Northwestern University Name: SOLUTIONS
Student ID:

# Math 290-2 Midterm 1

## Winter Quarter 2013

## Tuesday, February 5, 2013

**Put a check mark next to your section:**

| | | | |
|---|---|---|---|
| Allen | | Cyr | |
| Canez | | Peters | |

| Question | Possible points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 14 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 16 | |
| TOTAL | 100 | |

**Instructions:**

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 10 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

**Good luck!**

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) Suppose $L$ is a line through the origin in $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$. The quantity $\vec{x} \cdot \text{proj}_L(\vec{x})$ must be negative.

False. If $L = \text{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$ then

$$\begin{pmatrix}1\\0\end{pmatrix} \cdot \text{proj}_L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 1 > 0.$$

$\begin{pmatrix}1\\0\end{pmatrix} \in L$

   (b) There is an orthogonal matrix that has 2 as an eigenvalue.

$\boxed{\text{FALSE}}$. Orthogonal matrices preserve length ie. if $A$ is orthogonal, then $\|Ax\| = \|x\|$ for all $x$.

(c) If $A$ is an $n \times n$ symmetric matrix such that $A^3 = I_n$, then $A = I_n$.

$\boxed{\text{TRUE}}$ $A$ is symmetric, so it's orthogonally diagonalizable $A = QDQ^T$ (here, $Q$ is orthogonal and $D$ is diagonal). Then $A^3 = QD^3Q^T$. But $A^3 = I_n$, so $D^3 = I_n$, so $D = I_n$. Therefore, $A = I_n$.

Remember $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$, [redacted] so $D^3 = \begin{pmatrix} \lambda_1^3 & & 0 \\ & \lambda_2^3 & \\ 0 & & \ddots \\ & & & \lambda_n^3 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$.

So $\lambda_1^3 = \lambda_2^3 = \cdots = \lambda_n^3 = 1$. This means $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$ (the only real number $x$ s.t. $x^3 = 1$ is $x = 1$).

(d) The parametric equations $x = 3 - t^3, y = 4 + 2t^3, z = \pi - 4t^3$ describe a line.

$\boxed{\text{TRUE}}$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ \pi \end{pmatrix} + t^3 \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$

Since $t^3$ can be any real number, the above curve is indeed a line, passing through the point $(3, 4, \pi)$ and having direction vector $(-1, 2, -4)$.

(What if $t^3$ was replaced by $t^2$?)

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) If $V = \text{Span}(\vec{v_1}, \vec{v_2})$ is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is in $\mathbb{R}^n$, then

$$\text{Proj}_V(\vec{x}) = \text{Proj}_{\text{Span}(\vec{v_1})}(\vec{x}) + \text{Proj}_{\text{Span}(\vec{v_2})}(\vec{x})$$

Sometimes.
- If $v_1$ and $v_2$ are orthogonal, this is the formula for orthogonal projection onto their span.

- If $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\1\end{pmatrix}$ then $V = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$, so
  - $\text{proj}_V\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$.

But
  - $\text{proj}_{\text{span}\{v_1\}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\end{pmatrix}$
  - $\text{proj}_{\text{span}\{v_2\}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$.

So in this case

$$\text{proj}_V\begin{pmatrix}1\\1\end{pmatrix} \neq \text{proj}_{\text{span}\{v_1\}}\begin{pmatrix}1\\1\end{pmatrix} + \text{proj}_{\text{span}\{v_2\}}\begin{pmatrix}1\\1\end{pmatrix}.$$

(b) If $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then the vector of $\mathcal{B}$-coordinates of a vector $\vec{u} \in \mathbb{R}^n$ is given by

$$[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} \vec{u}\cdot\vec{v}_1 \\ \vec{u}\cdot\vec{v}_2 \\ \vdots \\ \vec{u}\cdot\vec{v}_n \end{pmatrix}.$$

Always. If we write

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \quad \text{then} \quad [\vec{u}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

For each $1 \le i \le n$,

$$\vec{u}\cdot\vec{v}_i = (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)\cdot\vec{v}_i = c_1(\vec{v}_1\cdot\vec{v}_i) + c_2(\vec{v}_2\cdot\vec{v}_i) + \cdots + c_n(\vec{v}_n\cdot\vec{v}_i).$$

Since $\mathcal{B}$ is orthonormal, RHS is $c_i$.

so $[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} \vec{u}\cdot\vec{v}_1 \\ \vec{u}\cdot\vec{v}_2 \\ \vec{u}\cdot\vec{v}_n \end{pmatrix}$.

(c) Let $a$ and $b$ be positive real numbers. There is an orthogonal $2 \times 2$ matrix $S$ such that

$$S^{-1}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} S = \begin{pmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{pmatrix}$$

Sometimes.

[redacted]

If $a \neq b$, then $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ is not symmetric

$\Rightarrow$ not orthogonally diagonalizable.

If $a = b$, then the eigenvalues of $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$

are $\lambda = \pm a = \pm\sqrt{a^2} = \pm\sqrt{ab}$.

Since $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ is symmetric, there is an orthogonal matrix $S$ such that

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} = S \begin{pmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{pmatrix} S^{-1}$$

eigenvalues.

(d) For a function $f(t)$ of $t$, the curve with parametric equations

$$x = 2e^t,\ y = f(t)\cos t,\ z = e^t$$

intersects the plane $x + 3z = 5$.

$5 = x(t) + 3z(t) = 2e^t + 3(e^t) = 5e^t \Rightarrow 1 = e^t \Rightarrow t = 0$

corresponds to the point $(2, f(0), 1)$ on the curve

CHECK: $2 + 3(1) \stackrel{\checkmark}{=} 5$

$\boxed{\text{ALWAYS TRUE}}$

3. Find an orthonormal basis for

$$\text{Span}\left\{\begin{pmatrix}0\\0\\-4\\0\end{pmatrix},\begin{pmatrix}2\\-1\\5\\0\end{pmatrix},\begin{pmatrix}3\\1\\7\\3\end{pmatrix}\right\}$$

*You may assume the vectors above are linearly independent.*

$$\vec{u}_1 = \frac{1}{\sqrt{16}}\cdot\begin{pmatrix}0\\0\\-4\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$$

$$\vec{V}_2^{\perp} = \begin{pmatrix}2\\-1\\5\\0\end{pmatrix} - \left(\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\5\\0\end{pmatrix}\right)\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\0\\0\end{pmatrix}$$

$$\Rightarrow \vec{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}$$

$$\vec{V}_3^{\perp} = \begin{pmatrix}3\\1\\7\\3\end{pmatrix} - \left(\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}3\\1\\7\\3\end{pmatrix}\right)\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} - \left(\frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3\\1\\7\\3\end{pmatrix}\right)\begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}3\\1\\7\\3\end{pmatrix} - \begin{pmatrix}0\\0\\7\\0\end{pmatrix} - \begin{pmatrix}2\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\0\\3\end{pmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\0\\3\end{pmatrix}$$

4. A satellite is floating in space (presently it is sitting at the origin in $\mathbb{R}^3$). It has two sets of thrusters that allow it to move in different directions:

the $1^{st}$ set let it move forward/backward in the direction determined by the vector $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$;

the $2^{nd}$ set let it move forward/backward in the direction determined by the vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

The owners of the satellite want to move it from its present location to the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Is it possible for the satellite to get there? If not, what point in $\mathbb{R}^3$ is the closest it can get to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

No: $\left(\begin{array}{cc|c} 3 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 1 & 1 \end{array}\right)$ is inconsistent.

use least-squares to find this point:

$A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$, $A^T = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $A^TA = \begin{pmatrix} 13 & 2 \\ 2 & 2 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A^T\vec{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

So want to solve $A^TA\vec{x} = A^T\vec{b}$, ie

$\left(\begin{array}{cc|c} 13 & 2 & 5 \\ 2 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -11 & -8 \end{array}\right) \to$ $x_2 = \frac{8}{11}$, $x_1 = 1 - x_2 = \frac{3}{11}$.

Then $A\vec{x}$, the point closest to $\vec{b}$, is

$$\begin{pmatrix} 3 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{11} \\ 8/11 \end{pmatrix} = \begin{pmatrix} 9/11 \\ 14/11 \\ 8/11 \end{pmatrix}.$$

5. Let $q : \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form:

$$q(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^3 + 4x_1x_3.$$

(a) Find the $3 \times 3$ symmetric matrix $A$ that represents $q$.

There is a typo, so the quadratic form is

$q(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 4x_1x_3$.

Then

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & a & 0 \\ 2 & 0 & 1 \end{pmatrix}$ is the matrix that represents $q$.

(b) For what real numbers $a$ is the quadratic form positive definite? For what values is it negative definite? For what values is it indefinite? Justify your answer.

One solution is to find the eigenvalues:

$$\det\begin{pmatrix} 1-\lambda & & 2 \\ & a-\lambda & \\ 2 & & 1-\lambda \end{pmatrix} = (1-\lambda)^2(a-\lambda) - 4(a-\lambda)$$
$$= ((1-\lambda)^2 - 4)(a-\lambda)$$
$$= ((1-\lambda)^2 - 2^2)(a-\lambda)$$
$$= (1-\lambda-2)(1-\lambda+2)(a-\lambda)$$
$$= -(1+\lambda)(3-\lambda)(a-\lambda)$$

So $\lambda_1 = -1$

$\lambda_2 = 3$

$\lambda_3 = a$

Since we have a positive eigenvalue, and a negative eigenvalue, $q$ is indefinite for all possible "$a$".

6. Consider the lines given by the parametric equations

$$\begin{cases} x = t \\ y = t \\ z = t \end{cases} \quad \text{and} \quad \begin{cases} x = 1 + 2t \\ y = 17 \\ z = 1 + 2t. \end{cases}$$

Find a parametric equation for the line which is perpendicular to both lines and passes through their point of intersection.

point of intersection

$t_1 = 1 + 2t_2$
$t_{\cancel{1}} = 17$
$t_1 = 1 + 2t_2$

when $t_1 = 17$,
$t_2 = 8$,
So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 17 \\ 17 \end{pmatrix}$

perpindicular to both lines: direction is cross-product of direction vectors:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 2\vec{i} + 0\vec{j} - 2\vec{k} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

So, line thru $\begin{pmatrix} 17 \\ 17 \\ 17 \end{pmatrix}$ in direction of $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$:

$x = 17 + 2t$
$y = 17$
$z = 17 - 2t$