Northwestern University Name: Key. Student ID:

Math 290-1 Midterm 1

Autumn Quarter 2012

Wednesday, October 24, 2012

Put a check mark next to your section:

Allen		Cyr (12pm)	
Canez		Peters	
Cyr (10am)			

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 12 pages, and 7 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

Question	Possible points	Score
1	20	
2	20	
3	12	
4	14	
5	8	
6	12	
7	14	
TOTAL	100	

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) If the augmented matrix of a system of equations can be put into reduced row echelon form, that system of equations has a unique solution.

False. Every matrix can be put into reduced echelon form, but not every linear system has a unique solution. For example, the augmented matrix of the system

$$x+y=0$$

is $\left(\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right)$ (which is in red. ech. form)

but the solution set is $\left\{ x\begin{pmatrix} -1 \\ 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$.

(b) $x = t, y = 3 - t, z = 2t + 1$ is a solution to the system of equations

$$\begin{aligned} 2x - z &= 1 \\ x + y &= 3 \\ x - y + z &= -2 \end{aligned}$$

False. If $\begin{aligned} x&=t \\ y&=3-t \\ z&=2t+1 \end{aligned}$ were a solution

to the system (for every t) then it would have to satisfy all of them. But if $x=t$ and $z=2t+1$, then $2x-z = 2t-(2t+1) = -1 \neq 1$

so this is not a solution for <u>any</u> values of t.

(c) If A is a 100×100 matrix with the property that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^{100}$, then the rank of A is 100.

True. Since we can solve $A\vec{x} = \vec{b}$ for every $\vec{b} \in \mathbb{R}^{100}$, there must be a pivot in every row of A. This means A has exactly 100 pivots (since there can be at most one pivot per row). The rank of A is the number of pivots.

(d) If A is an $n \times m$ matrix, and T is the linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$T(\vec{x}) = A\vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^m,$$

then the first column of A is

$$T\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

True. If $A = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_m)$, then

$$T\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_m)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= 1 \cdot \vec{a}_1 + 0 \cdot \vec{a}_2 + \cdots + 0 \cdot \vec{a}_m$$

$$= \vec{a}_1.$$

this is the first column of A.

(e) Orthogonal projection onto a line in $\mathbb{R}^3$ is an invertible linear transformation.

False. If L is a line in $\mathbb{R}^3$, then $\text{proj}_L(\vec{x}) \in L$ for all $\vec{x} \in \mathbb{R}^3$.

If $\vec{b} \in \mathbb{R}^3$ is any vector that isn't on the line L, the equation $\text{proj}_L(\vec{x}) = \vec{b}$ has no solutions.

This means that the transformation $\text{proj}_L : \mathbb{R}^3 \to \mathbb{R}^3$ is not onto.

All invertible linear transformations are both onto and one-to-one.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) A system of 2 linear equations in 3 unknowns has a unique solution.

Never.

The augmented matrix of this linear system would be 2×4. Since there is at most one pivot per row, there are at most two pivots. So one of the ~~first~~ three columns doesn't contain a pivot, and we know the system has a free variable. A linear system with a free variable has either zero or infinitely many sol.

(b) For fixed values of a, b, c, d, e, f, g, h, i, j, k, and l, if the system

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \end{aligned}$$

has infinitely many solutions, then the system

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

has a unique solution.

Sometimes.

True in this case

$\begin{cases} x+z=0 \\ y+z=0 \end{cases}$ has ∞'ly many sol.

$\begin{cases} x+z=0 \\ y+z=0 \\ z=0 \end{cases}$ has exactly one sol.

False in this case

$\begin{cases} x+z=0 \\ y+z=0 \end{cases}$ has ∞'ly many sol.

$\begin{cases} x+z=0 \\ y+z=0 \\ y+z=17 \end{cases}$ has no sol.

(c) If T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^m$, and a and b are scalars, then

$$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y}).$$

Always.

Since T is linear, we know

(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^m$

(2) $T(\lambda \vec{u}) = \lambda \cdot T(\vec{u})$ for all $\vec{u} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$.

$$T(a\vec{x}+b\vec{y}) \overset{(1)}{=} T(a\vec{x}) + T(b\vec{y}) \underset{(2)}{=} a\cdot T(\vec{x}) + b\cdot T(\vec{y}).$$

(d) If T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then there is a vector $\vec{v}$ in $\mathbb{R}^m$ such that

$$T(\vec{x}) = \vec{v} \cdot \vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^m.$$

Sometimes.

If $T: \mathbb{R}^2 \to \mathbb{R}$ is $T\begin{pmatrix} x \\ y \end{pmatrix} = x+y$,

True in this case: $T\begin{pmatrix} x \\ y \end{pmatrix} = x+y = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{v}.} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$.

False in this case: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Then $T\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ but if $\vec{v} \in \mathbb{R}^2$ then $\vec{v} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}$ (so $T\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{v} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$).

(e) Suppose A is a 2×2 matrix such that $A\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$ has no solutions. Then the equation $A^2\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$ does not have any solutions either.

Always.

we know $A\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$ has no solutions.

If there were some $\vec{u} \in \mathbb{R}^2$ such that

$$A^2\vec{u} = \begin{pmatrix}2\\3\end{pmatrix},$$

then $A(A\vec{u}) = A^2\vec{u} = \begin{pmatrix}2\\3\end{pmatrix}$, so $A\vec{u}$ would be a solution to $A\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$. Since we know that can't happen, we know there can't be a solution to $A^2\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$.

3. For which values of k (if any) does the system

$$\begin{aligned} 2x + 4y + 6z &= 8 \\ x + 3y + 5z &= 9 \\ 4x + 10y + 20z &= k \end{aligned}$$

(a) have a unique solution?

(b) have no solution?

(c) have infinitely many solutions?

The augmented matrix of the linear system is:

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 9 \\ 4 & 10 & 20 & k \end{array}\right).$$

Doing Gauss-Jordan elimination,

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 9 \\ 4 & 10 & 20 & k \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 8 \\ 4 & 10 & 20 & k \end{array}\right)$$

$$\xrightarrow{R_2 - 2 \cdot R_1} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -2 & -4 & -10 \\ 4 & 10 & 20 & k \end{array}\right) \xrightarrow{R_3 - 4 \cdot R_1} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -2 & -4 & -10 \\ 0 & -2 & 0 & k-36 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} \boxed{1} & 3 & 5 & 9 \\ 0 & \boxed{-2} & -4 & -10 \\ 0 & 0 & \boxed{4} & k-26 \end{array}\right).$$

pivots

So there is never a pivot in the far right column (i.e. the system is always consistent) and there are never any free variables (i.e. the system always has at most one solution).

(a) All values of k.

(b) None.

(c) None.

4. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\2\end{pmatrix},\quad T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix},\quad T\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}4\\7\\0\end{pmatrix}.$$

Find the matrix of T.

we want to find the matrix of T. we know the columns of this matrix are the vectors $T\begin{pmatrix}1\\0\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix}, T\begin{pmatrix}0\\0\\1\end{pmatrix}$. Since we only know how to apply T to $\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}$ we want to express $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ as linear combinations of $\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&0&1&1\\1&1&2&0\\1&0&0&0\end{array}\right)\xrightarrow{R_2-R_1}\left(\begin{array}{ccc|c}1&0&1&1\\0&1&1&-1\\1&0&0&0\end{array}\right)\xrightarrow{R_3-R_1}\left(\begin{array}{ccc|c}1&0&1&1\\0&1&1&-1\\0&0&-1&-1\end{array}\right)$$

$$\xrightarrow{R_3+R_2}\left(\begin{array}{ccc|c}1&0&1&1\\0&1&0&-2\\0&0&-1&-1\end{array}\right)\xrightarrow{R_1+R_3}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&-2\\0&0&-1&-1\end{array}\right)$$

$$\xrightarrow{-R_3}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&-2\\0&0&1&1\end{array}\right).\quad \text{So}\quad \begin{pmatrix}1\\0\\0\end{pmatrix}=-2\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}1\\2\\0\end{pmatrix}.$$

That means $T\begin{pmatrix}1\\0\\0\end{pmatrix}=T\left(-2\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}1\\2\\0\end{pmatrix}\right)=-2T\begin{pmatrix}0\\1\\0\end{pmatrix}+T\begin{pmatrix}1\\2\\0\end{pmatrix}$

$$=-2\begin{pmatrix}0\\2\\0\end{pmatrix}+\begin{pmatrix}4\\7\\0\end{pmatrix}=\begin{pmatrix}4\\3\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&0&1&0\\1&1&2&0\\1&0&0&1\end{array}\right)\xrightarrow{R_2-R_1}\left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\1&0&0&1\end{array}\right)\xrightarrow{R_3-R_1}\left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&-1&1\end{array}\right)$$

$$\xrightarrow{R_2+R_3}\left(\begin{array}{ccc|c}1&0&1&0\\0&1&0&1\\0&0&-1&1\end{array}\right)\xrightarrow{R_1+R_3}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&-1&1\end{array}\right)\xrightarrow{-R_3}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&-1\end{array}\right).$$

That means $T\begin{pmatrix}0\\0\\1\end{pmatrix}=T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix}\right)=T\begin{pmatrix}1\\1\\1\end{pmatrix}+T\begin{pmatrix}0\\1\\0\end{pmatrix}-T\begin{pmatrix}1\\2\\0\end{pmatrix}$

$$=\begin{pmatrix}3\\1\\2\end{pmatrix}+\begin{pmatrix}0\\2\\0\end{pmatrix}-\begin{pmatrix}4\\7\\0\end{pmatrix}=\begin{pmatrix}-1\\-4\\2\end{pmatrix}.$$

So $A=\left(T\begin{pmatrix}1\\0\\0\end{pmatrix}\ T\begin{pmatrix}0\\1\\0\end{pmatrix}\ T\begin{pmatrix}0\\0\\1\end{pmatrix}\right)=\boxed{\begin{pmatrix}4&0&-1\\3&2&-4\\0&0&2\end{pmatrix}}$.

5. Let L be the line in the (x_1, x_2)-plane consisting of all scalar multiples of $\begin{pmatrix}2\\1\end{pmatrix}$, and let T be the orthogonal projection onto L. Find a nonzero vector $\vec{x}$ such that

$$T(\vec{x}) = \begin{pmatrix}0\\0\end{pmatrix}.$$

Justify your answer.

we know $\operatorname{proj}_L \begin{pmatrix}x\\y\end{pmatrix} = \dfrac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}}{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}} \begin{pmatrix}2\\1\end{pmatrix}$

$$= \frac{2x+y}{5}\begin{pmatrix}2\\1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{4x+2y}{5}\\ \frac{2x+y}{5}\end{pmatrix}.$$

Any vector $\begin{pmatrix}x\\y\end{pmatrix}$ for which $2x+y=0$ will have the property that $\operatorname{proj}_L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

So, for example, $\begin{pmatrix}x\\y\end{pmatrix} = \boxed{\begin{pmatrix}-1\\2\end{pmatrix}}$ is a solution to $\operatorname{proj}_L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

6. Let A be the matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & -1 & -4 \end{pmatrix}.$$

Determine whether A is invertible. If it is, calculate A^{-1}. If not, find all solutions to $A\vec{x} = \vec{0}$.

we know that A is invertible if and only if $\operatorname{rref}(A) = I_3$. If A is invertible, we know

$$\operatorname{rref}(A \mid I_3) = (I_3 \mid A^{-1}).$$

If A isn't invertible,

$$\operatorname{rref}(A \mid I_3) = \left(\begin{matrix}\text{something} \\ \text{that isn't} \\ I_3\end{matrix} \middle| \begin{matrix}\text{some} \\ \text{matrix}\end{matrix}\right).$$

so we will answer both questions in one step by row reducing $(A|I_3)$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 3 \cdot R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right) \xrightarrow{R_2 - 2 \cdot R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 6 & -1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 6 & -1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right).$$

$\uparrow$

I_3

since the left 3 colums are I_3, the right 3 columns are A^{-1}. so A is invertible and

$$\boxed{A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 6 & -1 & -2 \\ -3 & 1 & 1 \end{pmatrix}}$$

7. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^5$ is the function

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1 \end{pmatrix}.$$

Show that T is a linear transformation and find its standard matrix A. Calculate A^{52}.

If T were linear, its standard matrix would be

$$A = \left(T\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \; T\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} \; T\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \; T\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \; T\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right)$$

$$= \begin{pmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \end{pmatrix}.$$

Let's check that

$$\begin{pmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1 \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

for <u>every</u> $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5$. So T is a matrix transformation and all matrix transformations are linear (and, in particular, T is). Since T is linear, we know the matrix A above is its standard matrix (we constructed it using the formula for the st. mat. of a lin. trans.).

Computing A^{52} directly would be very <u>difficult</u>! Instead, let's use the fact that A^k is the st. mat. of the linear trans. $\underbrace{T(T(\cdots T(T}_{k \text{ times}}(\vec{x}))\cdots))$. We can see that $T(T(T(T(T(\vec{x}))))) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^5$, so $A^5 = I_5$. That means

$$\boxed{A^{52} = (A^5)^{10} A^2 = A^2 = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \end{pmatrix}}$$