Northwestern University

Name: ____________________
Student ID:

Math 290-1 Midterm Exam 1

Fall Quarter 2013

Monday, October 21, 2013

Put a check mark next to your section:

Allen		Cañez	
Broderick 10:00		Davis	
Broderick 12:00			

Question	Possible points	Score
1	18	
2	24	
3	18	
4	10	
5	15	
6	15	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 9 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

Solutions

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) Every diagonal matrix is invertible. (Recall that a diagonal matrix is one of the form $A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$, where a_{ii} are real numbers.)

False

If $n=2$ and $a_{11}=a_{22}=0$, then

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

rank$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq 2$, so it's not invertible.

   (b) Any 2×2 matrix that commutes with $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, also commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

False

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

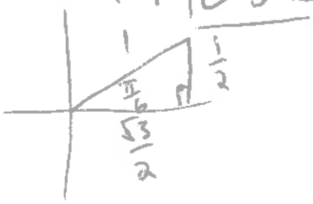
$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

So $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ commutes with $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, but doesn't commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^{30} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

FALSE:



$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix},$$

so A describes a counterclockwise rotation by $\frac{\pi}{6}$ radians.

So A^6 is a counterclockwise rotation by π, i.e. $A^6 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

and A^{12} is a counterclockwise rotation by 2π, i.e. $A^{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Hence

$$A^{30} = A^{12} A^{12} A^{6} = I_2 I_2 A^6 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \neq I$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

   (a) For a number k, the transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by

$$T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} kx + k^2y - z + k(k-1) \\ x - y + z - w \\ (k^2 - k)x^2 - x + z \\ 8z \end{pmatrix}$$

   is linear.

SOMETIMES If $k=0$, we have $T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -z \\ x-y+z-w \\ -x+z \\ 8z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$. So, T is linear with matrix $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 \end{pmatrix}$.

If $k \neq 0, 1$, then $T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k(k-1) \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. So, T is not linear if $k \neq 0, 1$.

   (b) For a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$, there is a nonzero vector $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$.

SOMETIMES

If $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then the only $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$ since $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ has rank 3.

If $T(\vec{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then all $\vec{x}$ in $\mathbb{R}^3$ satisfy $T(\vec{x}) = \vec{0}$.

(c) For 2×2 matrices A and B with rank 1, the product AB also has rank 1.

$\boxed{\text{SOMETIMES}}$

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$, $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ so true in this case

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so false here.

(d) For a reflection T of $\mathbb{R}^2$ across a line through the origin, the only vector $\vec{x}$ satisfying $T(\vec{x}) = \vec{x}$ is $\vec{x} = \vec{0}$.

$\boxed{\text{NEVER}}$

Any vector on the line we're reflecting across will satisfy this equation, since any such vector is left unchanged after reflecting.

3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

(a) Find A^{-1}, if it exists; otherwise, show that it doesn't.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right]$$

Check: $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ ✓

(b) Find a vector $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

$$x = A^{-1}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{9}{2} \\ \frac{1}{2} \end{pmatrix}$$

Check $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{2} \\ \frac{9}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ ✓

(c) Are there other vectors $\vec{v} \in \mathbb{R}^3$ such that $A\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$? Why or why not?

No, because A is invertible.

4. Let T_1, T_2, T_3 be the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$T_1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}, \quad T_2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}, \quad T_3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

For each $i = 1, 2, 3$, let A_i denote the matrix for T_i. Find the product

$$A_1A_2A_3.$$

$A_1A_2A_3$ represents the linear transformation

$$T_1 \circ T_2 \circ T_3$$

Now $T_1 \circ T_2 \circ T_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T_1\left(T_2\left(T_3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)\right) = T_1\left(T_2\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}\right) = T_1\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Since $T_1 \circ T_2 \circ T_3(\vec{x}) = \vec{0}$ for all $\vec{x}$, we have

$$A_1A_2A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ which first rotates the xy-plane counterclockwise by $\pi/4$, then applies the shear determined by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and finally reflects the xy-plane across the line $y = x$. Find the matrix of T.

Method 1

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{rotation}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \xrightarrow{\text{shear}} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 3/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\xrightarrow{\text{reflection}} \begin{pmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \xrightarrow{\text{shear}} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\xrightarrow{\text{reflect}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

So matrix of T = $\boxed{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}$

Method 2 rotation $= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ reflection $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So matrix of T

= reflection · shear · rotation

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \boxed{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}$$

6. Alex, Hillary and Jan each have some money. All together they have \$69. Alex's money, minus \$3, is half of Hillary's money. Alex and Hillary's money together is \$9 more than Jan's money. How much does each have?

Alex: a, Hillary: b, Jan: c

$$a+b+c=69$$
$$a-3-\tfrac{1}{2}b=0$$
$$a+b-c=9$$

$$\Rightarrow \begin{array}{r} a+b+c=69 \\ -\ \ a+b-c=9 \\ \hline 2c=60 \end{array}$$

$$\Rightarrow \boxed{c=30}$$

Thus,

$$\begin{array}{r} a-\tfrac{1}{2}b=3 \\ -\ \ a+b=39 \\ \hline -\tfrac{3}{2}b=-36 \end{array}$$

$$\Rightarrow \boxed{b=24}$$

$$a+b=39$$
$$\Rightarrow a+24=39$$
$$\Rightarrow \boxed{a=15}$$

So, Alex has \$15
Hillay has \$24
Jane has \$30.