1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

   (a) Suppose that the system

$$ax_1 + bx_2 + cx_3 = d$$
$$ex_1 + fx_2 + gx_3 = h$$

is consistent. (The variables are x_1, x_2, x_3.) If the system

$$ax_1 + bx_2 + cx_3 = d$$
$$ex_1 + fx_2 + gx_3 = h$$
$$ix_1 + jx_2 + kx_3 = l$$

has the same set of solutions as the first system, then the equation

$$ix_1 + jx_2 + kx_3 = l$$

is a multiple of one of the two equations in the first system.

Answer: FALSE

The third equation could be the sum of the first two for instance.

Example $x_1 + x_3 = 0$, $x_2 + x_3 = 0$ is consistent

and $x_1 + x_3 = 0$, $x_2 + x_3 = 0$, $x_1 + x_2 + 2x_3 = 0$ has same solutions

but $x_1 + x_2 + 2x_3 = 0$ is not a multiple of $x_1 + x_3 = 0$ nor $x_2 + x_3 = 0$.

(b) There is a scalar k which makes the function $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y\cos((k^2-1)x) \\ 3x + (2k-1)^2 y \\ 3 \\ (k+1)x^2 - 4y \end{bmatrix}$$

a linear transformation.

Answer: FALSE

$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ but a linear transformation always satisfies $T(\vec{0}) = \vec{0}$.

(c) If A and B are $n \times n$ invertible matrices, then AB has rank n.

Answer:

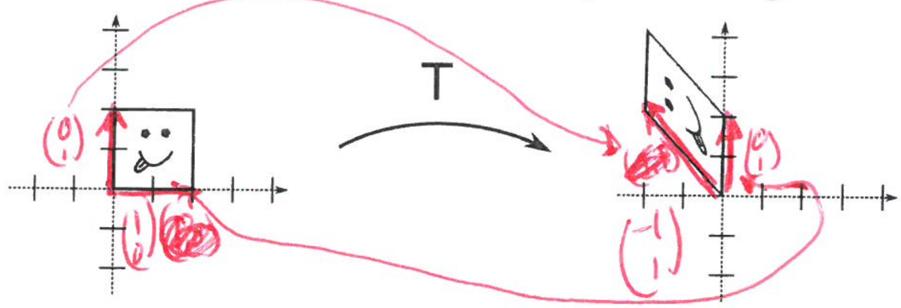
Since A and B are invertible then so is AB.

~~$\$$~~ AB is invertible implies that $\text{rank}(AB) = n$.

(d) Applying the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix product

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

transforms the smiley face on the left into the smiley face on the right:



Answer:

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a horizontal shear and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a ccw rotation by $90°$.

Note the shear occurs and then the rotation.

To check that the linear transformation depicted is correct check where the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ get sent.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

So $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

   (a) Let k be a real number. The matrix

$$\begin{bmatrix} 1 & 1+k^2 & 1+2k^2 \\ 0 & 1+k^2 & 1+2k^2 \\ 0 & 0 & 1+2k^2 \end{bmatrix}$$

   has rank 3.

   Answer: Always

We can row-reduce

$$\begin{bmatrix} 1 & 1+k^2 & 1+2k^2 \\ 0 & 1+k^2 & 1+2k^2 \\ 0 & 0 & 1+2k^2 \end{bmatrix} \begin{matrix} \xrightarrow{r_1-r_2} \\ \xrightarrow{r_2-r_3} \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 0 \\ 0 & 0 & 1+2k^2 \end{bmatrix}$$

this has rank 3 if and only if $1+k^2 \neq 0$ and $1+2k^2 \neq 0$. But $k^2 \geq 0$, so these never are 0.

Alternatively, you can observe that the matrix is in upper-triangular form, so it has full rank 3 as long as all the diagonal entries are nonzero. $1 \neq 0$ clearly, and $1+k^2 \neq 0$ and $1+2k^2 \neq 0$ as above.

   (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. The equation $T(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions.

   Answer: Sometimes

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. Then $T(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if, and only if, $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So, this is an example for which the statement is false.

On the other hand, if $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$, then $T(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if, and only if, $\vec{x} = \begin{bmatrix} 0 \\ y \end{bmatrix}$, where y is an arbitrary scalar. This is an example for which the statement is true.

In general, if $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$, then $T(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions if, and only if, $\operatorname{rank}(A) < 2$.

(c) Suppose that $\vec{u}$ and $\vec{v}$ are nonzero perpendicular vectors in $\mathbb{R}^2$. Then any vector $\vec{b}$ in $\mathbb{R}^2$ is a linear combination of $\vec{u}$ and $\vec{v}$.

Answer: ALWAYS

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Because $\vec{u} \neq \vec{0}$, either $u_1 \neq 0$ or $u_2 \neq 0$. Assume $u_1 \neq 0$. (The other case is similar.) Since $\vec{u} \perp \vec{v}$, $u_1v_1 + u_2v_2 = \vec{u}\cdot\vec{v} = 0$. Thus, $v_1 = -\frac{u_2}{u_1}v_2$. Consider $\det[\vec{u}\ \vec{v}] = \det\begin{bmatrix} u_1 & -\frac{u_2}{u_1}v_2 \\ u_2 & v_2 \end{bmatrix} = u_1v_2 + \frac{u_2^2}{u_1}v_2 = \frac{u_1^2+u_2^2}{u_1}v_2$. If $\det[\vec{u}\ \vec{v}] = 0$, then $u_1^2+u_2^2 = 0$ or $v_2 = 0$. If $u_1^2+u_2^2=0$, then $\vec{u}=\vec{0}$. If $v_2 = 0$, then $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$. Since $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, $\det[\vec{u}\ \vec{v}] \neq 0$. Thus, $[\vec{u}\ \vec{v}]$ is invertible. Let $\vec{b} \in \mathbb{R}^2$. Consequently, there is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ such that $[\vec{u}\ \vec{v}]\vec{x} = \vec{b}$. That is,

$$\vec{b} = x_1\vec{u} + x_2\vec{v}.$$

Therefore, $\vec{b}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

(d) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be an invertible matrix. Then the matrix $B = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}$ is also invertible.

Answer: Always

If A is invertible, then the equation $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has only the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

This equation is equivalent to

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= 0 \\ dx_1 + ex_2 + fx_3 &= 0 \\ gx_1 + hx_2 + ix_3 &= 0 \end{aligned}$$

, which is equivalent to

$$\begin{aligned} bx_2 + ax_1 + cx_3 &= 0 \\ ex_2 + dx_1 + fx_3 &= 0 \\ hx_2 + gx_1 + ix_3 &= 0 \end{aligned}$$

. Because addition is commutative, this system has only the solution $\begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, or rather $B\begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has only $\begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as a solution.

This last statement implies that B is invertible.

3. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$. Find conditions on the scalars b_1, b_2, b_3 so that $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is **not** a linear combination of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$.

$\vec{b}$ is not a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ if and only if $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{b}$ has no solution for x_1, x_2, x_3.

$$\left[\vec{v}_1, \vec{v}_2, \vec{v}_3 \,\middle|\, \vec{b}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -2 & -2 & b_2 \\ -1 & 1 & 3 & b_3 \end{array}\right] +(I)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -2 & -2 & b_2 \\ 0 & 2 & 2 & b_1+b_3 \end{array}\right] +(II)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -2 & -2 & b_2 \\ 0 & 0 & 0 & b_1+b_2+b_3 \end{array}\right]$$

so $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has no solution if and only if $b_1 + b_2 + b_3 \neq 0$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first rotates $\mathbb{R}^2$ by π radians, then applies the shear determined by $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, then reflects across the line $y = -x$, and finally scales by a factor of 3. Find the matrix of T.

rotation by π $= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

shear $= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

reflection across $y = -x$ $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Scale by 3 $= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

So T = Scale · reflection · Shear · rotation

matrix of T $= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} -6 & 3 \\ 3 & 0 \end{pmatrix}}$

5. Let $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$. Find a 2×2 matrix B such that B is not the zero matrix and AB is the zero matrix.

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AB = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a-6c & 3b-6d \\ -2a+4c & -2b+4d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives a system of linear equations that simplifies to $\begin{cases} a = 2c \\ b = 2d \end{cases}$

So, any matix of the form $B = \begin{pmatrix} 2c & 2d \\ c & d \end{pmatrix}$

where c & d are not both zero works.

6. (This problem has **two** parts.) Let A be the following matrix.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

(a) Find the inverse of A.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

(b) Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T\begin{bmatrix}-1\\0\\2\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix}, \quad T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\3\\0\end{bmatrix}.$$

Hint: Notice that the given input vectors are precisely the columns of the matrix A defined previously. Try using the inverse you found in part (a).

Let M denote the matrix of T. Then

$$MA = \begin{bmatrix}3 & 2 & 0\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}.$$

Multiplying on the right by A^{-1} gives

$$M = \begin{bmatrix}3 & 2 & 0\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\2 & -1 & 1\\-2 & 2 & -1\end{bmatrix} = \begin{bmatrix}7 & -5 & 5\\-4 & 5 & -2\\0 & 0 & 0\end{bmatrix}$$