Northwestern University   Name:

Math 290-2: Final Exam

Winter Quarter 2015

Monday, March 16, 2015

Put a check mark next to your section:

Davis		Canez	
Alongi		Peterson	

Question	Possible points	Score
1	18	
2	24	
3	10	
4	10	
5	8	
6	10	
7	10	
8	10	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 14 pages, and 8 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have two hours to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **six** parts.)

   (a) If f is a function of the form $f(x, y) = a + bx + cy + dx^2 + exy + hy^2$ where a, b, c, d, e, h are real numbers and at least one of d, e, h is nonzero, then the level sets of f are either ellipses, hyperbolas, or a pair of crossed lines.

Answer:

   (b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^3$. If $\mathbf{a} \times \mathbf{b} - 2\mathbf{c} = \mathbf{0}$, then $\mathbf{a} \cdot \mathbf{c} = 0$.

Answer:

(c) If $D(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$ is the zero matrix for each $\mathbf{x}$ in the domain of $\mathbf{g}$, then either $\mathbf{f}$ or $\mathbf{g}$ is a constant function.

Answer:

(d) Suppose that $\mathbf{a}$ is a saddle point of a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$. Then there exist linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^2$ such that $Hf(\mathbf{a})\mathbf{v}_1 = c_1\mathbf{v}_1$ and $Hf(\mathbf{a})\mathbf{v}_2 = c_2\mathbf{v}_2$ where c_1 and c_2 are both positive.

Answer:

(e) If the second order Taylor polynomial of a function $f(x, y, z)$ at the origin is

$$p_2(x, y, z) = 3 - 2x - x^2 - y^2 - z^2,$$

then f has a local maximum at the origin.

Answer:

(f) If $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are C^1 functions such that $\nabla f(1, 2, 3) = (1, 2, 3)$ and $\nabla g(1, 2, 3) = (2, 4, 3)$, then $(1, 2, 3)$ is a critical point of f subject to the constraint $g(x, y, z) = g(1, 2, 3)$.

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **six** parts.)

   (a) For $k > 0$, the surface defined by the spherical equation $\rho = 10 + \sin\phi\cos\theta$ lies between the sphere of radius k centered at the origin and the sphere of radius $2k$ centered at the origin.

Answer:

   (b) For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ of x and y and differentiable functions of two variables $x = x(s,t)$ and $y = y(s,t)$, we have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$.

Answer:

(c) For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point (a, b) in $\mathbb{R}^2$, there is an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of $\mathbb{R}^2$ such that $D_{\mathbf{u}_1} f(a, b) = D_{\mathbf{u}_2} f(a, b)$.

Answer:

(d) For a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $\mathbf{a}$ in $\mathbb{R}^2$ such that $\det Hf(\mathbf{a}) = 0$, f has a saddle point at $\mathbf{a}$.

Answer:

(e) For a real number k and a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ whose second-order Taylor polynomial at $(0, 1)$ is

$$p_2(x, y) = 5 + 3x^2 + 8x(y - 1) + k(y - 1)^2,$$

the function f has a local minimum at $(0, 1)$.

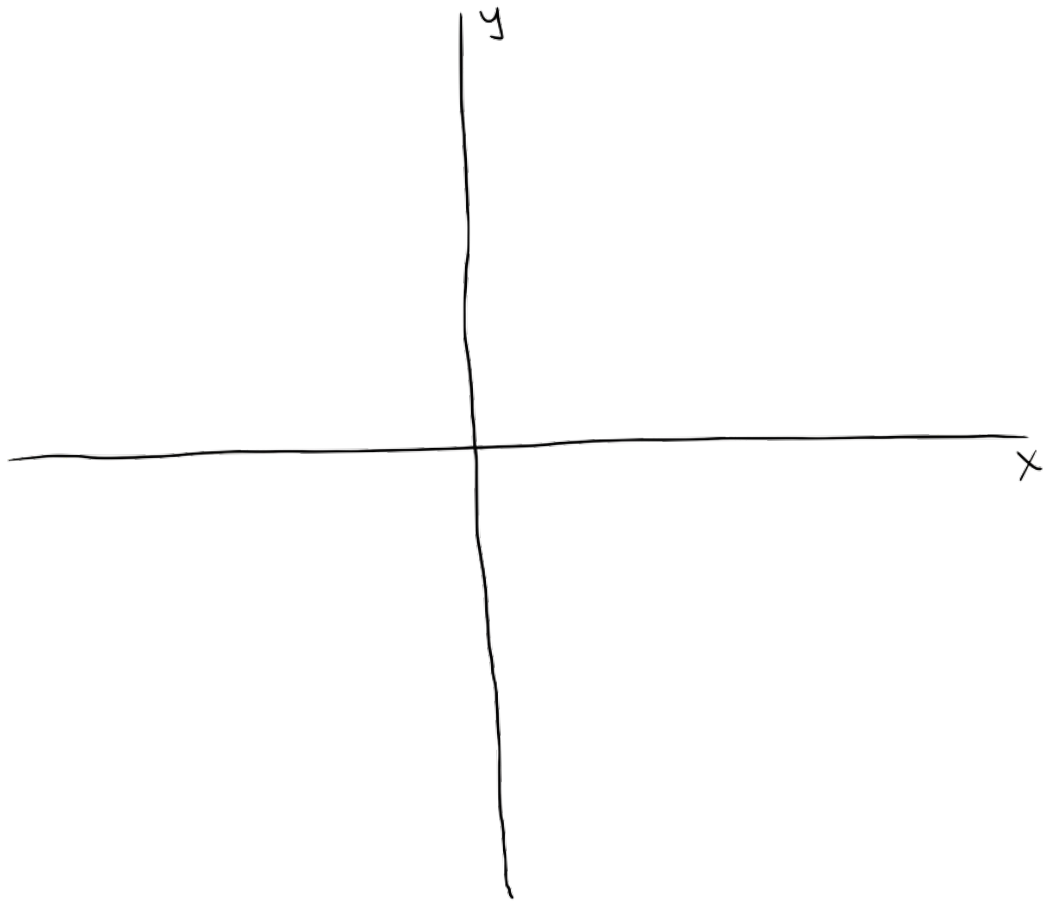
Answer:

(f) For a number k, the function $f(x, y, z) = x^2y^4 + (1 + k^2)z^3$ attains a maximum value over the region described in cylindrical coordinates by $0 \le r \le \sin\theta$.

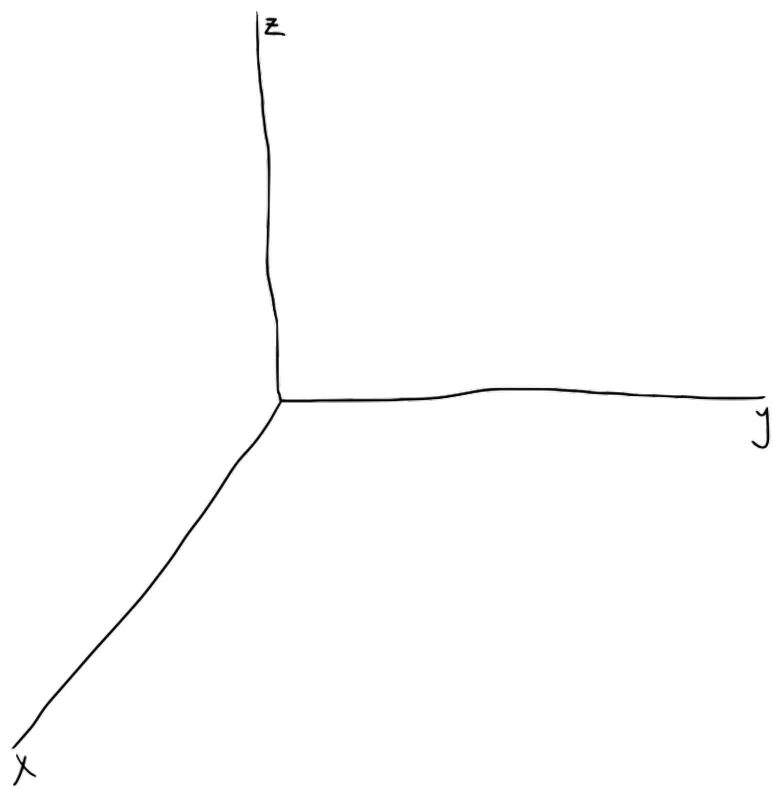
Answer:

3. (This question has **two** parts.) Let f be the function $f(x, y) = x^2 + y$.

(a) Sketch at least five level curves of f and label the levels.



(b) Sketch the graph of f. You may use words to help describe your sketch.



4. The tangent plane to a point $(a, b, f(a, b))$ on the graph of $f(x, y) = x^2 + xy + y^2$ is perpendicular to the line given by the parametric equations

$$x = 4 + 8t, \qquad y = 4 + t, \qquad z = -1 - t, \qquad -\infty < t < \infty.$$

Find the values of a and b.

5. Let

$$f(u,v) = \frac{u^2+v^2}{u^2-v^2}, \quad u(x,y) = e^{-x-y}, \quad \text{and} \quad v(x,y) = e^{xy}$$

Define $h(x,y) = f(u(x,y), v(x,y))$. Find $\partial h/\partial x$. You may leave your answer in terms of u, v, x, and y.

6. ConeCo™ manufactures conical waffle cones which are supposed to hold exactly $20\pi/3$ cubic inches of ice-cream. To accomplish this, they have designed their cones to have a base radius of 2 inches and a height of 5 inches. Due to imprecision in the manufacturing process, ConeCo can only guarantee that the radius is between 1.99 and 2.01 inches, and that the height is between 4.95 and 5.05 inches. Use differentials to estimate the maximum error that CupCo can expect in the volume of the cup. (Hint: The volume of a cone with base radius r and height h is $V = \frac{\pi}{3}r^2h$.)

7. Find and classify the critical points of the function

$$f(x, y, z) = -x^2 + e^z(z^2 - y^2).$$

8. An aquarium in the shape of an open rectangular box without a top is to hold 81 cubic feet of water and is to be built using slate for the rectangular base and glass for the sides. Slate costs \$12 per square foot and glass costs \$2 per square foot. Find the dimensions of the aquarium which minimize the cost. You may assume that the cost function has a global minimum.