Northwestern University

Name: ______________________

Student ID:

Math 290-3 Midterm Exam 1

Spring Quarter 2014

Thursday, May 1, 2014

Solutions

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) If D is the region in $\mathbb{R}^3$ above the xy-plane and below the parabola $z = 2-2x^2-2y^2$, then

$$\iiint_D x^2 z\sqrt{x^2+y^2}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_0^{2-2r^2} r^3 z\cos^2(\theta)\,dz\,dr\,d\theta.$$

FALSE: Converting to cylindrical coordinates, our region is described by $0 \leq z \leq 2-2r^2$, for $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$. Since the cylindrical volume element is $r dr\,dz\,d\theta$, we have

$$\begin{aligned}\iiint_D x^2 z\sqrt{x^2+y^2}\,dx\,dy\,dz &= \int_0^{2\pi}\int_0^1\int_0^{2-2r^2} r^2 z\cos^2(\theta)\sqrt{r^2}\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1\int_0^{2-2r^2} r^4 z\cos^2(\theta)\,dz\,dr\,d\theta,\end{aligned}$$

Now, this can only equal the stated integral if

$$\int_0^{2\pi}\int_0^1\int_0^{2-2r^2}(r^3-r^4)z\cos^2(\theta)\,dz\,dr\,d\theta = 0.$$

But since that integrand is nonnegative on our region, and is not constantly zero, that is impossible.

(b) If $D = \{(x,y) : x^2+y^2 \leq 1\}$ is the unit disk, then

$$\iint_D e^{x^2-3y}(5+3x^3+2x^4-15x^5)\,dA \geq 0.$$

TRUE: First note that

$$\iint_D e^{x^2-3y}(5+3x^3+2x^4-15x^5)\,dA = \iint_D e^{x^2-3y}(5+2x^4)\,dA + \iint_D e^{x^2-3y}(3x^3-15x^5)\,dA$$

In the first of these integrals, the integrand is always nonnegative, so the integral is at least 0. In the second, the integrand is an odd function of x since

$$e^{(-x)^2-3y}(3(-x)^3-15(-x)^5) = e^{x^2-3y}(-3x^3+15x^5) = -e^{x^2-3y}(3x^3-15x^5),$$

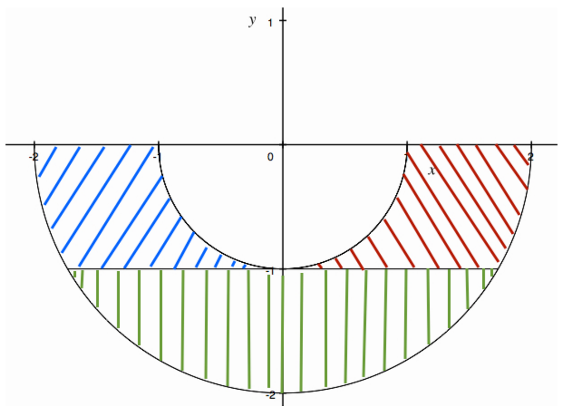
so since the unit disk is symmetric around the y-axis, the second integral is 0.

(c) The expression

$$\int_{-2}^{-1}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy + \int_{-1}^{0}\int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x,y)\,dx\,dy + \int_{-1}^{0}\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$

can be written as a single iterated integral in polar coordinates.

TRUE: The three regions of integration are pictured below.



The green region corresponds to the first integral, the blue to the second, and the red to the third. Since we integrate over all three, we can write this as $\iint_D f(x,y)\,dxdy$, where D is the region below the x-axis and between the circles $x^2+y^2=1$ and $x^2+y^2=4$. So in polar coordinates the integral is $\int_{\pi}^{2\pi}\int_{1}^{2} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

   (a) For a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(u, v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (au + bv, cu + dv),$$

   where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a rotation matrix, we have $\iint_D f(x, y)\, dx\, dy = \iint_D f(T(u, v))\, du\, dv$, where D is the disk centered at the origin of radius 1, in either the xy-plane or the uv-plane.

   ALWAYS: Since the matrix represents a rotation, its expansion factor is 1, so

$$|\det DT| = \left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = 1.$$

   Furthermore, rotations preserve the unit disk since they do not change length, so $T(D) = D$. Hence, since T is C^1 and one-to-one (it's given by an invertible matrix), we have

$$\iint_D f(x, y)\, dx\, dy = \iint_D f(T(u, v))|\det DT|\, du\, dv = \iint_D f(T(u, v))\, du\, dv.$$

   (b) For a positive integer k,

$$\iiint_E x^{k+4} y^{k+3} e^{z^{100}} \cos(y^{30} z)\, dV = 0$$

   where E is the solid enclosed by the bottom half of the unit sphere and the xy-plane.

   ALWAYS: First suppose k is odd. Then $k + 4$ is odd, so $(-x)^{k+4} = -x^{k+4}$ so the integrand is an odd function of x. Since E is symmetric around the yz-plane, the integral must be 0. Now, if k is even, then $k + 3$ is odd, so $(-y)^{k+3} \cos((-y)^{30} z) = -y^{k+3} \cos(y^{30} z)$, so the integrand is an odd function of y. Since E is symmetric around the xz-plane, the integral must be 0.

(c) Given a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\int_0^1 \int_0^2 f\,dy\,dx \leq 6$, we have $f(x, y) \leq 3$ for all points (x, y) in $[0, 1] \times [0, 2]$.

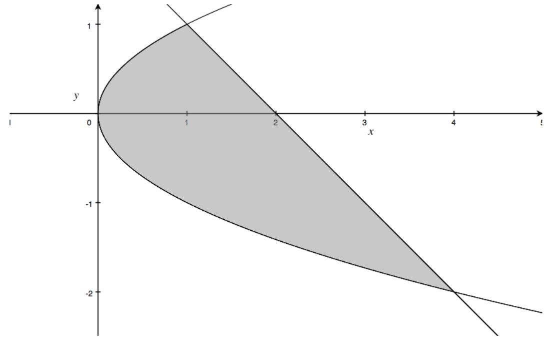
SOMETIMES: Clearly if $f = 3$ is a constant function, the statement holds. But if $f(x, y) = 3 + (y - 1) = 2 + y$, then

$$\int_0^1 \int_0^2 f\,dy\,dx = \int_0^1 [2y + y^2/2]_{y=0}^2\,dx = \int_0^1 6\,dx = 6,$$

but $f(1, 2) = 4 > 3$.

3. (a) Let D be the region enclosed by the parabola $x = y^2$ and the line $y = 2 - x$. Find $\iint_D 6y\, dA$.

ANSWER: The region of integration is:



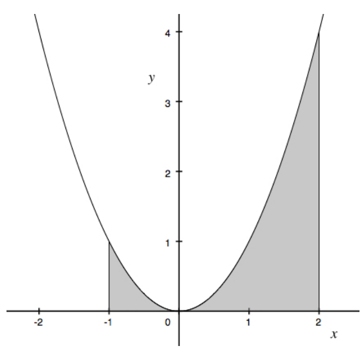
So our integral is

$$\begin{aligned} \int_{-2}^{1}\int_{y^2}^{2-y} 6y\,dx\,dy &= \int_{-2}^{1} 6y(2-y-y^2)\,dy = \int_{-2}^{1}(12y - 6y^2 - 6y^3)\,dy \\ &= [6y^2 - 2y^3 - \frac{3}{2}y^4]_{y=-2}^{1} = (6-2-3/2) - (6+16-6) \\ &= -12 - 3/2 = 13.5. \end{aligned}$$

(b) Evaluate the following expression.

$$\int_0^1 \int_{-1}^{-\sqrt{y}} e^{x^3}\,dx\,dy + \int_0^4 \int_{\sqrt{y}}^{2} e^{x^3}\,dx\,dy.$$

ANSWER: The region of integration is:



So changing the order of integration gives

$$\int_{-1}^{2}\int_{0}^{x^2} e^{x^3}\,dy\,dx = \int_{-1}^{2}(x^2 - 0)e^{x^3}\,dx = \left[\frac{1}{3}e^{x^3}\right]_{x=-1}^{2} = \frac{1}{3}(e^8 - e^{-1}).$$

4. Find the volume of the solid outside the double cone $z^2 = 3(x^2 + y^2)$ and inside the sphere $x^2 + y^2 + z^2 = 9$.

ANSWER: The cone can be written as $z = \sqrt{3}r$, so when $r = 1$ the cone is at height $\sqrt{3}$, which means the cone makes an angle of $\varphi = \pi/6$ with the z-axis. Thus, the region outside the cone corresponds to $\pi/6 < \varphi < 5\pi/6$. The sphere is centered at the origin and has radius 3, so the region inside the sphere corresponds to $\rho \leq 3$. Thus, our region is $0 \leq \theta < 2\pi$, $\pi/6 < \varphi < 5\pi/6$, and $\rho \leq 3$. Since the volume element for spherical coordinates is $\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$, the volume is

$$\begin{aligned} \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_0^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta &= 2\pi \int_{\pi/6}^{5\pi/6} \left[\frac{\rho^3}{3} \sin\varphi \right]_{\rho=0}^3 d\varphi = 18\pi \int_{\pi/6}^{5\pi/6} \sin\varphi \, d\varphi \\ &= 18\pi(-\cos(5\pi/6) + \cos(\pi/6)) = 18\pi\sqrt{3} \end{aligned}$$

5. Evaluate

$$\iint_D \frac{xy}{y^2 - x^2}\, dA$$

where D is the region in the first quadrant bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $\frac{x^2}{4} + y^2 = 1$, and $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

ANSWER: The region of integration is naturally described in terms of the variables $u = x^2 - y^2$ and $v = \frac{x^2}{4} + y^2$, since then it simply corresponds to the square $1 \le u \le 4$, $1 \le v \le 4$. We need to find the area element for this change of variables.

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} -2x & 2y \\ \frac{x}{2} & 2y \end{bmatrix} = -4xy - xy = -5xy,$$

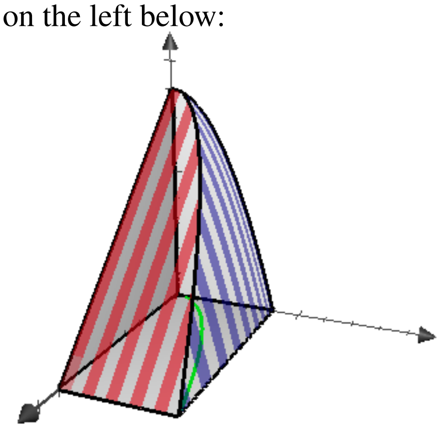
so since our region lies in the first quadrant $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{5xy}$. Thus,

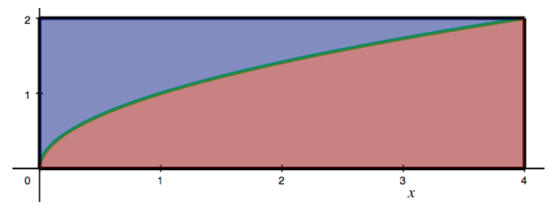
$$\iint_D \frac{xy}{y^2 - x^2}\, dA = \int_1^4 \int_1^4 \frac{xy}{-u} \frac{1}{5xy}\, du\, dv = -\frac{1}{5} \int_1^4 \int_1^4 \frac{1}{u}\, du\, dv = -\frac{1}{5} \int_1^4 \ln(4)\, dv = -\frac{3}{5} \ln(4).$$

6. Rewrite the following triple integral with respect to the given orders.

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_0^{4-z} dx\, dy\, dz$$

(a) $dy\, dz\, dx$ **ANSWER**: Our region lies between the surfaces $x = 4 - z$ and $x = 0$. The yz-shadow is the region $0 \leq y \leq \sqrt{4-z}$, $0 \leq z \leq 4$, so our region is also contained between the surfaces $y = 0$ and $z = 4 - y^2$. We need to see how the two cylinders intersect. If a point lies on both surfaces then $4 - x = z = 4 - y^2$, so $x = y^2$. Thus, the intersection is a curve extending from $(0, 0, 4)$ to $(4, 2, 0)$ and sitting above the curve $x = y^2$ in the xy-plane. So our picture looks like the image on the left below:





Here the red-striped face is a chunk of the plane $z = 4 - x$, the blue-striped face is a chunk of the parabolic cylinder $z = 4 - y^2$, and the grey faces behind them are all contained in coordinate planes. The xz-shadow of this region is simply the grey triangular face in the xy-plane, which is given by $0 \leq z \leq 4 - x$, $0 \leq x \leq 4$. If a line parallel to the y-axis intersects this region, it will enter in the xz-plane and exit through the blue-striped face, so the new integral is

$$\int_0^4 \int_0^{4-x} \int_0^{\sqrt{4-z}} dy\, dz\, dx.$$

(b) $dz\, dx\, dy$ **ANSWER**: The xy-shadow is the rectangle $[0, 4] \times [0, 2]$. If a vertical line intersects our region, it will enter through the xy-plane and exit through the red-striped surface if it lies to the right of the green curve, and it will exit through the blue-striped surface if it lies to the left of the green curve. So the xy-shadow is comprised of two parts as shown in the image above on the right. The red-shaded region below the parabola corresponds to values of x and y for which $0 \leq z \leq 4-x$, while the blue region corresponds to those values for which $0 \leq z \leq 4-y^2$. Hence, our expression is:

$$\int_0^2 \int_{y^2}^4 \int_0^{4-x} dz\, dx\, dy + \int_0^2 \int_0^{y^2} \int_0^{4-y^2} dz\, dx\, dy.$$