Northwestern University Name: SOLUTIONS
Student ID:

Math 290-3 Midterm 1

Spring Quarter 2013

Thursday, May 2, 2013

Put a check mark next to your section:

Allen		Canez	
Peters			

Question	Possible points	Score
1	18	
2	18	
3	16	
4	16	
5	16	
6	16	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 9 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) Suppose $f, g : \mathbb{R}^3 \to \mathbb{R}$ are differentiable. If ∇f and ∇g are never zero, and $\nabla f(\vec{x}) \cdot \nabla g(\vec{x}) = 0$ for all $\vec{x}$, then the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$ has no maximum or minimum value.

TRUE if $\nabla f(\vec{x}_0) \neq 0$, $\nabla g(x_0) \neq 0$

and $\nabla f(\vec{x}_0) \cdot \nabla g(\vec{x}_0) = 0$ then

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$$

can have no solutions

(two non-zero vectors cannot be both parallel and perpindicular).

Since f subject to g has no critical points, it can have no max or min.

   (b) The expression

$$\int_{-1}^{0} \int_{0}^{\sqrt{1+x}} \sin x \cos y \, dy \, dx + \int_{0}^{1} \int_{0}^{\sqrt{1-x}} \sin x \cos y \, dy \, dx$$

is equal to 0.

TRUE: $\sin x \cos y$ is odd (anti-symmetric) with respect to x, and the region of integration

is symmetric with respect to $x=0$ / the y-axis.

(c) If f and g are integrable functions from $\mathbb{R}^2$ to $\mathbb{R}$, and R is a region in $\mathbb{R}^2$, then the following expressions are always equal:

$$\iint_R f(x,y)g(x,y)\, \mathrm{d}A = \iint_R f(x,y)\, \mathrm{d}A \cdot \iint_R g(x,y)\, \mathrm{d}A$$

FALSE: IF $f(x,y)=1$, $g(x,y)=1$ and R is a rectangle of area 2, then

$$\iint_R f(x,y)\cdot g(x,y)\,dA = \iint_R 1\,dA = 2$$

while

$$\iint_R f(x,y)\,dA \cdot \iint_R g(x,y)\,dA = \left(\iint_R 1 \cdot dA\right)^2 = 2^2 = 4.$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) Let R be a closed rectangle in $\mathbb{R}^2$ with area A, and let $f : R \to \mathbb{R}$ be a function such that $M \leq f(x, y) \leq N$ for all $(x, y) \in R$, where M and N are some fixed constants. Then, any Riemann sum S for f satisfies

$$MA \leq S \leq NA.$$

ALWAYS

If S is a Riemann sum for f, there is a partition,

$a = x_0 < x_1 < \cdots < x_n = b$

$c = y_0 < y_1 < \cdots < y_m = b$ writing $R = [a,b] \times [c,d]$

and points $\vec{c}_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ such that

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\vec{c}_{ij}) \Delta x_i \Delta y_j$$

Since $M \leq f(x,y) \leq N$ for all $(x,y) \in R$ we have

$$MA = \sum_{i=1}^{n} \sum_{j=1}^{m} M \Delta x_i \Delta y_j \leq S \leq \sum_{i=1}^{n} \sum_{j=1}^{n} N \Delta x_i \Delta y_j = NA$$

(b) Consider the Cobb-Douglas production function $Q(K, L) = AK^a L^{1-a}$, constrained by the cost function $pK + wL = M$. If the Cobb-Douglas production function is maximized at a point where $K = L$, then p (the price of capital) and w (the price of labor) are equal.

SOMETIMES ~~A~~ Let $C(K,L) = pK + wL$. Then

$\nabla Q(K,L) = (aK^{a-1}L^{1-a}, (1-a)K^a L^{-a})$ and $\nabla C(K,L) = (p, w)$.

These both exist and $\nabla C \neq \vec{0}$ on the domain $K > 0, L > 0$. So, we have $\lambda \in \mathbb{R}$ such that

$$\left.\begin{aligned} aK^{a-1}L^{1-a} &= \lambda p \\ (1-a)K^a L^{-a} &= \lambda w \end{aligned}\right\} \Rightarrow \frac{a}{1-a}\frac{L}{K} = \frac{p}{w}$$

If $L = K$, then we get $\frac{a}{1-a} = \frac{p}{w}$. So $p = w \Leftrightarrow a = 1-a$
$\Leftrightarrow a = \frac{1}{2}$.

(c) For an integrable function $f(x,y)$ of two variables, the expressions

$$\int_{-1}^{0}\int_{-x-1}^{1} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\int_{0}^{1}\int_{-1}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{0}^{1}\int_{-\sqrt{1-x^2}}^{0} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{-1}^{0}\int_{-1-y}^{0} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

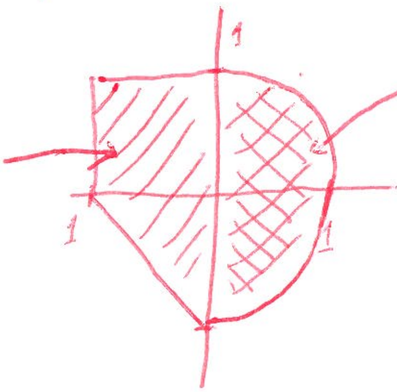
are equal.

ALWAYS

The region of integration for the 1st sum is

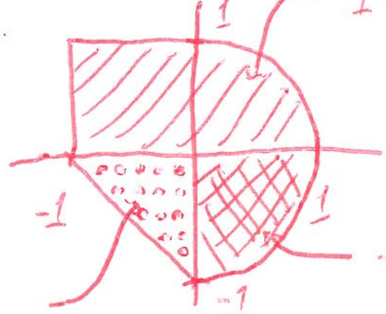
$0 \le x \le \sqrt{1-y^2}$, $-1 \le y \le 1$

$-x-1 \le y \le 1$, $-1 \le x \le 1$



The region for the 2nd sum is

$-1 \le x \le \sqrt{1-y^2}$, $0 \le y \le 1$



$-\sqrt{1-x^2} \le y \le 0$, $0 \le x \le 1$

$-1-y \le x \le 0$, $-1 \le y \le 0$.

These regions are the same.

3. If the point $(2, 1, 2)$ takes on the minimum value of $f(x, y, z) = x^2 + ay^2 + z^2$ subject to the constraint $xyz = 4$, what must a be?

Let $g(x,y,z) = xyz$.

Then $\nabla f = (2x, 2ay, 2z)$ and

$\nabla g = (yz, xz, xy)$.

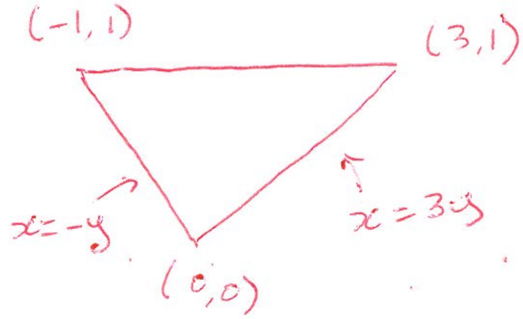
For $(2,1,2)$ to be a critical point, need

$\nabla f(2,1,2) = \lambda \cdot \nabla g(2,1,2)$, ie

$$\begin{cases} 4 = \lambda \cdot 2 \\ 2a = \lambda \cdot 4 \\ 4 = \lambda \cdot 2 \end{cases} \Rightarrow \lambda = 2 \Rightarrow 2a = 8 \Rightarrow a = 4.$$

4. Let D be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(-1,1)$, and $(3,1)$. Find

$$\iint_D 2xy\,dA.$$



$$\iint_D 2xy\,dA = \int_0^1 \int_{-y}^{3y} 2xy\,dx\,dy$$

$$= \int_0^1 y x^2 \Big|_{x=-y}^{x=3y} dy$$

$$= \int_0^1 8y^3\,dy$$

$$= 2y^4 \Big|_{y=0}^{y=1} = \boxed{2}$$

Alternate setup:

$$\iint_D 2xy\,dA = \int_{x=-1}^{x=0} \int_{y=-x}^{y=1} 2xy\,dy\,dx + \int_{x=0}^{x=3} \int_{y=\frac{x}{3}}^{y=1} 2xy\,dy\,dx$$

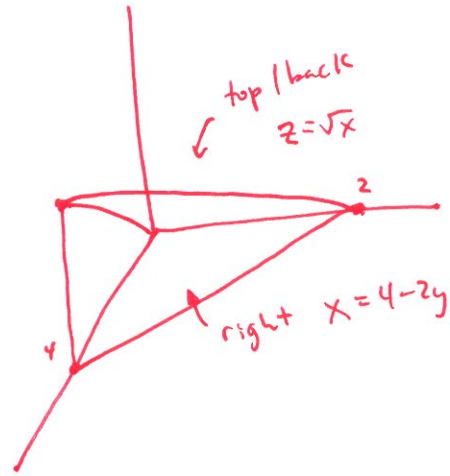
5. Rewrite the triple integral

$$\int_0^2 \int_0^{4-2y} \int_0^{\sqrt{x}} f(x,y,z)\, \overset{dz\,dx\,dy}{\cancel{dx\,dz\,dy}}$$

in the following different orders:

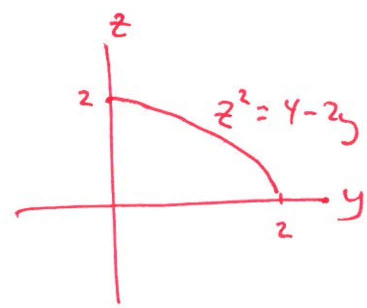
(a) In the order $dy\,dz\,dx$:

$$\int_0^4 \int_0^{\sqrt{x}} \int_0^{\frac{4-x}{2}} f(x,y,z)\,dy\,dz\,dx$$

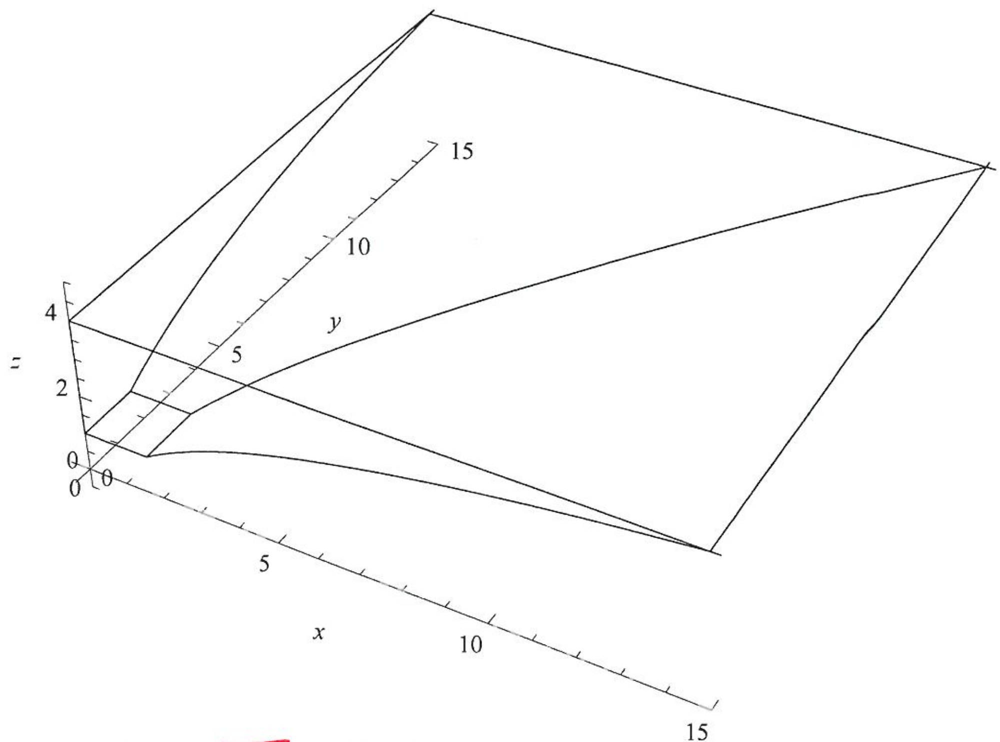


(b) In the order $dx\,dy\,dz$:

$$\int_0^1 \int_0^{\frac{4-z^2}{2}} \int_{z^2}^{4-2y} f(x,y,z)\,dx\,dy\,dz$$



6. Find the volume of the region (depicted below) bounded by the surfaces $y = 0$, $y = \sqrt{ze^z}$, $x = 0$, $x = \sqrt{ze^z}$, $z = 1$ and $z = 4$.



$$\text{Volume} = \int_1^4 \int_0^{\sqrt{ze^z}} \int_0^{\sqrt{ze^z}} 1 \, dx \, dy \, dz$$

$$= \int_1^4 \int_0^{\sqrt{ze^z}} \sqrt{ze^z} \, dy \, dz$$

$$= \int_1^4 ze^z \, dz$$ by parts: $u = z$, $v = e^z$, $du = dz$, $dv = e^z dz$

$$= ze^z \Big|_1^4 - \int_1^4 e^z dz$$

$$= 4e^4 - e - (e^4 - e) = \boxed{3e^4}$$