1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

   (a) If A is any square matrix and $q(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$ is a quadratic form, then A must be diagonalizable.

   Answer: FALSE

   Counterexample: $q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
   $= \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x+y \\ y \end{bmatrix} = x^2 + xy + y^2$

   is a quadratic form, but

   $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

   (b) If A is a symmetric matrix and $\mathbf{v}$ and $\mathbf{w}$ and vectors with $\mathbf{v}$ in the kernel of A and $\mathbf{w}$ in the image of A, then $\mathbf{v} \cdot \mathbf{w} = 0$.

   Answer: TRUE

   Since $\vec{w}$ is in the image of A, $\vec{w} = A\vec{x}$ for some $\vec{x}$.

   So $\vec{v} \cdot \vec{w} = \vec{v} \cdot (A\vec{x}) = (A^T\vec{v}) \cdot \vec{w} = (A\vec{v}) \cdot \vec{w}$ since A is symmetric
   $= \vec{0} \cdot \vec{w} = 0.$
   since $\vec{v} \in \ker A$.

   Alternatively, we know $\operatorname{Ker}(A^T) = (\operatorname{im} A)^\perp$, so

   $\vec{v} \in \operatorname{Ker}(A)$
   $\Rightarrow \vec{v} \in \operatorname{Ker}(A^T)$ since A is symmetric
   $\Rightarrow \vec{v} \in (\operatorname{im} A)^\perp$
   $\Rightarrow \vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in \operatorname{im} A$, as desired.

(c) If **a**, **b**, **c** are any nonzero vectors in $\mathbb{R}^3$, then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}$.

Answer: FALSE

Let $\vec{a} = \vec{e}_1$, $\vec{b} = \vec{e}_2$ and $\vec{c} = \vec{e}_3$.

Then $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_3 = 1$.

$(\vec{c} \times \vec{b}) \cdot \vec{a} = (\vec{e}_3 \times \vec{e}_2) \cdot \vec{e}_1 = - \vec{e}_1 \cdot \vec{e}_1 = -1$.

So, $(\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3 \neq (\vec{e}_3 \times \vec{e}_2) \cdot \vec{e}_1$

(d) If V is a nonzero subspace of $\mathbb{R}^n$, then V has an orthonormal basis.

Answer: TRUE

Let V be a nonzero subspace of $\mathbb{R}^n$. There exists a basis $\{\vec{v}_1, \ldots, \vec{v}_m\}$ for V. Applying the Gram-Schmidt process to $\vec{v}_1, \ldots, \vec{v}_m$, we obtain an orthonormal set $\{\vec{u}_1, \ldots, \vec{u}_m\}$ such that

$$\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = V.$$

Each orthonormal set is linearly independent. Therefore, $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is an orthonormal basis for V.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

(a) Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set of vectors in $\mathbb{R}^n$. Then

$$\{\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_3, \mathbf{u}_1 - \mathbf{u}_3\}$$

is also an orthonormal set.

Answer: NEVER

Note that $\|\vec{u}_1 - \vec{u}_2\|^2 = (\vec{u}_1 - \vec{u}_2)\cdot(\vec{u}_1 - \vec{u}_2) = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 = 2$, so the vectors do not have length 1.

(b) For a square matrix Q whose columns are perpendicular to one another, $Q^T Q = I$.

Answer: SOMETIMES

Example: $Q = I$, so $Q^T Q = II = I$

Counterexample: $Q = 0$, so $Q^T Q = 00 = 0 \neq I$

(c) For an $n \times m$ matrix A and a vector $\mathbf{b}$ in $\mathbb{R}^n$, there is a vector $\mathbf{x}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \text{proj}_{\text{im}\,A}\,\mathbf{b}$.

Answer: ALWAYS

$\text{proj}_{\text{im}A}\vec{b}$ is in im A, and anything in the image of A can be written as $A\vec{x}$ for some $\vec{x}$

or the solution of $A^TA\vec{x} = A^T\vec{b}$, which always exists, satisfies $A\vec{x} = \text{proj}_{\text{im}A}\vec{b}$

(d) For a number k, the line with parametric equations

$$x = 4 - 2t,\ y = k + t,\ z = 2k - 1$$

intersects the surface described by the equation $(x-4)^2 + (y-k)^2 + (z+1)^2 = 1$.

Answer: SOMETIMES

Plugging in (x,y,z) values:

$$\underbrace{(-2t)^2}_{x-4} + \underbrace{(t)^2}_{y-k} + \underbrace{(2k)^2}_{z+1} = 1 \quad \text{becomes}$$

$$5t^2 = 1 - 4k^2 \quad \text{or} \quad t^2 = \frac{1-4k^2}{5}.$$

This has solution for t when $k=0$

but not when $k = 10$

3. At O'Hare airport, the average low temperature (in degrees Celsius) t months into the year is listed below:

t	°C
0	-8
3	5
6	20
9	8

Find the function of the form $f(t) = c_0 + c_1 \sin(\frac{\pi t}{6}) + c_2 \cos(\frac{\pi t}{6})$ which best fits the data above, using least squares.

$f(0) = c_0 + c_2 = -8$

$f(3) = c_0 + c_1 = 5$

$f(6) = c_0 - c_2 = 20$

$f(9) = c_0 - c_1 = 8$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(A^TA)^{-1}A^T b = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -8 \\ 5 \\ 20 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 25 \\ -3 \\ -28 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{4} \\ -\frac{3}{2} \\ -14 \end{bmatrix}$$

so $f(t) = \frac{25}{4} - \frac{3}{2}\sin(\frac{\pi}{6}t) - 14\cos(\frac{\pi}{6}t)$

4. (This problem has **three** parts.) Define the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ by

$$q(x, y) = 4x^2 + 4xy + 7y^2.$$

(a) Determine whether q is positive definite, negative definite, or indefinite.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

$$\det(A - xI) = \det\begin{pmatrix} 4-x & 2 \\ 2 & 7-x \end{pmatrix}$$
$$= (x-4)(x-7) - 4$$
$$= x^2 - 11x + 28 - 4$$
$$= x^2 - 11x + 24$$
$$= (x-8)(x-3)$$

$x = 8, 3$

positive definite as all eigenvalues > 0.

(b) Find a set of principal axes for q.

E_3: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x = -2y$

get $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

E_8: $\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$2x = y$

get $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So principal axes

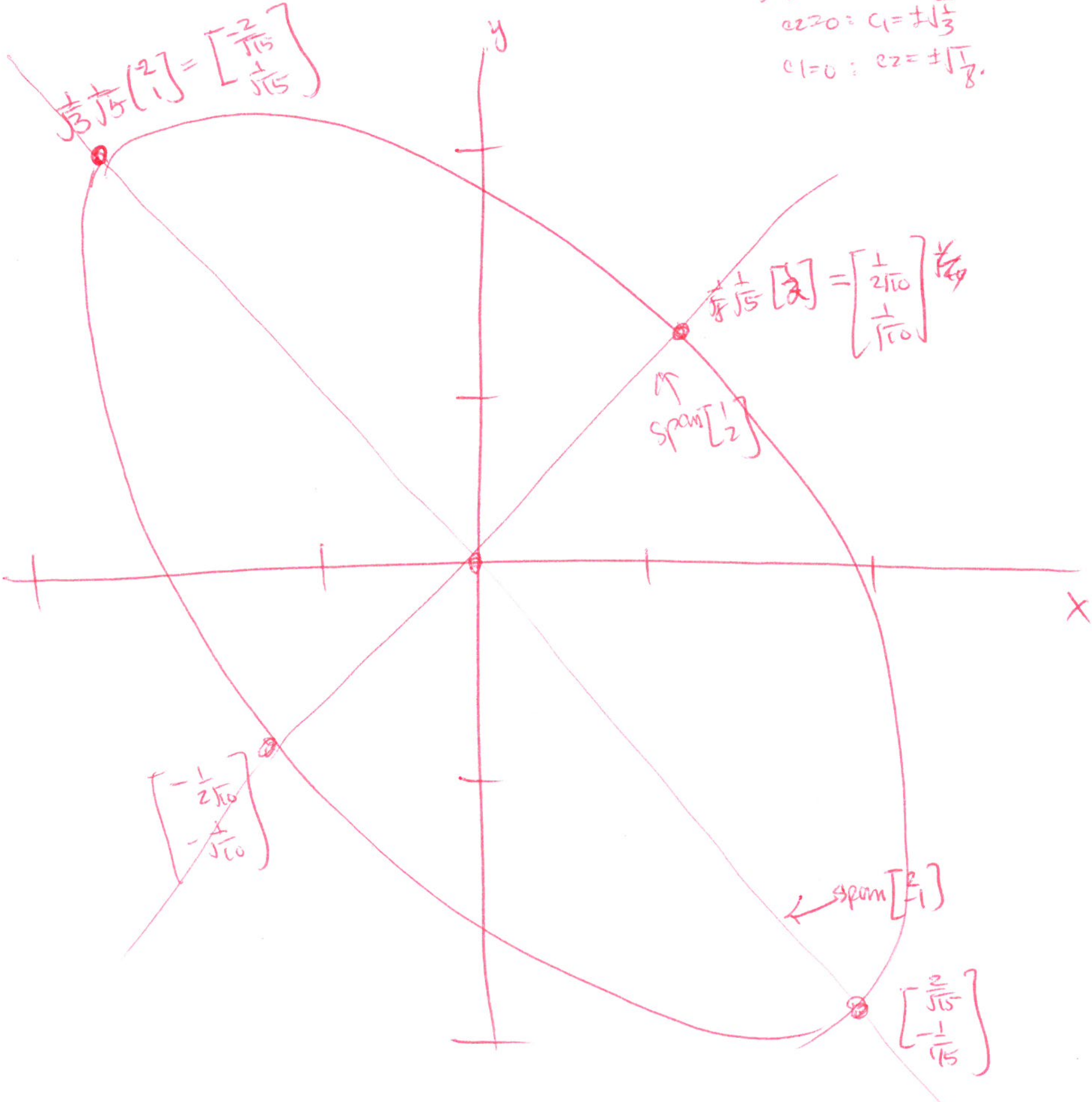
$\operatorname{span}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ & $\operatorname{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Draw the curve whose equation is $4x^2 + 4xy + 7y^2 = 1$, labeling the principal axes and the intercepts of the curve with these axes. The intercepts should be labeled by their standard (x, y) coordinates.

$3c_1^2 + 8c_2^2 = 1$

$c_2 = 0: \ c_1 = \pm\sqrt{\frac{1}{3}}$

$c_1 = 0: \ c_2 = \pm\sqrt{\frac{1}{8}}.$



5. Let P be the plane $x + 2y + 4z = 0$ and let $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Find the distance from $\mathbf{x}$ to P.

Method 1

$$\text{distance} = \left\| \text{proj}_{\vec{n}} \vec{x} \right\| \quad \text{where } \vec{n} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \text{ is normal vector}$$

$$\text{proj}_{\vec{n}} \vec{x} = \left(\frac{\vec{x} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \right) \vec{n} = \frac{12}{21} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

$$\text{distance} = \left\| \frac{12}{21} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\| = \frac{12}{21} \sqrt{21}$$

Method 2

$$\text{plane} = \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\text{Set } A = \begin{pmatrix} -2 & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

$$\text{Solve } A^T A \vec{y} = A^T \vec{x} \longrightarrow \begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix} \vec{y} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}$$

Same

$$\longrightarrow \vec{y} = \begin{pmatrix} -15/7 \\ 5/7 \end{pmatrix}$$

$$\text{Then } \text{proj}_P \vec{x} = A\vec{y} = \begin{pmatrix} 10/7 \\ -15/7 \\ 5/7 \end{pmatrix}$$

$$\text{and distance} = \left\| \vec{x} - \text{proj}_P \vec{x} \right\| = \frac{4\sqrt{21}}{7}$$

6. (This problem has **two** parts.) Suppose that A is a 4×4 matrix with eigenvalues 2 and -1, and associated eigenvectors

$$\underset{v_1}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-2\\0\\0\\-2\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}1\\4\\2\\1\end{bmatrix}} \text{ for 2 and } \underset{v_4}{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}} \text{ for } -1.$$

(a) Find a basis of $\mathbb{R}^4$ consisting of orthonormal eigenvectors of A.

Apply GS to $\{v_1, v_2, v_3\}$:

orthogonalize: $w_1 = v_1$

$$w_2 = \begin{pmatrix}-2\\0\\0\\-2\end{pmatrix} - \frac{-4}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}$$

$$w_3 = \begin{pmatrix}1\\4\\2\\1\end{pmatrix} - \frac{8}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{4}{4}\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

normalize:

$$\boxed{u_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad u_2 = \frac{1}{2}\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \quad u_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}}$$

v_4 is already $\perp$ to v_1, v_2, v_3.

(b) Compute $A^3\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}$ explicitly.

$$\begin{pmatrix}1\\2\\-1\\1\end{pmatrix} = 3\cdot\frac{1}{4}\underbrace{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}_{w_1} - 1\cdot\frac{1}{4}\underbrace{\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}}_{w_2} + 3\cdot\frac{1}{2}\underbrace{\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}}_{w_3} + 0\cdot\frac{1}{2}\underbrace{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}}_{w_4}$$

$$A^3\begin{pmatrix}1\\2\\-1\\1\end{pmatrix} = \frac{3}{4}A^3w_1 - \frac{1}{4}A^3w_2 + \frac{3}{2}A^3w_3$$

$$= \frac{3}{4}2^3w_1 - \frac{1}{4}2^3w_2 + \frac{3}{2}\cdot 2^3\cdot w_3$$

$$= 2^3\cdot\begin{pmatrix}1\\2\\-1\\1\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}8\\16\\-8\\8\end{pmatrix}}$$