So Tangential component = Normal component.

(d) For different values of the constant a, consider the vector field

$$\vec{F}(x,y,z) = \left(x^3yz - 2axy,\ z^2 + ay^2 - xy,\ \frac{a}{\sqrt{x^2+y^2}}\right).$$

Then div $\vec{F}$ has a local maximum at $(1,1,1)$.

NEVER

$\operatorname{div}\vec{F} = \dfrac{\partial}{\partial x}(x^3yz - 2axy) + \dfrac{\partial}{\partial y}(z^2 + ay^2 - xy) + \dfrac{\partial}{\partial z}\left(\dfrac{a}{\sqrt{x^2+y^2}}\right)$

$= 3x^2yz - 2ay + 2ay - x + 0 = 3x^2yz - x$

$\nabla \operatorname{div}\vec{F}(x,y,z) = \left(\dfrac{\partial}{\partial x}(3x^2yz - x), \dfrac{\partial}{\partial y}(3x^2yz - x), \dfrac{\partial}{\partial z}(3x^2yz - x)\right)$

$= (6xyz - 1, 3x^2z, 3x^2y)$

$\nabla \operatorname{div}\vec{F}(1,1,1) = (5,3,3)$. So $(1,1,1)$ is not a critical point.

(e) If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function whose second order partial derivatives are continuous (a.k.a. a C^2 function) and whose second order Taylor polynomial at $(1, 2)$ is

$$p_2(\vec{x}) = 3 + (x-1) + (x-1)^2 + 3(y-2)^2,$$

then f has a local minimum at $(1, 2)$.

NEVER Since

$$p_2(\vec{x}) = f(1,2) + Df(1,2)\begin{bmatrix} x-1 \\ y-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x-1 & y-2 \end{bmatrix} Hf(1,2)\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

$$= 3 + (x-1) + (x-1)^2 + 3(y-2)^2$$

We see that $Df(1,2) = \begin{bmatrix} 1 & 0 \end{bmatrix}$. So, $(1,2)$ is not a critical point.

3. Find the distance between the **line of intersection** of the planes (in $\mathbb{R}^3$)

$$2x + y = 5$$
$$x - y = -2$$

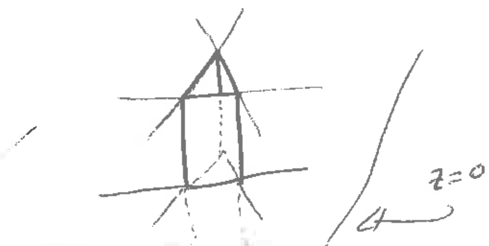
and the plane $x + y = 0$.

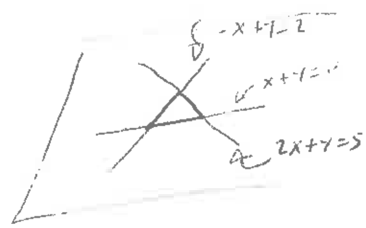
first convince yourself that:

the distance between the line of intersection of the planes (in $\mathbb{R}^3$) $2x+y=5$, $x-y=2$ and the plane $x+y=0$

is the same as

the distance between the point of intersection of the lines (in $\mathbb{R}^2$) $2x+y=3$, $x-y=2$ and the line $x+y=0$.





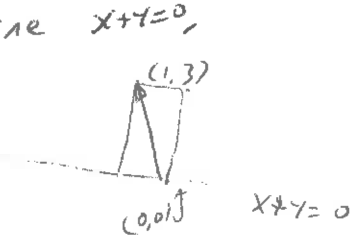
(you can show that the intersection of the planes $2x+y=5$, $x-y=-2$ in $\mathbb{R}^3$ ~~is does~~ doesn't intersect the plane $x+y=0$)

So $(1,3)$ is the intersection of $2x+y=0$, $x-y=-2$. (solve the system)

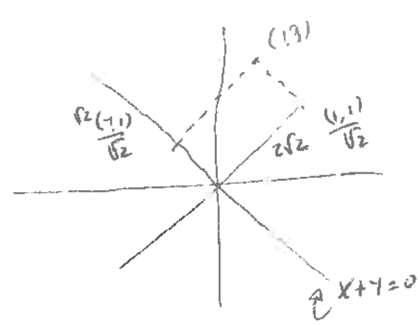
now to find the distance of $(1,3)$ to the line $x+y=0$,

notice that $(1,3) = 2(1,1) + (-1,1)$

$= 2\sqrt{2}\,\frac{(1,1)}{\sqrt{2}} + \sqrt{2}\,\frac{(-1,1)}{\sqrt{2}}$



So the distance is the $\mathrm{proj}_{\frac{(1,1)}{\sqrt{2}}}(1,3) = 2\sqrt{2}$.



4. This problem has four parts in total and is continued on the next page.

+5 (a) Let $q(x, y) = 7x^2 - 2\sqrt{3}xy + 5y^2$. Find the matrix A that represents this quadratic form and orthogonally diagonalize it.

$$A = \begin{pmatrix} 7 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{pmatrix} \qquad \det(A - \lambda I) = \lambda^2 - 12\lambda + 32$$

$$= (\lambda - 4)(\lambda - 8)$$

eigenvalue $4, 8$

$\underline{\lambda = 4}$ $\quad A - 4I = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ $\quad$ eigenvector $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

$\underline{\lambda = 8}$ $\quad$ eigenvector $\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$

+3 (b) Use your answer from part (a) to find a change of coordinates of the form

$$c_1 = ax + by, c_2 = cx + dy$$

such that $q(c_1, c_2) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ for some scalars λ_1 and λ_2.

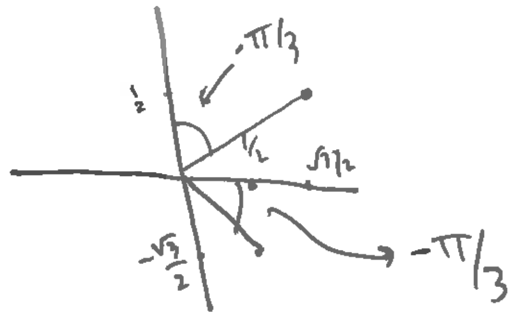
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$c_1 = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

$$c_2 = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y \qquad \cancel{\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}}$$

then $q(c_1, c_2) = 4c_1^2 + 8c_2^2$

+2 (c) This change of coordinates is the image of (x, y) under a rotation matrix. What is the angle of the rotation?



S. angle = $\boxed{-\pi/3}$

+2 (d) The surface $z = q(x, y)$ is a rotated version of one of the following types of surfaces:

- an elliptic paraboloid opening upward, $z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$
- an elliptic paraboloid opening downward, $z = -\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$
- a hyperbolic paraboloid, $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$ or $z = -\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$

Determine which one.

elliptic paraboloid opening upward,

matrix positive definite.

5. If g is a differentiable function of three variables (x, y, z) with $\nabla g(0, 0, -1) = (4, e^{\pi}, -1)$ and

$$x = rst,\ y = tr^2 e^{t^2},\ z = \cos(rse^t),$$

find the value of $\frac{\partial g}{\partial s}$ when $(r, s, t) = (1, \pi, 0)$.

Note that $x(1,\pi,0)=0$, $y(1,\pi,0)=0$, $z(1,\pi,0)=-1$.

$\frac{\partial x}{\partial s} = rt$, $\frac{\partial y}{\partial s} = 0$, $\frac{\partial z}{\partial s} = -\sin(rse^t)\cdot re^t$

By the chain rule,

$$\begin{aligned}\frac{\partial g}{\partial s} \text{ at } (1,\pi,0) &= g_x(0,0,-1)\, x_s(1,\pi,0) \\ &\quad + g_y(0,0,-1)\, y_s(1,\pi,0) \\ &\quad + g_z(0,0,-1)\, z_s(1,\pi,0) \\ &= (4)(0) + (e^{\pi})(0) + (-1)(-\sin(\pi)) \\ &= \sin(\pi) \\ &= 0.\end{aligned}$$

6. A parcel is dropped from a helicopter at a height of 250 meters with an acceleration of $\vec{a}(t) = -10\vec{j}$ and an initial velocity of $\vec{v}(0) = \vec{0}$. When it reaches a speed of 50 m/s, its parachute will deploy. Does it deploy its parachute before it hits the ground?

$\vec{v}(t) = -10t\vec{j} + \vec{v}(0) = -10t\vec{j}$

$\vec{x}(t) = -5t^2\vec{j} + \underbrace{\vec{x}(0)}_{=250\vec{j}} = (-5t^2 + 250)\vec{j}$

$\|\vec{v}(t)\| = 10t = 50$ at $t = 5$

$\vec{x}(5) = 125\vec{j}$ so above ground.

<u>Yes</u>

7. Let $\vec{F}(x,y,z) = \left(\sin x^2yz, e^{xy^2z}, \frac{xy}{z^2}\right)$.

(a) Compute curl $\vec{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(x^2yz) & e^{xy^2z} & \frac{xy}{z^2} \end{vmatrix} =$$

$$= \left(\frac{x}{z^2} - xy^2e^{xy^2z}\right)\vec{i} + \left(x^2y\cos(x^2yz) - \frac{y}{z^2}\right)\vec{j}$$

$$+\left(y^2ze^{xy^2z} - x^2z\cos(x^2yz)\right)\vec{k}$$

(b) Compute div $\vec{F}$.

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3 =$$

$$= 2xyz\cos(x^2yz) + 2xyze^{xy^2z} - \frac{2xy}{z^3}$$

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = -x^3 + y^2 + 3x$. Find all critical points of f and classify them.

$f_x = -3x^2+3, \quad f_y = 2y$

$f_x = 0 \iff 3x^2 = 3 \iff x^2 = 1 \iff x = \pm 1$

$f_y = 0 \iff 2y = 0 \iff y = 0$

$f_{xx} = -6x, \quad f_{yy} = 2, \quad f_{xy} = f_{yx} = 0$

So $f_{xx}f_{yy} - f_{xy}^2 = -12x.$ ← determinant of Hessian matrix

(i) $(+1, 0)$ $\quad -12(1) = -12 < 0$

$(1, 0, 2)$ is a saddle point.

(ii) $(-1, 0)$ $\quad -12(-1) = +12 > 0$

$f_{xx}(-1, 0), f_{yy}(-1, 0) > 0$

$(-1, 0, -2)$ is a local minimum.

The critical points are $(1, 0)$, a saddle point, and $(-1, 0)$, a local minimum.

9. Let X be the closed and bounded region defined by $-1 \leq x, y \leq 1$, and let $f : X \to \mathbb{R}$ be given by

$$f(x, y) = x^2y - 2xy.$$

Find the points in X where f attains a global maximum and a global minimum.

$\nabla f = (2xy - 2y,\ x^2 - 2x)$

$\nabla f = (0,0) \Rightarrow 2y(x-1) = 0$ and $x(2-x) = 0$

$\Rightarrow x = 0$ or $\underset{\text{not in } X}{2} \Rightarrow x = 0, y = 0$

$(0,0)$ is critical point

$f(-1, y) = y + 2y = 3y$: no critical point on this edge

$f(1, y) = y^2 - 2y = -y$: no critical point on this edge

$f(x,-1) = -x^2 + 2x$: $\frac{d}{dx} f(x,-1) = -2x + 2$: critical point at $(1,-1)$

$f(x,1) = x^2 - 2x$: $\frac{d}{dx} f(x,1) = 2x - 2$: critical point at $(1,1)$

also, must check corners!

(x,y)	$f(x,y)$	
$(0,0)$	0	
$(-1,-1)$	-3	global min
$(-1,1)$	3	global max
$(1,-1)$	1	
$(1,1)$	-1	