Northwestern University **Name:**

Math 290-3: Final Exam

Spring Quarter 2015

Tuesday, June 9, 2015

Put a check mark next to your section:

Davis (10am)		Canez	
Peterson		Davis (12pm)	

Question	Possible points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 13 pages and 8 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **five** parts.)

   (a) Let S be the surface $z = \sqrt{x^2 + y^2}$ for $z \leq 4$ with upward orientation and let $\mathbf{F} = (-ye^z, xe^z, 0)$. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

Answer:

   (b) Let D be the unit disk $x^2 + y^2 \leq 1$ in the xy-plane with upward orientation, and let S be the top half of the unit sphere $x^2 + y^2 + z^2 = 1$ with inward orientation. Then

   $$\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

   for $\operatorname{curl} \mathbf{F} = (x, -y, 1)$.

Answer:

(c) The surface area of the "spherical cap" which is the part of the sphere of radius 2 centered at the origin that is above the plane $z = 1$ is

$$\int_0^{2\pi} \int_0^{\pi/2} 2 \sin \phi \, d\phi \, d\theta.$$

Answer:

(d) Let $\mathbf{F} = (yz + ze^{xz}, z^2 + xz, 2yz + xy + xe^{xz})$, and let C be the part of the curve $\mathbf{x}(t) = (t^3 \sin t, 2t, 1 - \cos^2 t)$ with $0 \leq t \leq \pi$. Then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

Answer:

(e) Let C_1, C_2, and C_3 be three circles in $\mathbb{R}^2$ oriented counterclockwise such that $(0,0)$ does not lie on any of them, and let $\mathbf{F} = \frac{y\mathbf{i}-x\mathbf{j}}{x^2+y^2}$. If $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_3} \mathbf{F} \cdot d\mathbf{s}$.

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **five** parts.)

   (a) For a number k, the vector field $\mathbf{F} = (6x^2y, 4y^2 + kx^3, ze^z)$ has path-independent line integrals in $\mathbb{R}^3$.

Answer:

   (b) Let C be the unit circle $x^2 + y^2 = 1$, oriented counterclockwise. For a number k,

   $$\oint_C (x^2 + k^2 + 1)dx + dy > 0.$$

Answer:

(c) For a number k, define $f(x, y, z) = e^{kx} \sin(ky) + k^2 z$. Then there is a C^1 vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $\nabla f = \operatorname{curl} \mathbf{G}$.

Answer:

(d) Suppose that S is the unit sphere $x^2 + y^2 + z^2 = 1$ with inward orientation. Then for a number k,

$$\iint_S (3xz^4, 2yz^4, (k^2 - 1)z^5 + z^3) \cdot d\mathbf{S} > 0.$$

Answer:

(e) Let S be the (outward oriented) portion of the sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = -1$, and let C be the circle $x^2 + y^2 = 4$ in the plane $z = -1$, oriented counterclockwise when viewed from above. For a C^1 vector field $\mathbf{F}$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \oint_C (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{s}.$$

Answer:

3. Determine the value of the vector line integral

$$\int_C \left(e^x - \frac{4}{3}y^3\right) dx + (3x^3 - \sin(\cos y))\, dy$$

where C is the ellipse $9x^2 + 4y^2 = 36$ oriented clockwise.

4. Let $\mathbf{F} = (xe^{\sin x}, \sin(\cos y) + y, z + e^{z^4})$. Determine the value of the vector line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

where C is the curve which starts at $(0, 0, 0)$, follows the spiral with parametric equations $\mathbf{x}(t) = (t \cos t, t \sin t, t^2)$ for $0 \leq t \leq 5\pi$, and then follows the line segment from $(-5\pi, 0, 25\pi^2)$ to $(0, 0, 0)$.

5. Compute the surface area of the piece of the paraboloid $z = 3 - x^2 - y^2$ where $z \geq 2$.

6. Let S be the portion of the plane $y+z=2$ which is enclosed by the cylinder $x^2+y^2=4$, oriented upward. Compute the vector line integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = (e^{x^2+y^2}, e^{\sqrt{x^2+y^2+1}}, z+y)$.

7. Let $\mathbf{F} = (y - x\cos(x^4), z + x - \cos(e^y), e^{z^3+z})$. Compute the vector surface integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where S is the piece of the sphere $x^2 + y^2 + z^2 = 5$ where $x \geq 1$, oriented with normal vectors pointing in towards the x-axis.

8. Determine the value of the vector surface integral

$$\iint_S (y\cos(e^z), yz + y^2, e^{x^2+x}) \cdot d\mathbf{S}$$

where S is the portion of the cylinder $x^2 + z^2 = 1$ with $1 \leq y \leq 4$, oriented with outward pointing normal vectors.