1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

6/6 (a) The line through the points $(2,1,1)$ and $(1,2,2)$ intersects the line through the points $(1,1,1)$ and $(-9,7,7)$.

Line 1
slope is $(-1,1,1)$
$x = 2-t$
$y = 1+t$
$z = 1+t$

Line 2
slope is $(-5,3,3)$
$x = 1-5t$
$y = 1+3t$
$z = 1+3t$

y and z coordinates are always the same for both lines, so we can disregard z.

$2-t_1 = 1-5t_2$
$1+t_1 = 1+3t_2 \Rightarrow t_1 = 3t_2$
$2-3t_2 = 1-5t_2 \Rightarrow t_2 = -\frac{1}{2}$
$t_1 = -\frac{3}{2}$

Line1 at $t=-\frac{3}{2}$ is $(\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2})$
Lin2 at $t=-\frac{1}{2}$ is $(\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2})$ True

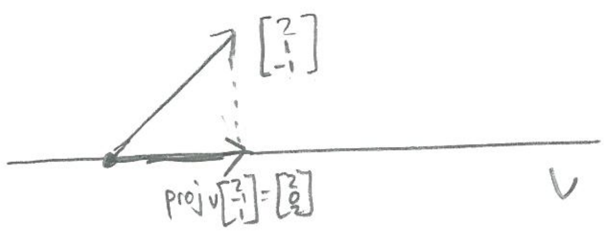
6/6 (b) There exist two unit vectors $\vec{v}, \vec{u} \in \mathbb{R}^n$ such that $\vec{u} \cdot \vec{v} = 2$.

False

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$ since $\vec{u}$ and $\vec{v}$ are both unit vectors, their magnitude is 1.
$= \cos\theta$
$-1 \leq \cos\theta \leq 1$, so $\vec{u}\cdot\vec{v} \neq 2$.

6/6 (c) There is a subspace $V \subset \mathbb{R}^3$ such that $\text{proj}_V \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

False, a projected vector cannot be of greater magnitude than the original vector,



$\left\| \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{6} < \left\| \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\| = \sqrt{8}$

cannot be true.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) Given a subspace V of $\mathbb{R}^4$, $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ is in both V and $V^\perp$.

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~~Sometimes.~~

~~True if $V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $V^\perp =$~~

~~False if $V = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$~~

Never.

the vector in V has to be orthogonal to the vector in $V^\perp$.
It could not be the same as $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \neq \vec{0}$.

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(b) Given a real number k, the curves $\begin{cases} x = t, \\ y = t^2 \end{cases}$ and $\begin{cases} x = t \\ y = k(t-1) \end{cases}$ intersect in two points.

Sometimes.

$\begin{cases} x = t_1 \\ y = t_1^2 \end{cases}$ $\begin{cases} x = t_2 \\ y = k(t_2 - 1) \end{cases}$

$\begin{cases} t_1 = t_2 \\ t_1^2 = k(t_2 - 1) \end{cases}$ $t_1^2 - kt_1 + k = 0$

$t_1 = \dfrac{k \pm \sqrt{k^2 - 4k}}{2}$

True, if $k^2 - 4k > 0$, for example $k = 5$, $t_1 = \dfrac{5 \pm \sqrt{5}}{2}$.
They intersect at $\left(\frac{5+\sqrt{5}}{2}, \frac{15+5\sqrt{5}}{2}\right)$ and $\left(\frac{5-\sqrt{5}}{2}, \frac{15-5\sqrt{5}}{2}\right)$

False, if $k^2 - 4k = 0$, $k = 4$, $t_1 = 2$, $\begin{cases} x = 2 \\ y = 4 \end{cases}$
They intersect only at $(2, 4)$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\frac{6}{6}$ (c) Given symmetric matrices A and B that have the same eigenvectors (but not necessarily the same eigenvalues), AB is symmetric.

Always.

Assume that $A = QD_1Q^{-1}$, $B = QD_2Q^{-1}$ where Q is orthogonal,

$AB = (QD_1Q^{-1})(QD_2Q^{-1}) = QD_1D_2Q^{-1}$.

Since D_1D_2 still equals to a diagonal matrix and Q is orthogonal,

$AB = QD_3Q^{-1}$ is orthogonally diagonalizable.

Therefore, it's symmetric.

3. Find the least-squares solution for the line best approximating the data points $(-1,6)$, $(0,0)$, $(2,10)$.

$f(-1)=6 \qquad c_0-c_1=6$

$f(0)=0 \qquad c_0=0$

$f(2)=10 \qquad c_0+2c_1=10$

$$\underbrace{\begin{pmatrix}1 & -1\\ 1 & 0\\ 1 & 2\end{pmatrix}}_{A}\begin{pmatrix}c_0\\ c_1\end{pmatrix}=\underbrace{\begin{pmatrix}6\\ 0\\ 10\end{pmatrix}}_{\vec{b}}$$

$$A^TA\vec{x}=A^T\vec{b}$$

$$\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 0\\ 1 & 2\end{pmatrix}\vec{x}=\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 2\end{pmatrix}\begin{pmatrix}6\\ 0\\ 10\end{pmatrix}$$

$$\begin{pmatrix}3 & 1\\ 1 & 5\end{pmatrix}\vec{x}=\begin{pmatrix}16\\ 14\end{pmatrix}$$

$$\vec{x}=\begin{pmatrix}\frac{33}{7}\\ \frac{13}{7}\end{pmatrix}$$

$$\therefore f(t)=\frac{33}{7}+\frac{13}{7}t$$

4. Suppose that A is a 3×3 matrix with eigenvectors

$$\begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}3\\-3\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\0\end{pmatrix}$$

corresponding respectively to the eigenvalues $2, 2, -3$. Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A, and use it to compute $A^3 \begin{bmatrix}1\\3\\2\end{bmatrix}$.

$$\vec{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix}3\\-3\\0\end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix}-1\\1\\2\end{bmatrix} - 0\begin{bmatrix}1\\1\\0\end{bmatrix} - \left(\frac{-6}{18}\right)\begin{bmatrix}3\\-3\\0\end{bmatrix}$$

$$\vec{b}_3 = \begin{bmatrix}-1\\1\\2\end{bmatrix} - \begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$\vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} \quad \vec{u}_2 = \frac{\vec{b}_2}{\|\vec{b}_2\|} = \begin{bmatrix}3/\sqrt{18}\\-3/\sqrt{18}\\0\end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\lambda = -3$ $\quad$ $\lambda = 2$ $\quad$ $\lambda = 2$

$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}3/\sqrt{18}\\-3/\sqrt{18}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \checkmark$$

$$\begin{bmatrix}1\\3\\2\end{bmatrix} = \left(\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right)\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} + \left(3/\sqrt{18} - 9/\sqrt{18}\right)\begin{bmatrix}3/\sqrt{18}\\-3/\sqrt{18}\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \left(\frac{4}{\sqrt{2}}\right)\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} + \left(-6/\sqrt{18}\right)\begin{bmatrix}3/\sqrt{18}\\-3/\sqrt{18}\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\\0\end{bmatrix} + \begin{bmatrix}-1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$A = (-3)\begin{bmatrix}2\\2\\0\end{bmatrix} + (2)\begin{bmatrix}-1\\1\\0\end{bmatrix} + (2)\begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$A^3 = (-3)^3\begin{bmatrix}2\\2\\0\end{bmatrix} + (2)^3\begin{bmatrix}-1\\1\\0\end{bmatrix} + (2)^3\begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$= (-27)\begin{bmatrix}2\\2\\0\end{bmatrix} + \begin{bmatrix}-8\\8\\0\end{bmatrix} + \begin{bmatrix}0\\0\\8\end{bmatrix}$$

$$= \begin{bmatrix}-54\\-54\\0\end{bmatrix} + \begin{bmatrix}-8\\8\\0\end{bmatrix} + \begin{bmatrix}0\\0\\8\end{bmatrix} = \boxed{\begin{bmatrix}-62\\-46\\8\end{bmatrix}} \checkmark$$

$q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$

5. Sketch the curve described by the equation

$$3x^2 - 2xy + 3y^2 = 1.$$

Label the principal axes and their points of intersection with the curve.

$q(\vec{x}) = 3x^2 - 2xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (3-\lambda)(3-\lambda) - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2) = 0$

$\hookrightarrow \lambda = 4, \lambda = 2$

$\lambda = 4 \rightarrow \ker \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \rightarrow \text{eigenvector} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\lambda = 2 \rightarrow \ker \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \rightarrow \text{eigenvector} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

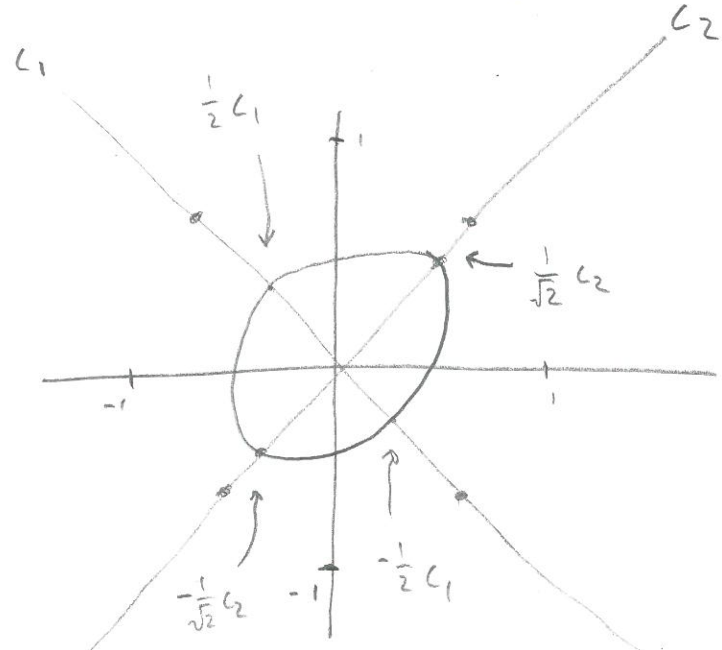
$q(\vec{x}) = 4c_1^2 + 2c_2^2 = 1 \rightarrow$ ellipse b/c $c_1 + c_2 > 0$

$4c_1^2 = 1$

$c_1 = \sqrt{\frac{1}{4}} = \pm \frac{1}{2}$

$2c_2^2 = 1$

$c_2 = \sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}}$



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6. Let $V = \text{span}\left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\4\end{bmatrix}, \begin{bmatrix}0\\0\\6\\-3\end{bmatrix}\right)$.

(a) Find an orthonormal basis for V.

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1 = \begin{bmatrix}2\\1\\0\\4\end{bmatrix} - \left(\begin{bmatrix}1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}2\\1\\0\\4\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\\0\\4\end{bmatrix} - 6/\sqrt{3}\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\\0\\4\end{bmatrix} - \begin{bmatrix}2\\0\\2\\2\end{bmatrix} = \begin{bmatrix}0\\1\\-2\\2\end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\-2\\2\end{bmatrix} = \begin{bmatrix}0\\1/3\\-2/3\\2/3\end{bmatrix}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1\cdot\vec{v}_3)\vec{u}_1 - (\vec{u}_2\cdot\vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix}0\\0\\6\\-3\end{bmatrix} - \left(\begin{bmatrix}1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}0\\0\\6\\-3\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} - \left(\begin{bmatrix}0 & 1/3 & -2/3 & 2/3\end{bmatrix}\begin{bmatrix}0\\0\\6\\-3\end{bmatrix}\right)\begin{bmatrix}0\\1/3\\-2/3\\2/3\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\6\\-3\end{bmatrix} - 3/\sqrt{3}\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} + 6\begin{bmatrix}0\\1/3\\-2/3\\2/3\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\6\\-3\end{bmatrix} - \begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\2\\-4\\4\end{bmatrix} = \begin{bmatrix}-1\\0\\5\\-4\end{bmatrix} + \begin{bmatrix}0\\2\\-4\\4\end{bmatrix} = \begin{bmatrix}-1\\2\\1\\0\end{bmatrix} \qquad \vec{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\\0\end{bmatrix} = \begin{bmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix}$$

orthonormal basis is $\left\{\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}0\\1/3\\-2/3\\2/3\end{bmatrix}, \begin{bmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix}\right\}$

(b) Write the standard matrix for $\mathrm{proj}_V : \mathbb{R}^4 \to \mathbb{R}^4$ as a product of matrices. (You do not have to multiply them out.)

$$QQ^T = \begin{bmatrix} 1/\sqrt{3} & 0 & -1/\sqrt{6} \\ 0 & 1/3 & 2/\sqrt{6} \\ 1/\sqrt{3} & -2/3 & 1/\sqrt{6} \\ 1/\sqrt{3} & 2/3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/3 & -2/3 & 2/3 \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} & 0 \end{bmatrix}$$

(c) Find the point on V closest to $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathrm{proj}_V \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2 + (\vec{u}_3 \cdot \vec{v})\vec{u}_3$$

$$= 3/\sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ 1/3 \\ 5/3 \end{bmatrix}$$

7. Let $A = \frac{1}{3}\begin{bmatrix} 2 & b & c \\ a & -b & b \\ 2 & -1 & -b \end{bmatrix}$. Find all values of a, b, and c, if any, such that $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^3$.

if $\|A\vec{x}\| = \|\vec{x}\|$

then A is orthogonal matrix +3

$$\left(\frac{2}{3}\right)^2 + \left(\frac{a}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = 1$$

$$\frac{8}{9} + \left(\frac{a}{3}\right)^2 = 1$$

$$a = 1 \text{ or } -1$$

$$\left(\frac{b}{3}\right)^2 + \left(\frac{-b}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 = 1$$

$$\frac{2}{9}b^2 = \frac{8}{9}$$

$$b^2 = 4$$

$$b = \pm 2$$

$$\left(\frac{c}{3}\right)^2 + \left(\frac{b}{3}\right)^2 + \left(\frac{-b}{3}\right)^2 = 1$$

$$c = \pm 1$$

$\begin{cases} a = 1 \\ b = 2 \\ c = 1 \end{cases}$ ~~$a = 1$, $b = 2$~~ ~~$a = 1$~~

+9

is the only solution