Northwestern University

Name: ______________________
Student ID:

Math 290-2 Final Exam Solutions

Winter Quarter 2014

Monday, March 17, 2014

Put a check mark next to your section:

Broderick		Cañez 12:00	
Cañez 10:00		Davis	

Question	Possible points	Score
1	20	
2	20	
3	10	
4	12	
5	8	
6	10	
7	10	
8	10	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 13 pages and 8 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function, and let $g(s,t) = f(x(s,t), y(s,t), z(s,t))$, where $x(s,t) = s+t$, $y(s,t) = s-t$, and $z(s,t) = st$. Then we have
   $$g_s(1,-1) + g_t(1,-1) = 2f_x(0,2,-1).$$

   TRUE: By the chain rule,
   $$g_s(1,-1) = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}\right)_{(s,t)=(1,-1)} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}t\right)_{(s,t)=(1,-1)}$$
   and
   $$g_t(1,-1) = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\right)_{(s,t)=(1,-1)} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(-1) + \frac{\partial f}{\partial z}s\right)_{(s,t)=(1,-1)}.$$
   Since $x(1,-1) = 0$, $y(1,-1) = 2$, and $z(1,-1) = -1$, this gives,
   $$g_s(1,-1) = f_x(0,2,-1)\cdot 1 + f_y(0,2,-1)\cdot 1 + f_z(0,2,-1)\cdot(-1)$$
   and
   $$g_t(1,-1) = f_x(0,2,-1)\cdot 1 + f_y(0,2,-1)\cdot(-1) + f_z(0,2,-1)\cdot 1,$$
   so $g_s(1,-1) + g_t(1,-1) = 2f_x(0,2,-1) + 0 + 0$.

   (b) The tangent plane to the surface $xe^{(x-y)z} + y^2z = 2$ at the point $(1,1,1)$ contains the lines $x = 1-t, y = 1+t, z = 1+t$ and $x = 1, y = 1-2t, z = 1+3t$.

   FALSE: The surface is a level set of the function $f(x,y,z) = xe^{(x-y)z} + y^2z$, so the tangent plane has normal vector
   $$\vec{n} = \nabla f(1,1,1) = \langle e^{(x-y)z} + xze^{(x-y)z}, -xze^{(x-y)z} + 2yz, (x-y)xe^{(x-y)z} + y^2\rangle|_{(x,y,z)=(1,1,1)} = \langle 2,1,1\rangle.$$
   For the plane to contain both lines, $\vec{n}$ must be orthogonal to both direction vectors $\vec{v_1} = \langle -1,1,1\rangle$ and $\vec{v_2} = \langle 0,-2,3\rangle$. But $\vec{n}\cdot\vec{v_2} = 0 - 2 + 3 = 1 \neq 0$.

(c) The polar curve $r = 1 + \sin(\theta)$ intersects the x-axis in exactly three points.

TRUE: For (r, θ) to lie on the x-axis, we must have either $\theta = 0$, $\theta = \pi$, or $r = 0$. When $\theta = 0$ or $\theta = \pi$, $r = 1 + 0 = 1$. And when $\theta = 3\pi/2$, $r = 1 + (-1) = 0$. Hence, the points $(x, y) = (1, 0), (-1, 0), (0, 0)$ comprise the intersection of the polar curve with the x-axis.

(d) There is a level curve of $f(x, y) = e^{x^2+y^2} - 9$ which consists of a single point.

TRUE: Let $k = -8$. Then the level curve $f(x, y) = k$ is the set of points (x, y) such that $e^{x^2+y^2} - 9 = -8$, i.e. $e^{x^2+y^2} = 1$, or $x^2 + y^2 = \ln 1 = 0$. Since the only such point is $(x, y) = (0, 0)$, the level set consists of a single point.

(e) Let $f(x, y) = \left(\frac{x^2}{\sqrt{2}} - \frac{y^2}{2\sqrt{2}}, \frac{x^2}{2} + \frac{\sqrt{2}y}{\sqrt{3}}\right)$. There is a point $\vec{a} \in \mathbb{R}^2$ at which the Jacobian $Df(\vec{a})$ (aka matrix of partials) preserves lengths, i.e. $\|Df(\vec{a})\vec{v}\| = \|\vec{v}\|$ for all $\vec{v} \in \mathbb{R}^2$.

TRUE: We claim that there is a point $\vec{a} \in \mathbb{R}^2$ at which the Jacobian $Df(\vec{a})$ is orthogonal. For this we need the columns to be orthonormal. Note that

$$Df(a, b) = \begin{bmatrix} \sqrt{2}a & -b/\sqrt{2} \\ a & \sqrt{2}/\sqrt{3} \end{bmatrix}.$$

The second column is a unit vector if $b^2/2 = 1 - 2/3$, or $b = \pm\sqrt{2}/\sqrt{3}$. The first column is a unit vector if $2a^2 + a^2 = 1$, or $a = \pm 1/\sqrt{3}$. The two columns are orthogonal if $0 = -ab + \sqrt{2}/\sqrt{3}a = a(\sqrt{2}/\sqrt{3} - b)$. Hence, the Jacobian is orthogonal at $(1/\sqrt{3}, \sqrt{2}/\sqrt{3})$ and at $(-1/\sqrt{3}, \sqrt{2}/\sqrt{3})$.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

   (a) If the temperature at (x, y) is given by $T(x, y) = x^2 + y^2$, then the hottest and coldest points achieved on the circle $(x + 1)^2 + (y + 2)^2 = k$ lie on the line $y = 2x$.

   ALWAYS: The extrema of T subject to the constraint $g(x, y) = (x + 1)^2 + (y + 2)^2 = k$ can only occur at critical points of T and g or at points where $\nabla T = \lambda \nabla g$. $\nabla T = \langle 2x, 2y\rangle$ and $\nabla g = \langle 2(x + 1), 2(y + 2)\rangle$. The only critical point of either function that satisfies the constraint is $(-1, -2)$, which lies on the line $y = 2x$. The Lagrange multiplier equation is satisfied only if $\dfrac{2x}{2(x+1)} = \lambda = \dfrac{2y}{2(y+2)}$, which implies $x(y + 2) = y(x + 1)$, or $xy + 2x = xy + y$, which gives $y = 2x$.

   (b) For a point $(a, b) \in \mathbb{R}^2$, the second-degree Taylor polynomial of $f(x, y) = e^{x^3+x+y}$ at (a, b) has the form $Q(x, y) = c_1x^2 + c_2y^2 + c_3x + c_4y + c_5$ for some constants $c_1, \ldots, c_5 \in \mathbb{R}$.

   NEVER: The quadratic approximation can only take this form if the mixed term $f_{xy}(a, b)(x - a)(y - b)$ is zero. But

$$f_{xy} = \frac{\partial}{\partial y}((3x^2 + 1)e^{x^3+x+y}) = (3x^2 + 1)e^{x^3+x+y}.$$

   This is positive for all $x, y \in \mathbb{R}$, so the mixed term in the quadratic approximation is never zero.

(c) Given a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $(a, b) \in \mathbb{R}^2$, the graph of f has a tangent plane at $(a, b, f(a, b))$.

SOMETIMES: True if f is differentiable, e.g. a polynomial; false if it isn't, e.g. $f(x, y) = \sqrt{x^2 + y^2}$, a function whose graph is a cone and thus has a sharp corner at the origin.

(d) Given a function $f : \mathbb{R}^3 \to \mathbb{R}$ with continuous first- and second-order partial derivatives, a point $\vec{a} \in \mathbb{R}^3$, and two *non-perpendicular* and *non-parallel* eigenvectors of $Hf(\vec{a})$ with eigenvalues λ and μ respectively, we have $\lambda = \mu$.

ALWAYS: Since f has continuous second-order partial derivatives, Clairaut's theorem implies that the Hessian is symmetric everywhere. Thus, any two eigenvectors from different eigenspaces must be orthogonal. So if two eigenvectors are not perpendicular, they must have the same eigenvalue.

(e) Given a differentiable function $f(x, y, z)$ such that $\nabla f(1, 1, 1) = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}$, the point $(1, 1, 1)$ is the global maximum of f subject to the constraint

$$(x-2)+(y-2)+2z = 0.$$

NEVER: Note that $(1, 1, 1)$ is not a critical point of f or of the function $g(x, y, z) = (x-2)+(y-2)+2z$. So an extreme value of f subject to the given constraint could occur at $(1, 1, 1)$ only if the Lagrange multiplier equation $\nabla f(1, 1, 1) = \lambda\langle 1, 1, 2\rangle$ is satisfied. But $\langle 2, 2, 0\rangle$ is clearly not parallel to $\langle 1, 1, 2\rangle$.

3. Find an equation for the plane containing the lines $x = 5 - 2t, y = t, z = 1 + 3t$ and $x = 2 + 4t, y = 1 - 2t, z = -6t$.

ANSWER: The plane must be parallel to the vector $\langle -2, 1, 3\rangle$ since this is a direction vector for both lines. Furthermore, since it contains the points $(5, 0, 1)$ and $(2, 1, 0)$, the plane must also be parallel to $\langle -3, 1, -1\rangle$. Thus a normal vector is given by $\vec{n} = \langle -2, 1, 3\rangle \times \langle -3, 1, -1\rangle = \langle -4, -11, 1\rangle$. Hence,

$$-4(x - 5) - 11(y - 0) + 1(z - 1) = 0$$

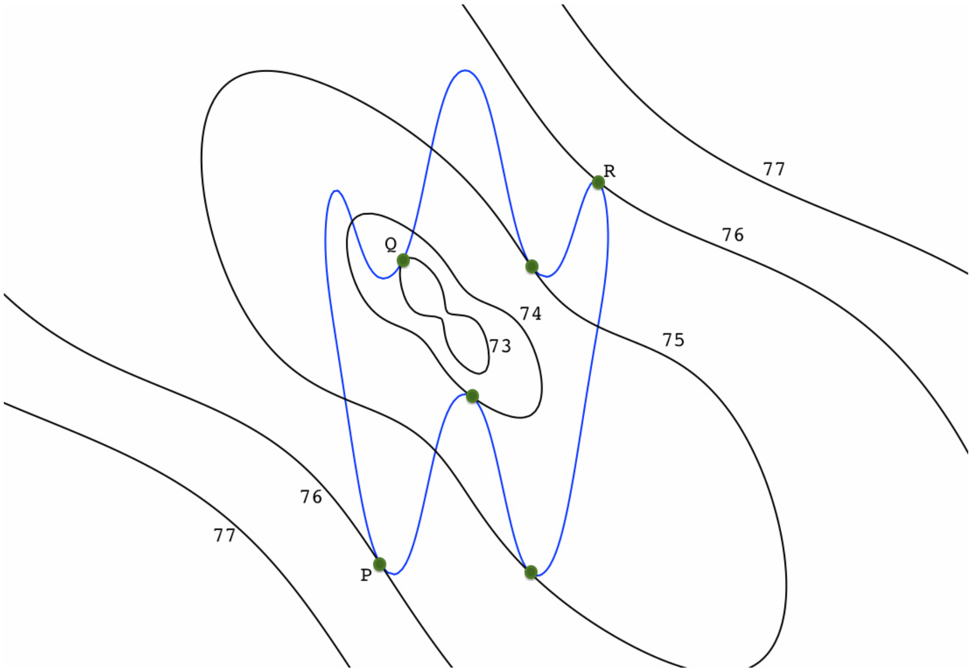
is an equation for the plane.

4. Find the maximum and minimum values of $f(x, y) = x^2y^2 - 4x^2 - 9y^2$ on the region described by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

   ANSWER: We have $\nabla f = \langle 2xy^2 - 8x, 2x^2y - 18y \rangle$. So the critical points of f are the points (x, y) satisfying $0 = 2xy^2 - 8x = 2x(y^2 - 4)$ and $0 = 2x^2y - 18y = 2y(x^2 - 9)$. If $x = 0$, then $0 = 2y(0 - 9) = -18y$, so $y = 0$. Otherwise, $y^2 - 4 = 0$, so $y = \pm 2$. But any such point is not in the specified region, so the only critical point we need to consider is $(0, 0)$.

   We now deal with each of the four line segments comprising the boundary. If $x = \pm 1$, then $f(x, y)$ becomes $y^2 - 4 - 9y^2$. Since $\frac{d}{dy}(-4 - 8y^2) = -16y = 0$ only when $y = 0$, the only candidates on the lines $x = 1$ and $x = -1$ are $(-1, -1), (-1, 0), (-1, 1), (1, -1), (1, 0)$, and $(1, 1)$. Similarly, if $y = \pm 1$, the function becomes $x^2 - 4x^2 - 9$, so since $\frac{d}{dx}(-3x^2 - 9) = -6x = 0$ only when $x = 0$, the only new candidates we obtain from these line segments are $(0, -1)$ and $(0, 1)$. Plugging in we see that $f(\pm 1, \pm 1) = 1 - 4 - 9 = -12$, $f(\pm 1, 0) = -4$, $f(0, \pm 1) = -9$, and $f(0, 0) = 0$, so the maximum value of f on this region is 0 and the minimum value on the region is -12.

5. Winston's Wheels, Inc. is sponsoring a bike race on a course in the shape of their corporate logo, which looks vaguely like a 'W.' The course is defined by the equation $W(x, y) = 10$ and is pictured below along with several level curves of the temperature function $T(x, y)$.



(a) Suppose that you solved the equation $\nabla T(x, y) = \lambda \nabla W(x, y)$. Mark the resulting solution points (x, y) on the picture.

ANSWER: The solutions would occur at points where the level curve $W(x, y) = 10$ is parallel to one of the level curves of f. Given the information we have, we can find six such points, marked as green dots on the picture above.

(b) Which points give the maximum and minimum values of $T(x, y)$ on the course, and what are the hottest and coldest temperatures on the course?

ANSWER: The minimum temperature is 73 and occurs at the point Q. The maximum temperature is 76 and it occurs at two points on the course: P and R.

6. Find the maximum and minimum values achieved by the function $f(x, y) = x^2y$ on the ellipse $x^2 + 2y^2 = 6$.

   ANSWER: Let $g(x, y) = x^2 + 2y^2$. Then $\nabla f = \langle 2xy, x^2 \rangle$ and $\nabla g = \langle 2x, 4y \rangle$. Both functions have only one critical point, the origin, which does not satisfy the constraint. (Note also that every ellipse is closed and bounded, so the function *does* attain a maximum and minimum value on the ellipse.) We now solve the Lagrange multipliers equation $\nabla f = \lambda \nabla g$.

   We must have $2xy = 2x\lambda$, which forces either $x = 0$ or $\lambda = y$. If $x = 0$ the constraint equation forces $y^2 = 3$, so $y = \pm\sqrt{3}$. If $\lambda = y$ then $x^2 = 4\lambda y = 4y^2$, so the constraint equation gives $4y^2 + 2y^2 = 6$, or $y^2 = 1$, which implies $x^2 = 4$. Thus, the only solutions are $(0, \pm\sqrt{3})$ and $(\pm 2, \pm 1)$. Plugging in, we get $f(0, \sqrt{3}) = f(0, -\sqrt{3}) = 0$, $f(2, 1) = f(-2, 1) = 4$, and $f(2, -1) = f(-2, -1) = -4$. So the maximum value is 4 and the minimum value is -4.

7. The altitude in inches around a pair of anthills is given by $A(x, y) = -x^4 - y^4 + 8y^2 + 4x$.

   (a) If a water droplet falls on the point $(2, 2)$ in which direction will it fall initially? (That is, what is the direction of steepest descent at that point?)

   ANSWER: The vector $-\nabla A(2, 2)$ points in the direction of steepest descent. We have $\nabla A = \langle -4x^3 + 4, -4y^3 + 16y\rangle$, so $-\nabla A(2, 2) = -\langle -28, 0\rangle = \langle 28, 0\rangle$.

   (b) As more water droplets fall the hills become partially flooded, and the water has risen to a height of 7 inches. If an ant is standing at the edge of the water at the point $(0, 1, 7)$, give a direction the ant can walk in to stay along the shoreline, i.e. to stay at a constant height of 7.

   ANSWER: Since A is a polynomial and therefore differentiable, we have $D_{\vec{u}}A(0, 1) = \nabla A(0, 1) \cdot \vec{u}$ for any unit vector $\vec{u}$. Hence, the directional derivative is zero in the direction of some vector $\vec{v}$ if and only if $\vec{v}$ is orthogonal to $\nabla A(0, 1) = \langle 4, 12\rangle$. One such vector is $\langle -12, 4\rangle$.

8. The function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous first- and second-order partials. The table below lists the values of these partials at several points.

(x,y)	$(1,2)$	$(2,4)$	$(-1,3)$	$(3,-2)$
f_x	2	0	0	0
f_y	0	0	0	0
f_{xx}	4	3	-2	2
f_{yy}	3	2	-1	2
f_{xy}	-3	3	1	2

On the table below, indicate what we can conclude about the point (x, y) from the information above. Check one box per column, and justify your answers below.

(x,y)	$(1,2)$	$(2,4)$	$(-1,3)$	$(3,-2)$
Local mimimum				
Local maximum			✓	
Saddle point		✓		
Not a critical point	✓			
More information is needed				✓

ANSWER:

$(1,2)$ This is not a critical point, since $\nabla f(1,2) = \langle 2, 0\rangle$ is defined and nonzero.

$(2,4)$ The gradient is zero, so this is a critical point. The Hessian is $\left[\begin{smallmatrix} 3 & 3 \\ 3 & 2 \end{smallmatrix}\right]$. Since the determinant is negative, $(2,4)$ is a saddle point by the second derivative test. Alternatively, one can check directly that the Hessian has one positive eigenvalue and one negative eigenvalue.

$(-1,3)$ The gradient is zero, so this is a critical point. The Hessian is $\left[\begin{smallmatrix} -2 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. The determinant is 1 and $f_{xx}(-1,3) < 0$, so $(-1,3)$ is a local maximum by the second derivative test. Alternatively, one can check directly that the Hessian has two negative eigenvalues.

$(3,-2)$ The gradient is zero, so this is a critical point. The Hessian is $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$. The determinant is zero, so more information is needed. Alternatively, one can check directly that the Hessian has 0 as an eigenvalue.